ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

ΣΥΝΕΔΡΙΑ ΤΗΣ 8ΗΣ ΔΕΚΕΜΒΡΙΟΥ 1983

ΠΡΟΕΔΡΙΑ ΜΕΝΕΛΑΟΥ ΠΑΛΛΑΝΤΙΟΥ

ΜΗΧΑΝΙΚΗ.— **Yield criteria depending on pressure and dilatancy,** *by P. S. Theocaris**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Περικλῆ Θεοχάρη.

ABSTRACT

Previously introduced yield criteria taking into consideration the influence of internal dilation of the materials during yielding were based on a conventional assumption of the influence of the hydrostatic component of stresses on the yielding process and therefore they were fitted to the actual macroscopic behaviour of the materials. They were proved experimentally to predict the overall plastic behaviour of a great number of substances. In this paper a yield criterion was tested which was based on the theory of void growth and coalescence in the vicinity of internal discontinuities of the material during yielding and fracture. The inverse dependence of yielding on the hydrostatic tension is incorporated, whose increase creates a rapid decrease of fracture ductility.

INTRODUCTION

The various yield criteria introduced in mechanics of isotropic and elastic-plastic materials should depend on the magnitudes of the three principal applied stresses and not on their directions. Therefore, any yield criterion should have the form [**1**] :

$$f(I_1, I_2, I_3) = 0 \qquad (1)$$

* Π. Σ. ΘΕΟΧΑΡΗ, **Κριτήρια διαρροῆς ἐξαρτώμενα ἀπὸ τὴν πίεσιν καὶ διόγκωσιν.**

where $I_1$, $I_2$ and $I_3$ are the first three invariants of the stress tensor $\sigma_{ij}$ defined by

$$I_1 = (\sigma_1 + \sigma_2 + \sigma_3), \quad I_2 = -(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) \quad \text{and} \quad I_3 = \sigma_1 \sigma_2 \sigma_3 \qquad (2)$$

Furthermore, yield criteria should satisfy the obligation to be expressed by functions, f, which must be symmetric in the three principal stresses.

The first and simplified forms of the yield criteria assumed the approximation that yielding is unaffected by the first stress invariant $I_1$ for cases of hydrostatic tensions or compressions either applied alone, or superimposed on some state of combined stress.

Then, the ideal plastic body was assumed to yield under the influence of only the deviatoric stress tensor expressed by :

$$s_{ij} = \sigma_{ij} - \sigma\delta_{ij} \qquad (3)$$

where $\sigma = \sigma_{ii}/3$, $\delta_{ij}$ the Kronecker delta, and the indices i, j running between 1, 2 and 3.

According to this assumption the yield criteria are reduced to the form :

$$f(J_2, J_3) = 0 \qquad (4)$$

where

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \quad \text{and} \quad J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki} \qquad (5)$$

A further restriction, which is only valid for ideally plastic bodies, has been introduced, according to which the magnitude of the yield stress is the same in simple tension and compression. This assumption is never met, except perhaps only for ideal single crystals. All polycrystalline materials present some small or large difference in yielding under simple tension and compression.

Furthermore, since reversal of one stress influences the sign of $J_3$ reversing also it, fact which violates the symmetry obligation of the yield criterion, it follows that $f(J_2, J_3) = 0$ should be an even function of $J_3$.

Finaly, because of the assumptions of isotropy and non-existence of the Bauschinger effect, an interchange in the order of succession of the principal stresses and their opposite values must not influence the yield locus. Therefore, the shape of the yield locus may be fully defined by one of its twelve 30-degree segments limited by a simple axial load and a pure shear loading.

Leaving aside Tresca's and Coulomb's yield criteria, which are depending either on the maximum shear stress or on a combination of the maximum shear stress and the hydrostatic stress, we are concentrating to yield criteria based on energy assumptions. Among them and the most ancient is the so-called Huber - Mises - Hencky criterion [**2**].

According to this criterion yielding occurs when the second stress invariant $J_2$ attains a critical value equal to this energy in simple tension ($\sigma_0$). Hencky [**3**] was the first who interpreted this criterion that it corresponds to the fact that yielding begins when the recoverable elastic distortional energy reaches a critical value equal to the same quantity in simple tension or in pure shear. Then, in this criterion the $J_3$-invariant is not involved and, furthermore, it is assumed that hydrostatic tension or pressure does not cause any yielding of the material. This may be approximately true for isotropic metallic specimens when yielding of the overall metallic body is anticipated.

A simpler than Mises's criterion is Tresca's criterion [**4**], according to which yielding occurs when the maximum shear stress in the body reaches a certain value, that is the respective maximum shear stress in simple tension. Although this criterion is basically not well founded, since it considers only shear stresses and not energies for yielding, where all stresses acting on the body participate for yielding, it is frequently used because of its simpicity.

However, for soils and other materials, where the hydrostatic component of stresses interferes in yielding and fracture and where yielding in simple tension is much different to yielding in simple compression, an improvement of the Tresca criterion is the criterion introduced by Coulomb [**5**] and Mohr [**6**].

According to this criterion elastic strains are neglected and plastic strain-increment is assumed depending only on the applied stress. The condition that there is no strain in the z-direction (along the thickness of plane-strain cases) implies a functional dependence of the $\sigma_z$-stress on the principal stresses $\sigma_1$ and $\sigma_2$ along the planes of flow. If the reversal of the stresses assumes a reversal of strain increments then the $\sigma_z$-stress is equal to

$(\sigma_1 + \sigma_2)/2$ and may be, therefore, eliminated from the yield criterion so that the yielding function is reduced to

$$f(p, \tau_{max}) = 0 \tag{6}$$

where

$$p = -\frac{1}{2}(\sigma_1 + \sigma_2) \quad \text{and} \quad \tau_{max} = \frac{1}{2}(\sigma_2 - \sigma_1) \tag{7}$$

that is p is the mean compressive stress taking the place of hydrostatic pressure and $\tau_{max}$ is the maximum shear stress. Then the yield criterion may be represented by a locus referred to coordinate-axes $1/2(\sigma_1 + \sigma_2)$ and $1/2(\sigma_2 - \sigma_1)$. This yield locus is closed on the tension side, if it is assumed that a pure hydrostatic tension may alone produce yielding. Then, Tresca's yield condition, where $\tau_{max} = \pm k$, where k is the maximum shear stress in yielding in a state of pure shear, may be assumed as a special case of Mohr's yield condition for the case when f is independent of p and the Mohr envelope degenerates into a pair of parallel lines to the normal-stress $\sigma$-axis.

Coulomb - Mohr's criterion is not the only criterion depending on the hydrostatic component of stresses. In this paper such criteria will be established and discussed and important results were derived concerning the behaviour in yielding and fracture for real materials.

## THE GENERATION OF EMPIRICAL YIELD CRITERIA DEPENDING ON DILATION

It was as early as 1904 that Huber has introduced his yield or brittle fracture criterion, where he distinguished two cases depending on whether the hydrostatic component of stress applied to the specimen was tensile or compressive. For compression he introduced a criterion based on the distortional component of the elastic energy, whereas for tension the criterion depended on the total elastic energy [**7**].

Afterwards, von Mises [**8**] and independently Schleicher [**9**] have introduced the notion of the equivalent critical yielding stress, instead of that of simple shear k, of an arbitrary function of the hydrostatic component of stresses. The criterion was convenient for materials whose yielding depended on hydrostatic tension or compression and therefore they presented different critical values for yielding under the different modes of loading.

This criterion was a general one, incorporating all previous criteria taking into account the influence of hydrostatic component of stress.

Although from the early tests for the determination of the mechanical properties of the materials it has been realized that very seldom the yield stress in simple tension coincided with the same quantity in simple compression, it was assumed at least for the ductile metals, where this difference was not so important, to exist a complete symmetry for the yield locus in the tension and compression spaces. Thus, Tresca's and Mises' yield conditions were accepted as describing universally the plastic behaviour of ductile substances.

On the contrary, in brittle materials, where the ratio of the yield stress in simple compression $\sigma_{oc}$ was always much different to the yield stress in simple tension $\sigma_{ot}$, it was accepted from the early beginnings that a Mohr - Coulomb type of yield locus was describing appropriately the plastic behaviour of these substances. Although the Mohr - Coulomb, or internal friction, criterion fitted satisfactorily the results for non-metallic, stony, earthy or concrete specimens, this acceptable coincidence between theory and experiments may be due to the large scattering of results with tests of such materials.

However, tests with brittle metallic materials, such as cast iron, bronze, etc. have also shown a strong dependence of yielding on the strength-differential effect of the materials, expressed by the ratio $R = \sigma_{oc}/\sigma_{ot}$. Experiments with gray cast-iron thin-walled tubes executed by Coffin [**10**], presenting a strength differential effect $R = 3.0$ ($\sigma_{oc} = 100 \times 10^3$ psi, $\sigma_{ot} = 33 \times \times 10^3$ psi), and Grassi and Cornet [**11**] (with $\sigma_{ot} = 28.5 \times 10^3$ psi and $\sigma_{oc} = = 96.0 \times 10^3$ psi) obeyed satisfactorily a hydrostatic-stress depending Mises criterion. According to this conception the strength-differential effect arises in initially isotropic materials as a result of the dependence of the yield criterion on the first stress invariant $J_1$. In this case, there exists also a predictable change of volume. Fig. 1 presents the yield loci of these materials normalized to the yield stress $\sigma_0$ in simple tension. It is clear from this figure that all experimental results fit excellently the stress-differential modified yield criterion (SDM-criterion). Although there are not sufficient data in the compression-compression quadrant, it is clear that the material follows such a form of criteria. If one considers further all discarding of results, which were assumed as non compatible with existing theories in the time

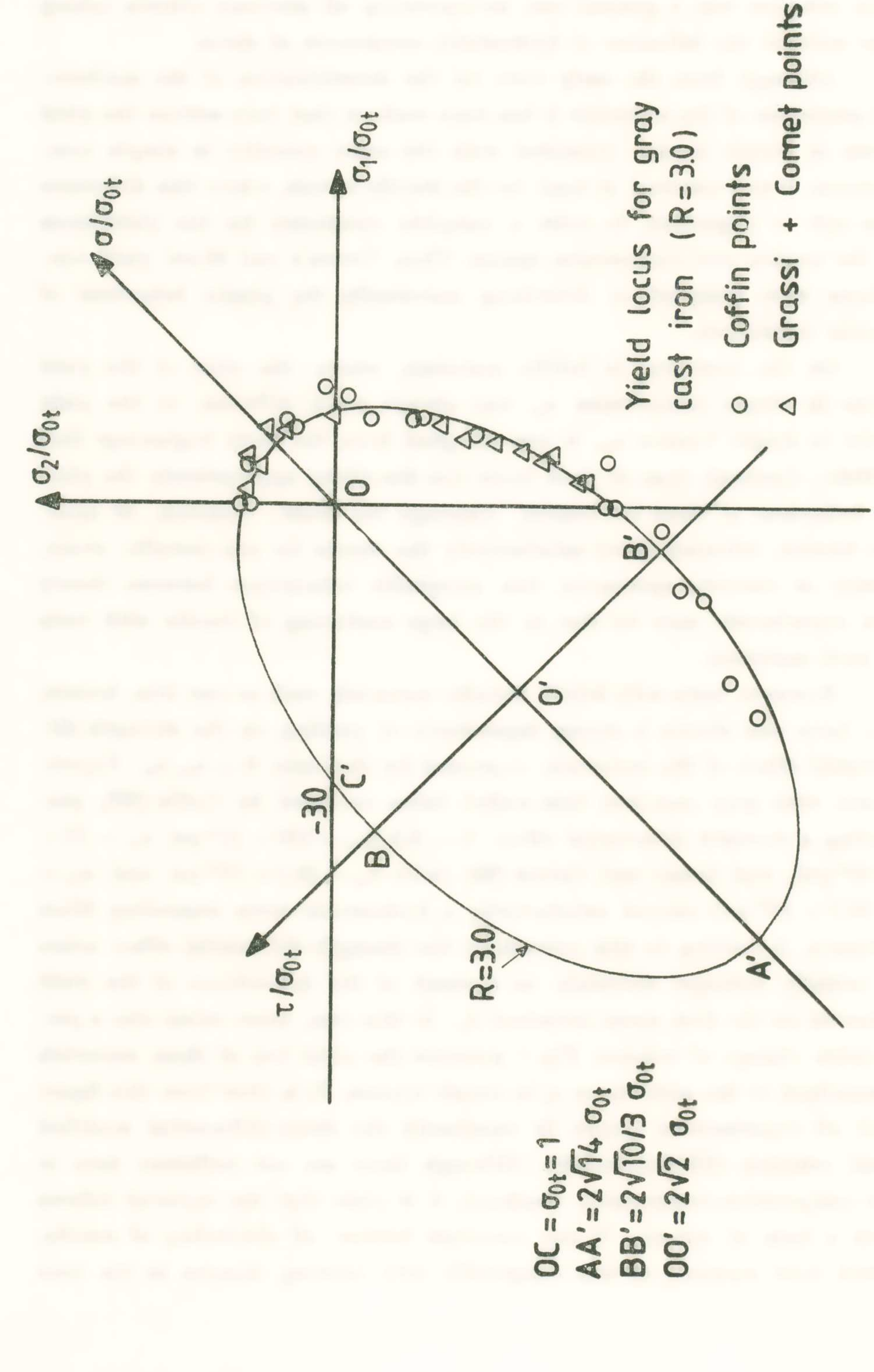



Fig. 1. The yield locus for gray cast-iron with R = 3.0 and the experimental points derived from tests of Coffin and Grassi and Cornet.

of execution of the experiments, one may assumes that the results given could not be avoided.

Furthermore, the reliable early experiments by Taylor and Quinney [**12**] indicate also clearly that, while aluminium and copper with R = 1.0 obey excellently the Mises yield criterion, mild-steel specimens deviate considerably all the experimental points, lying outside the Mises yield locus. It can be readily proven that these values obey a hydrostatic-stress depending Mises criterion with R = 1.3. Fig. 2 presents the results of Taylor and Quinney, as well as those of Lode [**13**] for various types of steels and copper, which again show an excellent agreement with the strength-differential modified criterion valid for R = 1.3.

Yield criteria based on the hydrostatic-stress modified Mises criterion were extended during the last twenty years to predict the yielding behaviour of high-polymers. This empirical criterion complements the typical Mises criterion by a term which depends on the hydrostatic component of stress and it is proportional to the difference of yield stresses in uniaxial compression and tension, so that if these two quantities are equal, the modified criterion reduces to the typical Mises criterion. This criterion is expressed by :

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6\,(\sigma_{oc} - \sigma_{ot})\,p = 2\sigma_{ot}\,\sigma_{oc} \qquad (8)$$

where p denotes the hydrostatic component of stresses given by :

$$p = \sigma_{ii} / 3 \quad (i = 1, 2, 3) \qquad (9)$$

This type of criterion in its three-dimensional form was suggested by Schleicher [**9**] and elaborated by Stassi d'Alia [**14**] and Tschoegl [**15**]. Raghava, Caddell and Yeh [**16**] have applied it to the yield behaviour of some polymers and compared it with other forms of thé same idea. Theocaris [**17**] has discussed its application and Gdoutos [**18**] used it for studying the initiation of plastic zones at the tips of cracks in infinite isotropic and elastic-plastic plates, whereas Theocaris et al. [**19**] used it just to show the influence of mechanical properties of a bi-material plate when a crack existing in the one phase approaches the interface.

The form of this criterion complies with the limiting cases of loading, that is when $\sigma_{oc} = \sigma_{ot}$, it reduces to the classical Mises yield condition, which is independent of the influence of the hydrostatic stress. Moreover, for very

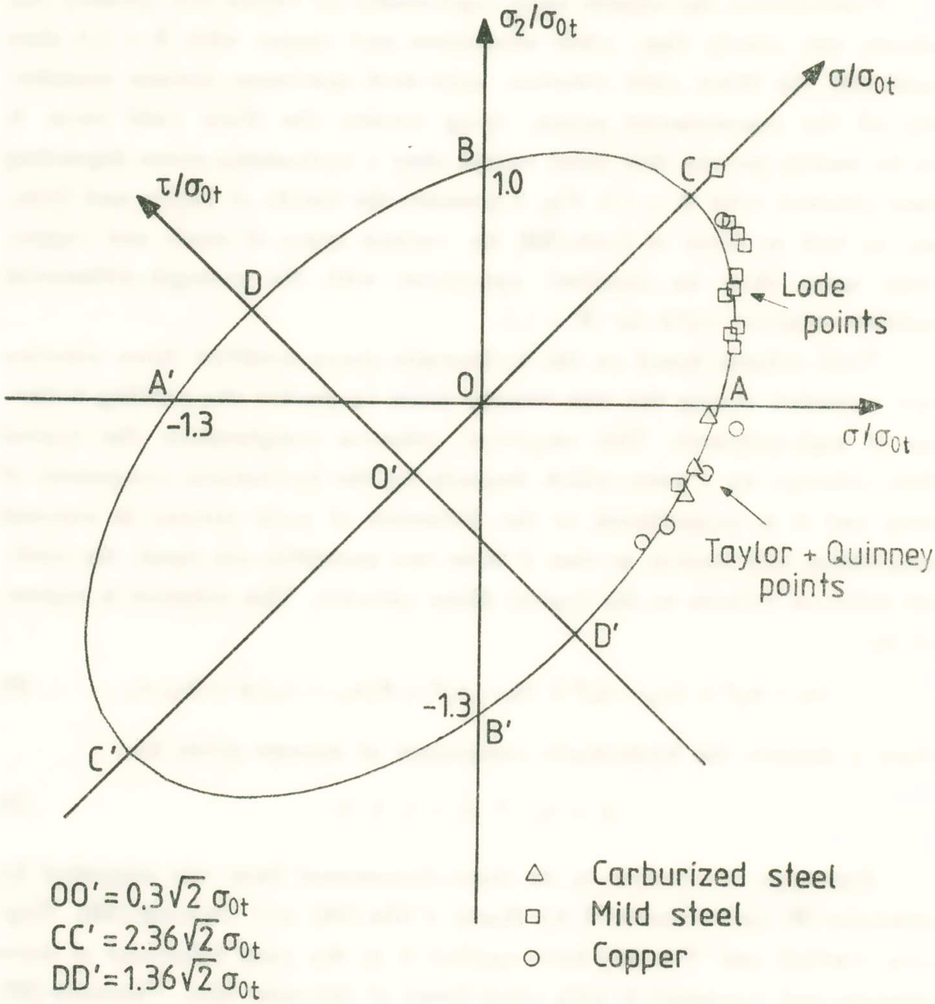



Fig. 2. The experimental results of Taylor and Quinney and Lode for the yield loci of various kinds of steels and coppers and their coincidence with the stress-differential modified yield criterion with R = 1.3.

brittle materials for which $\sigma_{oc} \gg \sigma_{ot}$, so that it may be assumed that $\sigma_{oc} / \sigma_{ot}$ tends to infinity, the yields locus degenerates to a form which is very similar to the Mohr - Coulomb criterion in its general form.

However, there is no explanation why the influence of the hydrostatic component of stress is linear, with a linearity factor depending on the difference $(\sigma_{oc} - \sigma_{ot})$ of the yield stresses in simple compression and tension. Moreover, there is no explanation why the square of the yield stress $\sigma_{ot}$ in simple tension, existing in the right-hand side of the Mises criterion, should be replaced by the geometric mean value of the two yield stresses $\sigma_{ot}$ and $\sigma_{oc}$.

Experimental evidence with metallic specimens, however, indicated that, whereas Tresca's yield condition is in general non-compatible with reality, Mises' yield condition constitutes a lower bound for the yield loci corresponding to R = 1.0. As soon as there is a slight difference between yield stresses in simple tension and compression the yield loci lie always on the one side of the Mises yield locus and they recede from it, as R is increasing. The figure 4 in the book of Hill [**1**] gives the yield loci for copper, aluminium and mild steel specimens, based on the experiments of Taylor and Quinney. This figure also shows implicitly, but clearly, the influence of R, which for mild steel is of the order of R = 1.3.

The *Schleicher - Stassi* criterion expressed by relation (8), which for plane stress conditions becomes :

$$(\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2) + 3 (\sigma_{oc} - \sigma_{ot})\, p = \sigma_{oc} \sigma_{ot} \qquad (10)$$

consists of three terms, from which the first expresses the distortional component of energy and corresponding to the classical Mises yield condition, the second term expresses an elastic energy depending on hydrostatic stress p and the difference in yield stresses for compression and tension, whereas the right-hand side term is the geometric mean of these two yield stresses.

Addition of the three terms in Eq. (10) is legitimate, since these terms express energy quantities. They tend to a limit when $\sigma_{oc} = \sigma_{ot}$ reducing to the classical Mises yield condition for ductile materials. Moreover for $\sigma_{oc} \gg \sigma_{ot}$, so that it may be assumed that $\sigma_{ot} / \sigma_{oc} \rightarrow 0$, Eq. (10) yields a limiting ellipse, which is equal in size with the Mises ellipse with $\sigma_{oc} = \sigma_{ot}$ and it passes through the origin of coordinates in a $(\sigma_1/\sigma_{oc}, \sigma_2/\sigma_{oc})$-diagram and the points (—1, 0) and (0, —1). All other ellipses referred to the same yield stress in simple compression $\sigma_{oc}$ are smaller in size than these two limit curves and they pass all of them through the points (—1, 0) and (0, —1). Fig. 3 indicates the family of SDM yield-loci referred to the same yield stress in simple compression. It is obvious from this figure that the ellipses for R = 1.0 and R = ∞ are equal.

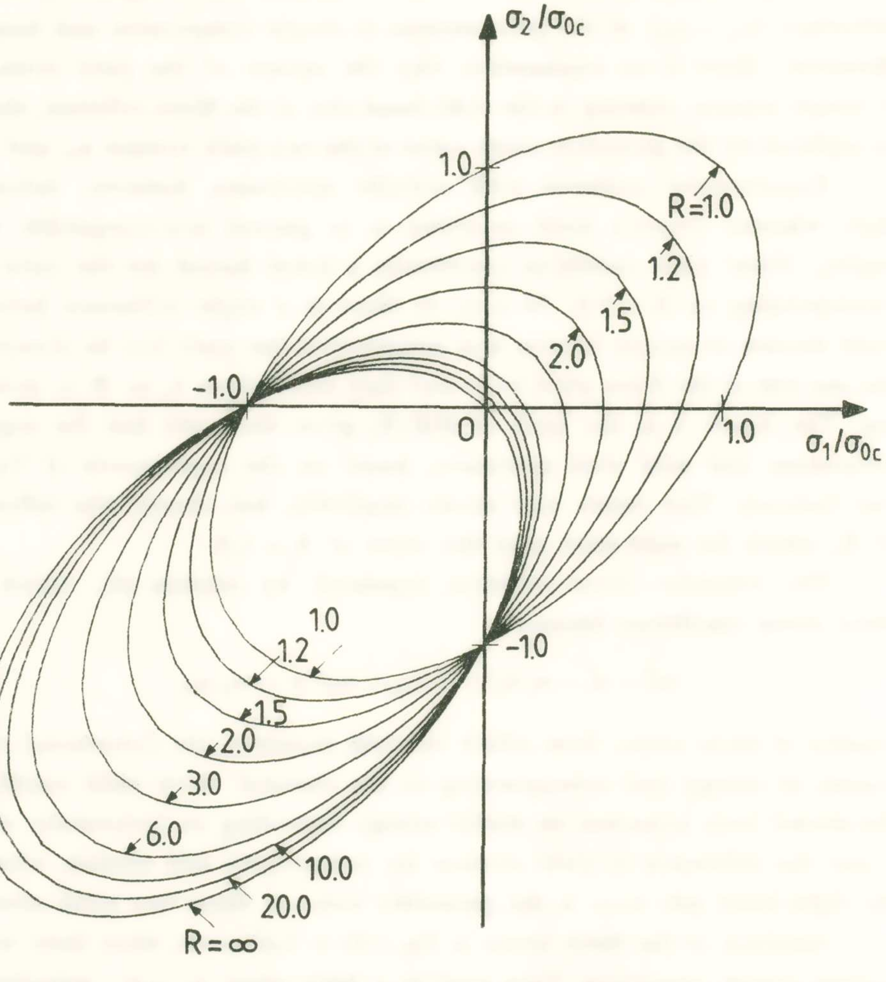



Fig. 3. The family of the stress-differential modified yield criteria with R varying between unity and infinity for the same yield-stress in uniaxial compression.

Fig. 4 presents a similar family of yield loci for the same yield stress in simple tension. Although both families are ellipses, the sizes and forms of these curves are much different in either family.

Although the Schleicher - Stassi criterion is a satisfactory general criterion, valid for the whole range and diversity of materials, since it con-

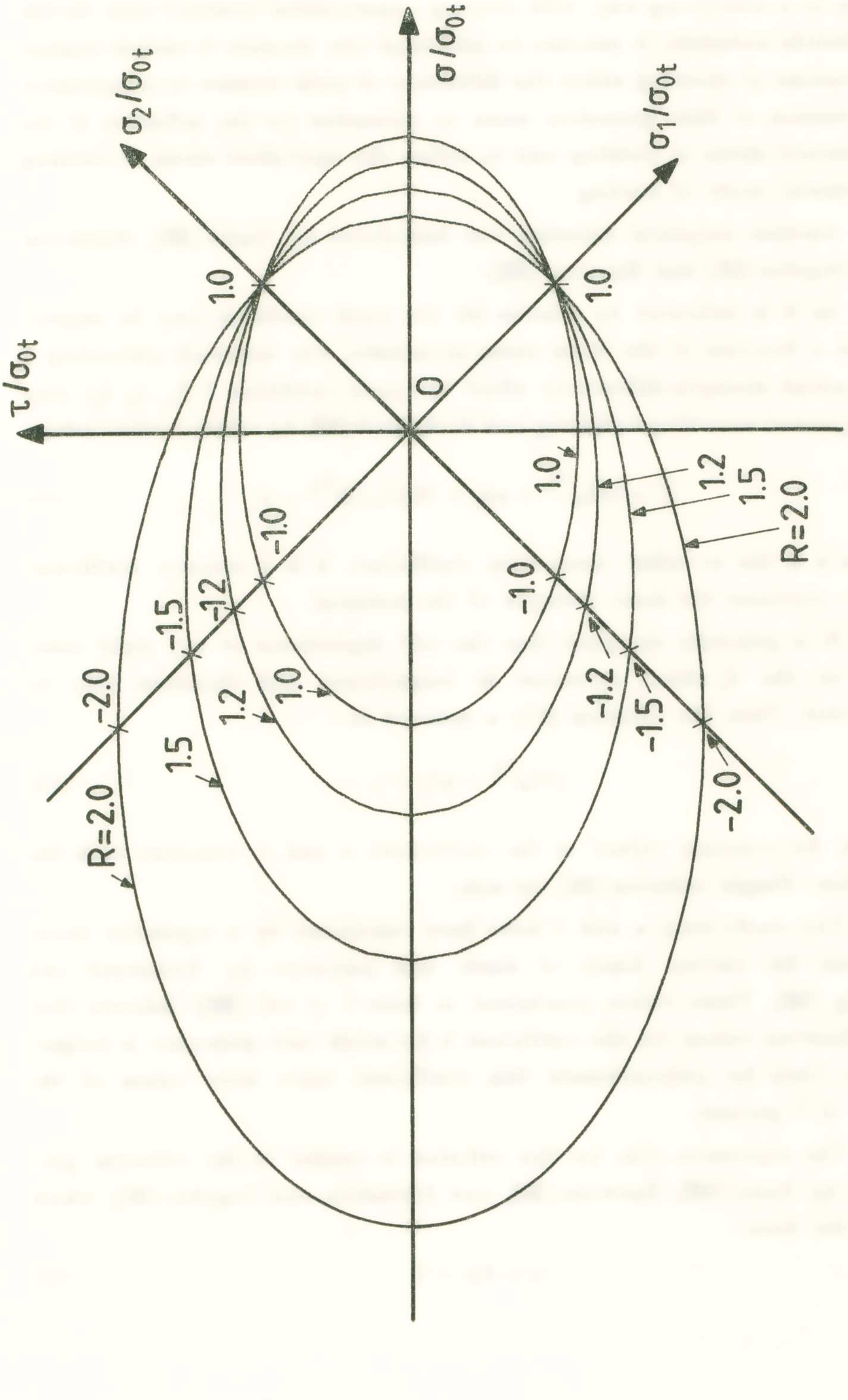



Fig. 4. The family of the stress-differential modified yield criteria, with R varying between unity and infinity for the same yield-stress in uniaxial tension.

forms, in a convincing way, with existing experimental evidence with ductile and brittle materials, it remains an empirical one, because it cannot explain the reasons of choosing either the difference of yield stresses in compression and tension or their geometric mean to encounter for the influence of the hydrostatic stress in yielding and to define the equivalent stress of yielding in uniaxial mode of loading.

Another empirical criterion was introduced by Nadai [**20**], Sternstein and Ongchin [**21**] and Bauwens [**22**].

As it is indicated by relation (8) the yield condition may be expressed as a function of the three stress invariants. For materials presenting a substantial strength-differential effect the yield condition $F(I_1, I_2, I_3)$ may be expressed according to Spitzig and Richmond [**23**] by a linear relationship:

$$F = (3J_2)^{1/2} + aJ_1 + 3b\,(J_3/2)^{1/3} = c \qquad (11)$$

where a is the so-called *mean-stress* coefficient, b is a *skewness* coefficient and c expresses the basic strength of the material.

It is generally accepted that the odd dependence of any yield criterion on the $J_3$-stress invariant is insignificant and therefore may be neglected. Then the criterion (11) is reduced to :

$$(3J_2)^{1/2} + aJ_1 = c \qquad (12)$$

which, for constant values of the coefficients a and c, coincides with the Drucker - Prager criterion [**24**] for soils.

The coefficients a and c have been calculated by a regression linear analysis for various kinds of steels and polymers by Richmond and Spitzig [**25**]. These values (contained in table I of ref. [**26**]) indicate that the observed values for the coefficient b for steels and polymers is insignificant. Only for polycarbonate this coefficient takes some values of the order of 5 percent.

The expression (12) for this criterion is similar to the criterion proposed by Nadai [**19**], Bauwens [**20**], and Sternstein and Ongchin [**21**], which is of the form :

$$\tau_n + Ap = C \qquad (13)$$

where $\tau_n$ is the octahedral shear stress, which is directly related to the second stress invariant $J_2$ and p is the mean normal stress and A and C real constants, the one of them (constant C) must have the dimensions of stress.

As it is pointed out by Raghava et al. [**16**], Eq. (13) is another expression of the Mises yield criterion modified by the influence of the hydrostatic component of stresses. However, this criterion is not equivalent with the criterion of Eq. (8).

Indeed, this criterion considers an algebraic addition of stresses, which are not collinear. The octahedral shear stress $\tau_n$ lies always on the deviatoric plane, whereas the hydrostatic component is always normal to this plane. Therefore any algebraic addition of these stresses is meaningless

The constants A and C were given by Raghava et al. [**16**] as :

$$A = \sqrt{2}\,\frac{\sigma_{oc} - \sigma_{ot}}{\sigma_{oc} + \sigma_{ot}} \quad \text{and} \quad C = \frac{2\sqrt{2}\sigma_{oc}^2}{3(\sigma_{oc} + \sigma_{ot})} \tag{14}$$

Comparing relations (14) with Raghava's equation (8) it can be readily shown that the constants A and C, from which the first expresses the strength differential effect and the second is some characteristic value for yielding of the material combining its behaviour in simple tension and compression, are identical with the relations :

$$A = \frac{\sigma_{oc} - \sigma_{ot}}{\sigma_{oc} + \sigma_{ot}}, \qquad C = \frac{\sigma_{ot}\,\sigma_{oc}}{\sigma_{ot} + \sigma_{oc}} \tag{15}$$

and therefore Eq. (13) is identical with Raghava's relation (5) which for plane-stress conditions is expressed as follows :

$$(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2)^{1/2} + \frac{\sigma_{oc} - \sigma_{ot}}{\sigma_{oc} + \sigma_{ot}}(\sigma_1 + \sigma_2) = \frac{2\sigma_{ot}\,\sigma_{oc}}{\sigma_{oc} + \sigma_{ot}} \tag{16}$$

It is worthwhile indicating that relation (16) may be written as follows :

$$\frac{(c_{oc} + c_{ot})}{\sqrt{2}}(J_2)^{1/2} + (c_{oc} - c_{ot})\,J_1 = 2c_{ot}\,c_{oc} \tag{17}$$

In this form Eq. (17) expresses the equivalence of energy components and therefore constitutes legitimate the addition of the terms of this equation.

However, the first and second terms of the left-hand side of this equation represents energies, which depend on the distortional and the dilatational components of energy respectively, but they are not strictly these energies, as they should be.

In order to show the weakness of this criterion we have plotted in Fig. 5 the yield loci derived from both criteria and for identical values of the strength-difference coefficient $R = \sigma_{oc} / \sigma_{ot}$. It can be readily seen from the corresponding loci that, whereas for values of R close to unity there is a small difference between the loci derived from both criteria, for larger values of $R (R > 1.1)$ the differences increase and they become significant, so that, for brittle materials with $R = 3.0$, the ellipses of the Nadai - Bauwens - Sternstein criterion degenerate into a parabola passing through the points (1, 0), (0, 1), and (0, —3), (—3, 0). For $R > 3.0$ these curves become hyperbolas.

However, comparing the yield loci resulting from the two models and the experimental data available for various materials, it may be concluded that whereas the Schleicher - Stassi criterion corroborates with experience, the Nadai - Bauwens - Sternstein criterion deviates significantly, especially in the critical compression-compression quadrant.

Fig. 6 shows the yield locus for a series of polymers plotted in the $(\sigma_1, \sigma_2)$-plane and taken from ref. [**17**]. The strength-difference effect for these materials found to be $R = 1.3$ approximately. In the same figure the DSM-yield criterion was plotted for $R = 1.3$ and represented by the continuous ellipse. It is clear from this figure that again the DSM-criterion corroborates with all experimental results.

On the other hand, experiments executed by Spitzig et al. [**26**] on various types of steels, presenting ratios $\sigma_{oc} / \sigma_{ot} = 1.055$, gave values for the constants A and C as follows : $A = 0.026$ and $0.028$, whereas $C = 1{,}480$ and $1{,}066$ MPa. From the respective values of $\sigma_{oc}$ and $\sigma_{ot}$ these quantities are 1,470 MPa and 1,070 MPa respectively. Therefore, the theory by Spitzig et al. and the Nadai - Sternstein and Bauwens criterion yield approximately identical results. However, these results, with ratios $\sigma_{oc} / \sigma_{ot}$ of the order of $R = 1.1$ correspond to yielding loci which differ only slightly between theories and therefore they are not decisive for the selection of the correct criteria (see Fig. 5).

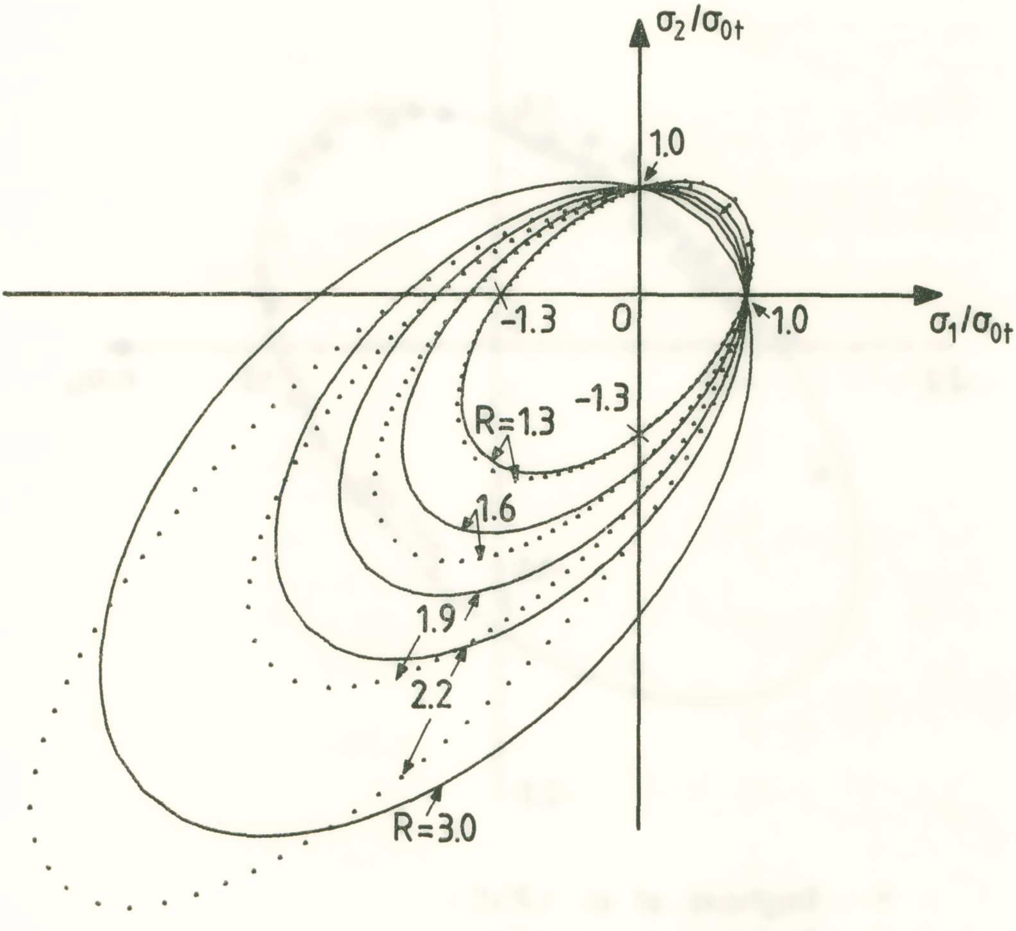



Fig. 5. The families of yield loci for two kinds of stress-differential modified criteria for constant yield stress in simple tension and R varying between R = 1.3 and R = 3.0. Full lines correspond to the SDM-criterion based on the addition of energy components (Schleicher - Stassi criterion), whereas dotted lines correspond to the SDM-criterion based on the addition of components of stresses (Nadai - Bauwens - Sternstein criterion).

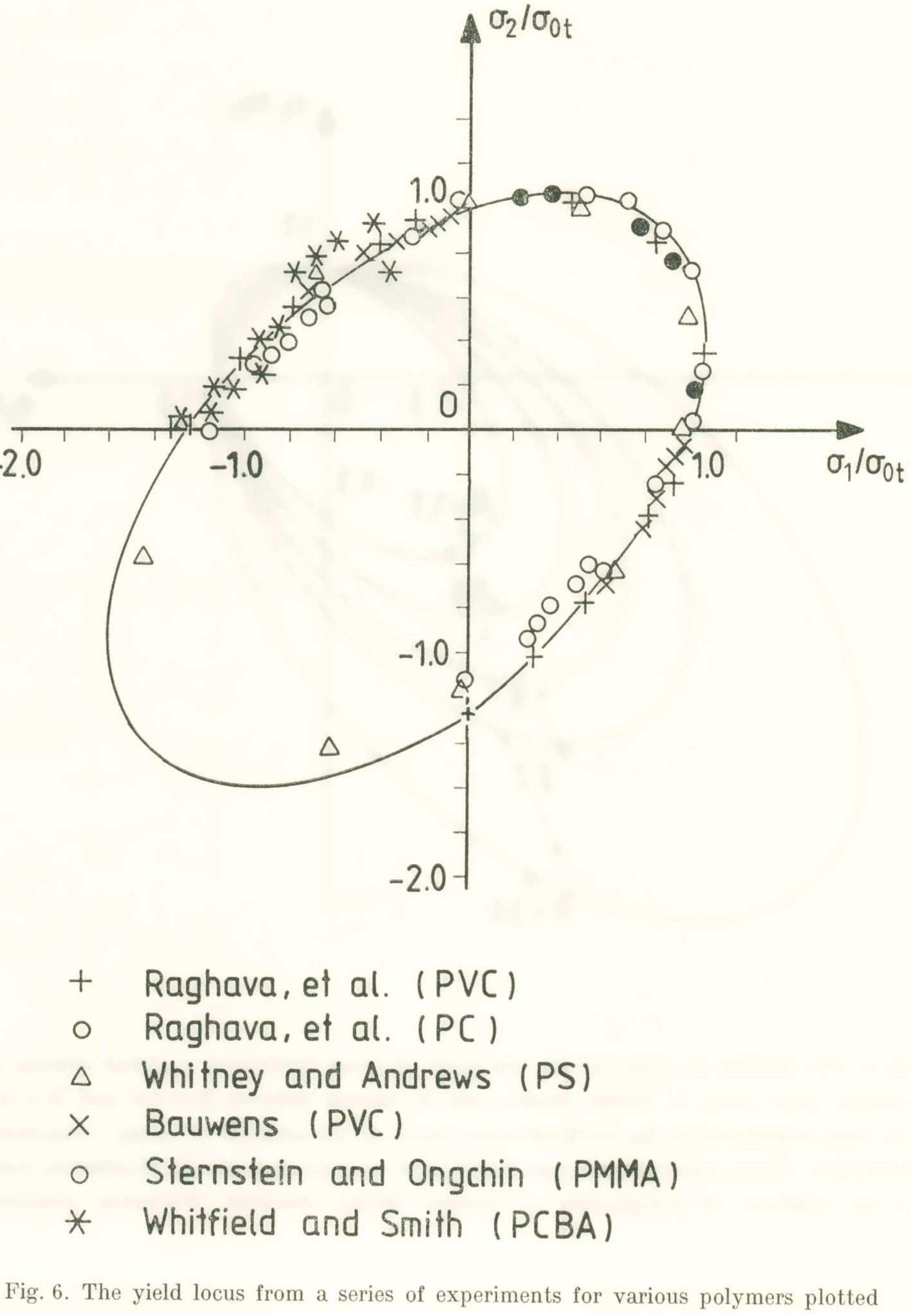



Fig. 6. The yield locus from a series of experiments for various polymers plotted in the $(\sigma_1, \sigma_2)$-plane and the corresponding SDM-locus for $R = 1.3$.

## CRITERIA DEPENDING ON PRESSURE AND DILATANCY BASED ON THEORY OF VOID FORMATION

It has been lately shown that the influence of the hydrostatic component of stresses on yielding is directly related to the mechanism of local void nucleation and growth, or the inverse, at regions of high stress-concentration of structures. This void nucleation and development is followed by a bulk dilatancy, or in materials containing initially voids due to their structure, void closure is followed by volume contraction. Both procedures have as a result to change drastically the yield behaviour of the material.

Many models have been recently introduced, which are based on the development of voids and their influence on the yield criterion of the materials. The Mc Clintock model [**27**] for ductile fracture assumes that a mechanism of localization of deformation, which starts from some discontinuity of the substance (macrovoid, grain boundary, crack) is developed along and within a narrow shear band, due to the progressive softening of the material during loading along a zone ahead of the discontinuity, because of the progressive softening of the material by an increasing porosity. Then, the material along this zone is damaged to such an extent that voids begin to appear with increasing load. Further loading induces the formation of a population of voids usually in an enclave, which ultimately coalesces with its neighbour zones and produces a propagation of the discontinuity.

This process is preceded by an incubation period for cavity nucleation, which is always short, followed by a rapid development and spreading out of the void zone. Thus, while the short period of incubation of voids is a transitional one and it does not influence directly the mode of plastic spreading, the period of cavity nucleation is very important for the subsequent development of regional yielding.

On the other hand, the thickness variation is another important factor because the hydrostatic tension $\sigma_{\kappa\kappa}/3$, and especially the intermediate principal stress, influences the fracture strain. McClintock [**27**] has shown that the three principal stresses should be tensile for void growth. Therefore, if both transverse stresses, or only one of them is positive, this fact creates a big difference with the role of the intermediate principal stress, which is more important for plane-stress conditions.

Experimental evidence with scanning electron microscopy yielded ample proof of the validity of the principle of the influence of void-nucleation and coalescence to the mode of yielding of all materials independently of their brittleness or ductility. Fig. 7 presents some convincing experimental evidence for the initiation and development of voids around a crack in Polycarbonate tensile specimens.

Assuming that, for a void-containing material, failure happens at some critical void-volume, $f_c$, the constitutive relations according to the model developed by Gurson [**28**] make use of an approximate yield condition of the form $\Phi(\sigma_{ij}, \sigma_m, f) = 0$, where $\sigma_{ij}$ is the average macroscopic Cauchy stress-tensor, $\sigma_m$ is the equivalent tensile flow stress, representing the actual macroscopic stress-state in the matrix material, and f is the current void volume-fraction.

For some loading level, when some cavities have already nucleated, they elongate for continuing loading along the major tensile axis, so that neighbouring voids coalesce when their length is of the order of magnitude of their spacing [**29**]. Then, local failure occurs by the development of slip planes between major voids, creating necking at the ligaments. It has estimated by Brown and Embury [**29**] that critical values of the void-volume fraction may vary between $f = 0.05$ and $0.20$, with a probable value of $f_c = 0.15$.

As soon as necking is established between ligaments the straining of the specimen is no more uniform and the stress-carrying capacity of its cross-section decreases rapidly, so that the average macroscopic stresses $\sigma_{ij}$ decay considerably. In order to take into account this decay, the void volume fraction notion is introduced in the model, so that this volume reduction takes care of the local decay of the macroscopic stresses.

Gurson [**28**] gave an approximate yield condition, which is based on an upper-bound rigid-plastic solution for spherically symmetric deformations, applied around a spherical inclusion. This condition is expressed by :

$$\sigma_e^2 + 2q_1 f\sigma_m^2 \cos h\left[\frac{q_2}{2}\ \frac{\sigma_{\varkappa\varkappa}}{\sigma_m}\right] - (1 + q_3 f^2)\,\sigma_m^2 = 0 \qquad (18)$$

where $\sigma_e$ is the macroscopic effective Mises stress given by $\sigma_e^2 = \frac{1}{2} s_{ij} s_{ij}$.

While Gurson assumed values for the constants $q_i$ $(i = 1, 2, 3)$ given by $q_1 = q_2 = q_3 = 1$, it was found by Tvergaard [**30**] that a better fitting

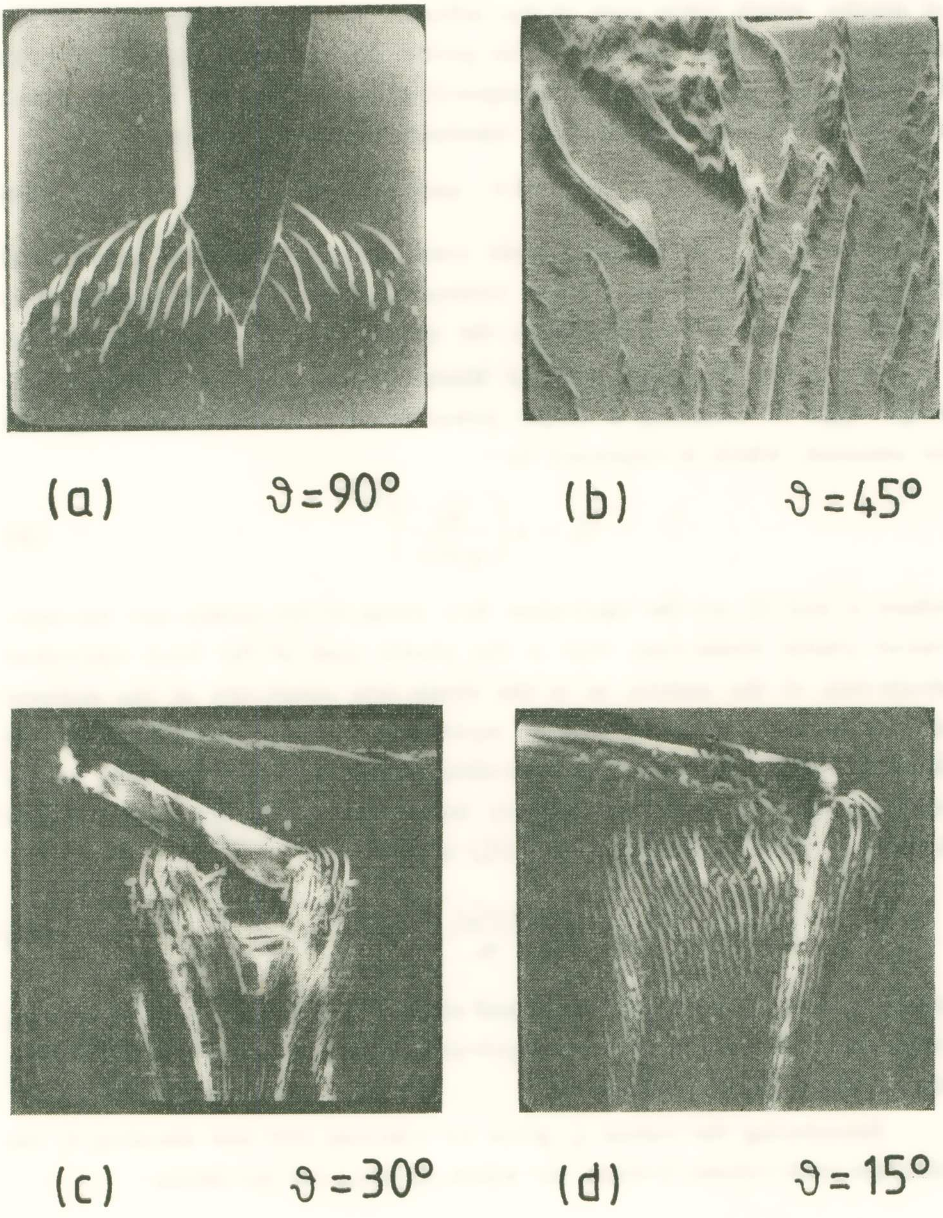



Fig. 7. A series of scanning electron micrographs showing the plastic zones developed around the crack-tips in polycarbonate plates containing oblique cracks with angles $\beta = 90°, 45°, 30°$ and $15°$. The development of voids in these plastic zones is obvious in some of these figures.

of results, which takes care of the influence of neighbouring voids to the central pair of voids considered, for periodically arranged cylindrical voids in a matrix, with the results corresponding to a continuum model without any voids, is ascertained, if these constants take the values :

$$q_1 = 1.5, \quad q_2 = 1.0 \quad \text{and} \quad q_3 = q_1^2 = 2.25 \tag{19}$$

The Gurson - Mc Clintock yield condition (18) reduces to the classical Mises yield surface, if the relative volume fraction f is taken equal to zero (f = 0) and then it corresponds to the special case of isotropic hardening.

For a rate-dependent type of Mises criterion (and for any other isotropic type of criterion) a simple power law for the rate hardening may be assumed, which is expressed by :

$$\dot{\bar{\varepsilon}}_m^p = \dot{a} \left[ \frac{\bar{\sigma}_m}{g(\bar{\varepsilon}_m^p)} \right]^{1/m} \tag{20}$$

where $\bar{\sigma}$ and $\dot{\bar{\varepsilon}}_m^p$ are the equivalent flow stress of the matrix and the equivalent plastic strain-rate, that is the plastic part of the total equivalent strain-rate of the matrix, m is the strain-rate sensitivity of the material and a is the reference plastic equivalent strain-rate. Moreover, the function $g(\bar{\varepsilon}_m^p)$ represents the equivalent tensile flow stress of the matrix material, derived from an ordinary tensile test at $\dot{\bar{\varepsilon}}_m^p = \dot{a}$. For a power hardening material the function $g(\bar{\varepsilon}_m^p)$ is given by :

$$\bar{\varepsilon}_m^p = \varepsilon_0 \left[ \frac{g(\bar{\varepsilon}_m^p)}{\sigma_0} \right]^{1/N} - \frac{g(\bar{\varepsilon}_p)}{E} \tag{21}$$

where $\sigma_0$ and $\varepsilon_0$ are the true stress and strain at yielding in uniaxial tension, conducted at the reference plastic equivalent strain-rate, and N is the hardening exponent of the material.

Introducing the values $q_i$ given by relations (19) and denoting $f_u$ the ultimate void-volume fraction for which $\sigma_e = \sigma_{xx} = 0$ we obtain :

$$f_u = \frac{1}{q_1} = 0.67 \tag{22}$$

For this value of $f_u$, which corresponds to a limiting step of deformation of the body, where no more macroscopic loading can be carried out by it, the value of $f_u = 0.67$ is quite high although it is below unity.

While an estimation of this quantity may be derived from close-packed arrays of spherical or cylindrical voids in a body, where this volume fraction is a limit for initiation of yielding, the value of f may be estimated experimentally from the overall strength differential effect for each material studied.

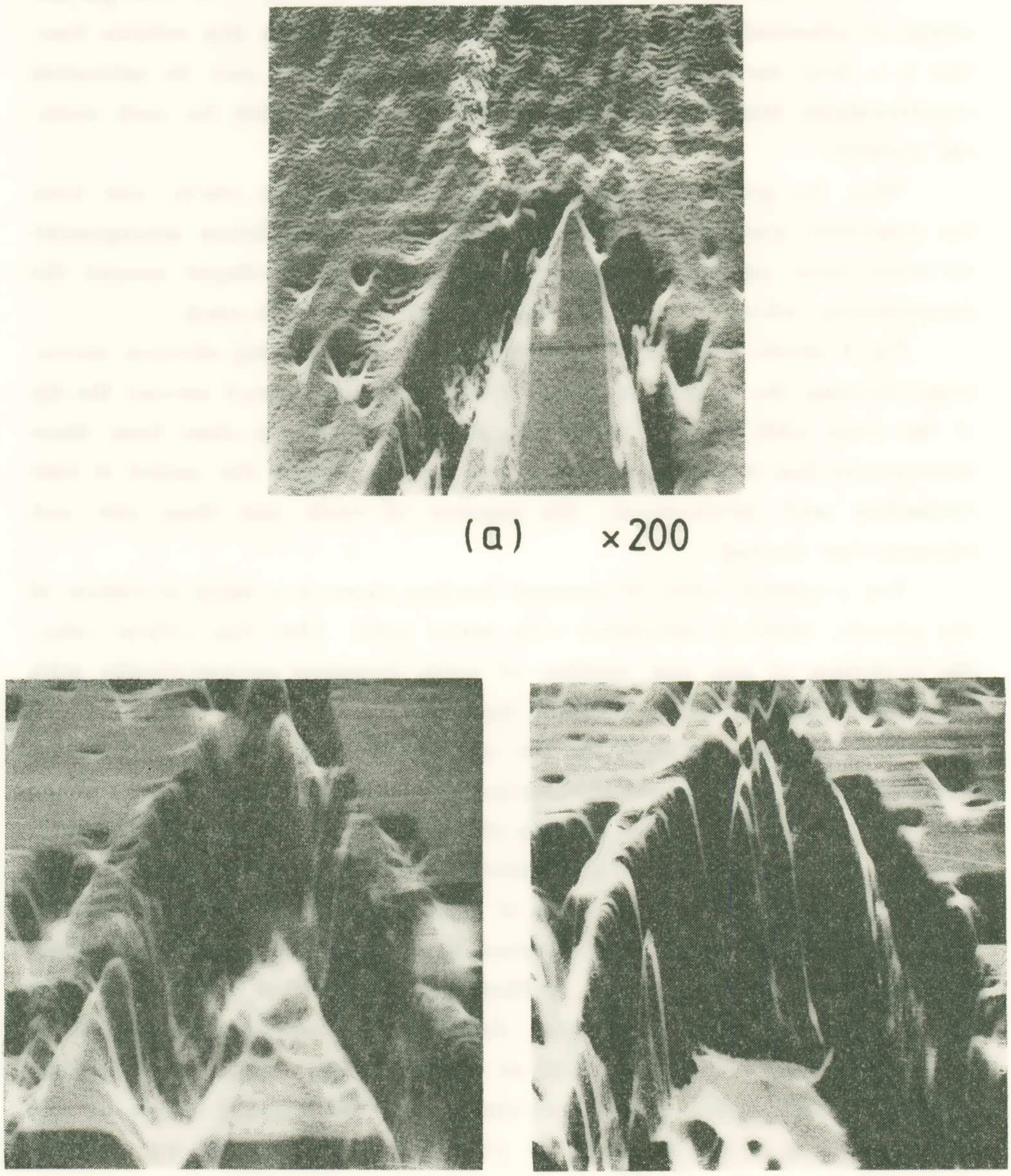
Thus, for polycarbonate specimens containing edge cracks and from the respective scanning electron micrographs in z-modulation arrangement we could have an overall view of the plastic zone developed around the discontinuity, which for our case was the tip of an edge-crack.

Fig. 8 shows a series of micrographs in the scanning electron microscope showing the nucleation and evolution of voids created around the tip of the crack with loading, inside the plastic zone. It is clear from these micrographs that in the beginning of the process, during the period of void nucleation and development, the number of voids and their size and intensity are limited.

For a critical value of external loading there is a rapid evolution of the process, which is associated with initial yield. After this critical value, the evolution of size and number of voids increases asymptotically with a minimum increase of the external loading.

We are now interested for the initial appearance of yielding, which corresponds to the threshold of initiation of instability of void-development. Therefore, in our case, $\sigma_m$ in relation (18) should be taken equal to the yield stress $\sigma_0$ in simple tension of the material. In order to evaluate the variation of the void-fraction, f, a series of tests with edge-cracked plates under conditions of plane-stress were executed. Care was taken to define the angle of obliqueness of the edge-crack, so that the stress-field in front of the crack tip corresponded to typical stress distributions yielding variable values for the individual principal stresses, as well as for their sum. For each principal stress distribution in front of the cracks, the limiting values for the void nucleation at the initiation of yielding were determined by counting the number and size of voids developed during each loading step.

By counting at these limiting values of loads the number and volume of voids we could calculate through a simple model of a cluster of cylindrical voids along the thickness of the specimen, the values of the constant f for different polar angles θ, corresponding to different values of the sum of principal stresses. These values of f for different points of



(a) ×200

(b) ×200

(c) ×200

Fig. 8. A series of micrographs in the scanning electron microscope showing the nucleation and evolution of voids in the plastic zone around the crack tip for increasing loading of the plate.

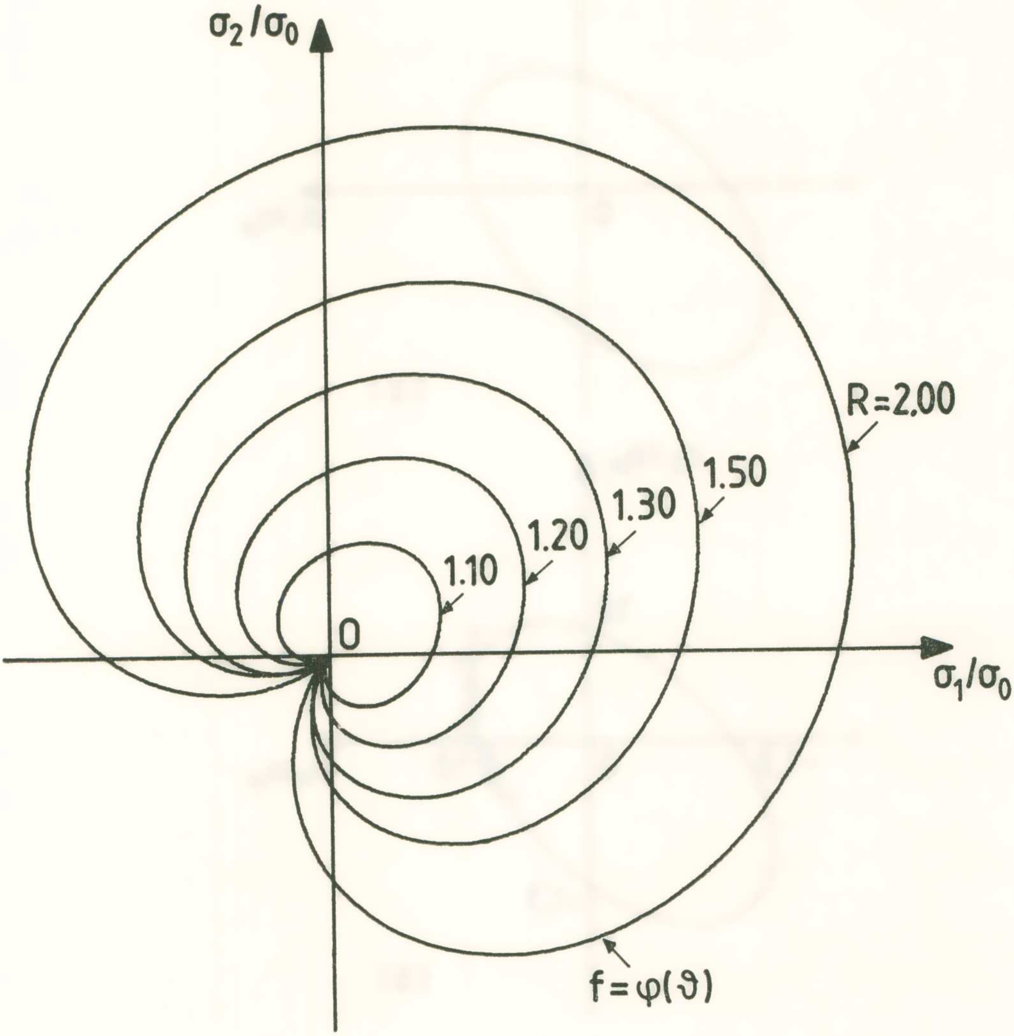



Fig. 9. The variation of the void-volume fraction for initial yielding, f, around a crack-tip, versus the polar-angle θ in the $(\sigma_1, \sigma_2)$-plane, for parametric values of stress-differential parameter R between R = 1.10 and R = 2.00.

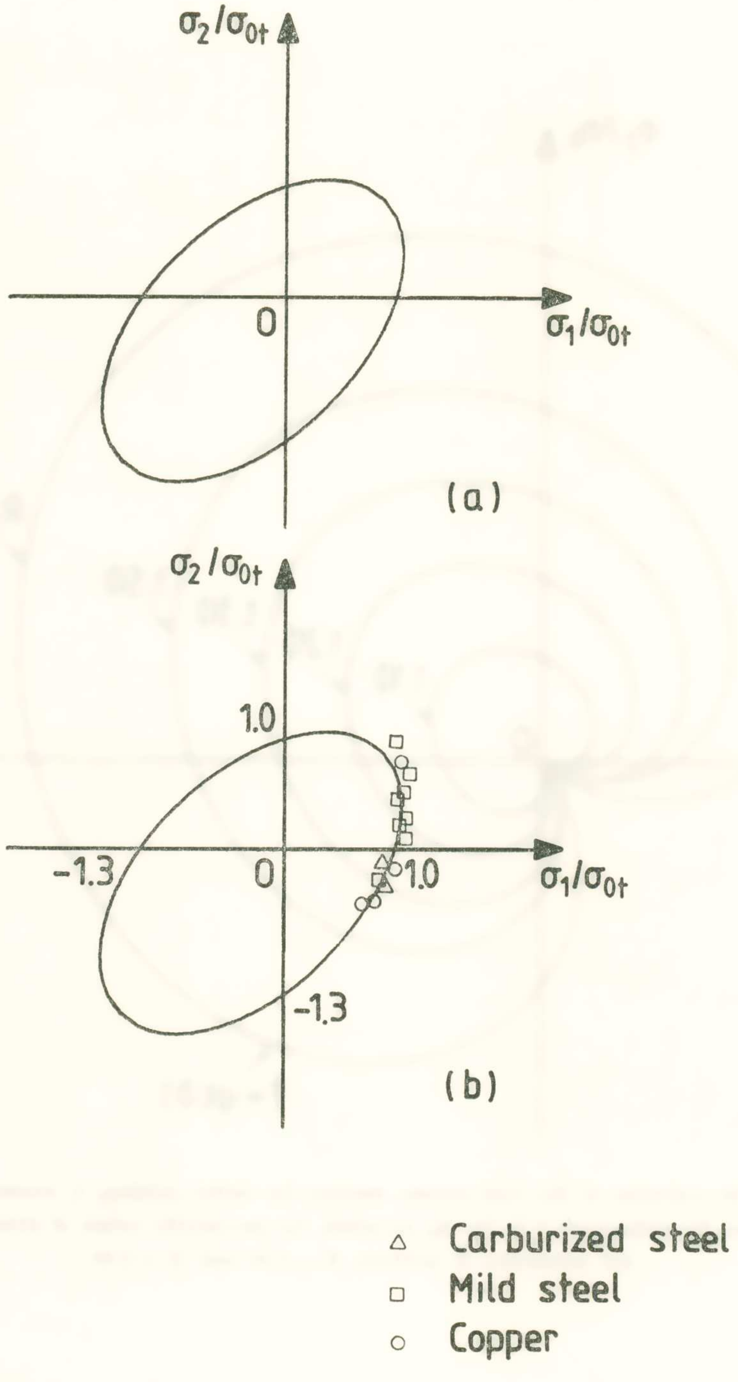



Fig. 10. (a) The initial yield locus as derived by the Gurson - McClintock type of yielding for R=1.30 and (b) The respective locus from the experimental evidence.

the initial yield locus and different values of R between 1.10 and 2.00 were plotted in Fig. 9. When these values were introduced into Eq. (18) for R = 1.3 gave a yield locus, which was plotted in Fig. 10. In the same figure the yield locus for the strength-differential modified criterion for R = 1.30 was plotted (Fig. 10b). Both curves are highly coincident. By taking into consideration that vast experimental evidence for yield loci with different materials, shown in Figs. 1, 2 and 6, indicated a good coincidence of the experiments with this criterion, it is reasonable to accept that the void-nucleation criterion is also valid and explains satisfactorily the real behaviour of the materials.

Finally, since the strength-differential modified criterion is a simple one, represented by a family of ellipses for values of R varying between R = 1.0 to R = $\infty$, it is reasonable to accept that this type of criterion constitutes a universal criterion valid, for every material from the brittlest to the most ductile one.

A c k n o w l e d g m e n t : The research contained in this paper was partly supported by a research grant allocated to the Laboratory by the Hellenic Aluminium Co.

The author acknowledges this financial support. He expresses also his thanks to his assistants Drs C. Stassinakis and Ch. Georgiadis for their help in executing the experiments.

## ΠΕΡΙΛΗΨΙΣ

Ὅλα τὰ κριτήρια διαρροῆς τῶν ὑλικῶν τὰ ὁποῖα ἔχουν παρουσιασθῆ εἰς τὴν διεθνῆ βιβλιογραφίαν μέχρι σήμερον καὶ τὰ ὁποῖα λαμβάνουν ὑπ' ὄψιν των τὴν ἐπίδρασιν τῆς ἐσωτερικῆς διογκώσεως τοῦ ὑλικοῦ κατὰ τὴν διαρροὴν βασίζονται εἰς τὴν συμβατικὴν παραδοχὴν τῆς ἐπιδράσεως τῆς ὑδροστατικῆς συνιστώσης τῶν τάσεων εἰς τὴν διαδικασίαν πλαστικῆς παραμορφώσεώς των, καὶ δι' αὐτοῦ τοῦ τρόπου προσαρμόζονται εἰς τὴν πραγματικὴν κατάστασιν παραμορφώσεως τῶν ὑλικῶν, ὅπως αὐτὴ παρουσιάζεται κατὰ τὴν μακροσκοπικὴν ἐξέτασίν των. Τοιουτοτρόπως ἡ ἰσχὺς καὶ ἡ ἀκρίβεια τῆς θεωρίας αὐτῆς τῶν κριτηρίων διαρροῆς τῶν ὑλικῶν ἀποδεικνύεται πειραματικῶς καὶ ἑπομένως ἐπιτρέπει τὴν ἀκριβῆ πρόβλεψιν τῆς πλαστικῆς συμπεριφορᾶς μεγάλου ἀριθμοῦ ὑλικῶν χρησιμοποιουμένων εἰς τὴν πρᾶξιν.

Ἐνῶ τὰ προϋπάρχοντα κριτήρια διαρροῆς, ἰσχύοντα δι' ἰσότροπα καὶ ἐλαστικῶς - πλαστικῶς παραμορφούμενα ὑλικά, βασίζονται εἰς συνθῆκας, αἱ ὁποῖαι ἐκφράζουν εἴτε τὸ ἀναλλοίωτον τῆς στροφικῆς συνιστώσης τῆς ἐλαστικῆς ἐνεργείας ἡ ὁποία, ὅταν ὑπερβῇ ὡρισμένον ὅριον, ἐπιτρέπει τὴν διαρροὴν τοῦ ὑλικοῦ, εἴτε τὴν μεγίστην διατμητικὴν τάσιν, εἶχε γίνει ἀντιληπτὸν εὐθὺς ἐξ ἀρχῆς ὅτι τὰ κριτήρια αὐτὰ μόνον ὡς πρῶται προσεγγίσεις ἠδύναντο νὰ ληφθοῦν ὑπ' ὄψιν, διότι τὰ πλεῖστα ἐκ τῶν ὑλικῶν τῶν χρησιμοποιουμένων εἰς τὴν πρᾶξιν παρουσιάζουν μορφὴν διαρροῆς κατὰ πολὺ πολυπλοκωτέραν τῆς περιγραφομένης ὑπὸ τῶν ἁπλῶν αὐτῶν κριτηρίων.

Τοιουτοτρόπως, πολλὰ ἐκ τῶν ψαθυρῶν ὑλικῶν τῶν χρησιμοποιουμένων εἰς τὰς κατασκευὰς τοῦ πολιτικοῦ μηχανικοῦ (λίθοι, σκυρόδεμα, κονιάματα κ.λπ.) ἀποκλίνουν σημαντικῶς ἐκ τῶν κριτηρίων αὐτῶν δημιουργήσαντα τὴν ἀνάγκην τῆς ἐξ ἀρχῆς ἀντιμετωπίσεώς των διὰ καταλλήλου κριτηρίου, ὅπου ὄχι μόνον ἡ μεγίστη διατμητικὴ τάσις ὑπεισέρχεται ὡς παράγων ἐπιδράσεως τῆς διαρροῆς τοῦ ὑλικοῦ, ἀλλὰ καὶ ἡ ὑδροστατικὴ συνιστῶσα τῶν τάσεων ἐκφραζομένη διὰ τοῦ ἀθροίσματος τῶν κυρίων τάσεων τῶν ἀναπτυσσομένων εἰς τὸ σῶμα, ἤτοι τῆς πρώτης ἀναλλοιώτου τοῦ τανυστοῦ τῶν τάσεων. Τὸ κριτήριον αὐτό, εἰσαχθὲν ἀπὸ τὸν Coulomb κατὰ τὸ ἔτος 1773, ἀποτελεῖ τὸν προάγγελον τῶν συγχρόνων κριτηρίων τὰ ὁποῖα βασίζονται τόσον εἰς τὴν δευτέραν ὅσον καὶ εἰς τὴν πρώτην ἀναλλοίωτον τοῦ τανυστοῦ τῶν τάσεων.

Τὸ φαινόμενον τῆς ἐπιδράσεως τῆς πρώτης ἀναλλοίωτου ἐπὶ τῆς διαρροῆς τῶν ὑλικῶν ἐμφαίνεται διὰ τοῦ φαινομένου τοῦ ἀποκαλουμένου ὡς φαινομένου διαφορικῆς ἀντοχῆς τῶν ὑλικῶν. Τὸ φαινόμενον διαφορικῆς ἀντοχῆς παρουσιάζεται χαρακτηριστικῶς εἰς ὅλα σχεδὸν τὰ ὑλικά, τὰ ὁποῖα παρουσιάζουν διάφορον ἀντοχὴν διαρροῆς εἰς ἐφελκυσμὸν καὶ εἰς θλῖψιν.

Πράγματι, ἤδη καὶ ἀπὸ τῶν πρώτων πειραματικῶν δοκιμῶν εἶχε ἀποδειχθῆ ὅτι ἡ διαφορὰ αὐτὴ εἰς τὰς τάσεις διαρροῆς εἰς ἐφελκυσμὸν $\sigma_{ot}$ καὶ εἰς θλῖψιν $\sigma_{oc}$ ἐκφραζομένη διὰ τοῦ λόγου $R = \sigma_{oc} / \sigma_{ot}$ ἦτο πάντοτε διάφορος τῆς μονάδος καὶ συνήθως μεγαλυτέρα αὐτῆς.

Κατὰ τὴν τελευταίαν εἰκοσαετίαν σειρὰ ἐμπειρικῶν σχέσεων κριτηρίων διαρροῆς παρουσιάσθη εἰς τὴν βιβλιογραφίαν, ὅπου ἀμφότεραι αἱ ἀναλλοίωτοι τοῦ τανυστοῦ τῶν τάσεων ὑπεισέρχονται κατὰ λογικὸν τρόπον εἰς τὰς σχέσεις τὰς ἐκφράζουσας τὰ κριτήρια διαρροῆς τῶν ὑλικῶν. Ἡ τιμὴ τοῦ λόγου R, ἡ ὁποία μπορεῖ νὰ μεταβάλλεται ἀπὸ τῆς μονάδος μέχρι τοῦ ἀπείρου, ρυθμίζει τὴν συσχέτισιν τῶν δύο ἀναλλοιώτων μεγεθῶν, ὥστε τὰ ἀποτελέσματα τῆς θεωρίας νὰ συμπίπτουν μὲ τὰς τιμὰς τῶν σχετικῶν πειραμάτων δι' ἓν ἕκαστον ἐκ τῶν ἐξεταζομένων ὑλικῶν. Ὁ λόγος R καλεῖται π α ρ ά μ ε τ ρ ο ς δ ι α φ ο ρ ί σ ε ω ς τ ῶ ν τ ά σ ε ω ν δ ι α ρ ρ ο ῆ ς τῶν ὑλικῶν.

Δύο κυρίως κριτήρια ἀνεπτύχθησαν, τὰ ὁποῖα λαμβάνουν ὑπ' ὄψιν των τὴν παράμετρον R. Τὸ πρῶτον, τὸ ὁποῖον βασίζεται εἰς τὸ κλασσικὸν κριτήριον Mises καὶ εἰς πειράματα τῶν Schleicher καὶ Stassi d'Alia καλούμενον κριτήριον Schleicher, καὶ τὸ ἕτερον τὸ ὁποῖον θεωρεῖται ὡς ἐπέκτασις τοῦ κριτηρίου τοῦ Tresca καὶ βασίζεται εἰς πειράματα τῶν Nadai, Bauwens καὶ Sternstein.

Τὸ πρῶτον κριτήριον δι' ἐπίπεδον ἐντατικὴν κατάστασιν ἐκφράζεται διὰ τῆς σχέσεως :

$$3J_2 + (\sigma_{oc} - \sigma_{ot})\, J_1 = \sigma_{ot}\, \sigma_{oc} \qquad \text{(E1)}$$

ὅπου $J_1$ καὶ $J_2$ εἶναι ἡ πρώτη καὶ ἡ δευτέρα ἀναλλοίωτος τοῦ τανυστοῦ τῶν τάσεων ἐκφραζόμεναι ὡς :

$$J_1 = \frac{\sigma_{\kappa\kappa}}{3} \quad \text{καὶ} \quad J_2 = \frac{1}{2}\left(\sigma_{ij} - \delta_{ij}\frac{\sigma_{\kappa\kappa}}{3}\right)\left(\sigma_{ij} - \delta_{ij}\frac{\sigma_{\kappa\kappa}}{3}\right) \qquad \text{E2}$$

καὶ $\sigma_{ot}$, $\sigma_{oc}$ ἐκφράζουν τὰς τάσεις διαρροῆς τοῦ ὑλικοῦ εἰς ἁπλοῦν ἐφελκυσμὸν καὶ θλίψιν ἀντιστοίχως.

Τὸ δεύτερον κριτήριον δίδεται ἀπὸ τὴν σχέσιν (πάλι δι' ἐπίπεδον ἐντατικὴν κατάστασιν) :

$$\frac{1}{\sqrt{2}}\,(\sigma_{oc} + \sigma_{ot})\, J_2^{1/2} + (\sigma_{oc} - \sigma_{ot})\, J_1 = 2\sigma_{ot}\, \sigma_{oc} \qquad \text{(E3)}$$

Εἰς τὴν ἐργασίαν αὐτὴν μελετᾶται ἡ ἐπίδρασις τῆς ὑδροστατικῆς συνιστώσης τῶν τάσεων καὶ τῆς παραμέτρου R εἰς τὴν μορφὴν τῶν τόπων διαρροῆς τῶν ὑλικῶν καὶ δίδονται διαγράμματα μεταβολῆς τῶν τόπων διαρροῆς διὰ διαφόρους τιμὰς τῆς παραμέτρου R, ἤτοι διὰ διαφόρους βαθμοὺς ὀλκιμότητος ἢ ψαθυρότητος τῶν ὑλικῶν.

Ἐκ τῆς μακρᾶς πειραματικῆς ἐρεύνης τῶν κριτηρίων διαρροῆς τῶν ὑλικῶν (μετάλλων, πλαστικῶν, μὴ μεταλλικῶν ὑλικῶν κ.λπ.) ἀποδεικνύεται ὅτι τὸ πρῶτον κριτήριον εἶναι πολὺ πλησίον τῶν πειραματικῶν ἀποτελεσμάτων, ἐνῶ τὸ δεύτερον κριτήριον δίδει καλὰ μὲν ἀποτελέσματα δι' ὅλκιμα ὑλικὰ (μὲ μικρὸν R) ἀλλὰ διὰ ψαθυρὰ ὑλικὰ (διὰ μεγάλα R) ἀποκλίνει ὅλον καὶ περισσότερον ἐκ τῶν πειραματικῶν τιμῶν, ὅπου τὸ R αὐξάνει.

Ἡ τοιαύτη ὑπεροχὴ τοῦ πρώτου κριτηρίου ἔναντι τοῦ δευτέρου, ἑρμηνεύεται ἐκ τοῦ γεγονότος ὅτι τὸ πρῶτον κριτήριον περιέχει ὅρους, οἱ ὁποῖοι ἀναφέρονται κεχωρισμένως εἴτε εἰς τὴν στροφικὴν ἐνέργειαν εἴτε εἰς τὴν ἐνέργειαν μεταβολῆς τοῦ ὄγκου, ἐνῶ τὸ δεύτερον κριτήριον περιέχει ὅρους ἐνεργείας μεικτούς.

Τέλος περιγράφεται τὸ νέον κριτήριον διαρροῆς ὀλκίμων ὑλικῶν, τὸ εἰσαχθὲν ὑπὸ τῶν McClintock καὶ Gurson, τὸ ὁποῖον βασίζεται εἰς τὴν θεωρίαν ἀναπτύξεως κενῶν περὶ τὴν καταπονουμένην ρωγμὴν τὸ ὁποῖον, διὰ καταλλήλου προσδιορισμοῦ τῆς ἀνηγμένης περιεκτικότητος κενῶν εἰς τὴν περιοχὴν διαρροῆς τοῦ ὑλικοῦ, ἀποδεικνύεται ὅτι συμπίπτει μὲ τὸ κριτήριον Schleicher.

## REFERENCES


1. R. Hill, "Mathematical Theory of Plasticity", Oxford, at the Clarendon Press (1956).
2. R. von Mises, Göttinger Nachrichten, math-phys-Klasse pp. 582 - 596 (1913).
3. H. Hencky, Zeitsch. ang. Math. und Mech., Vol. 4, p. 323 - 334 (1924).
4. H. Tresca, Comptes Rendus Acad. Sci. Paris, Vol. 59, pp. 754 - 760 (1864) and Vol. 64, p. 809 (1867).
5. C. A. Coulomb, Mémories Mathématiques et Physiques, Vol. 7, p. 343 (1773).
6. O. Mohr, Zeitsch. Verein Deitsch - Ing., Vol. 44, p. 1524 (1900).
7. M. T. Huber, Czasopismo Techniczne, Vol. 22, p. 81 (1904).
8. R. von Mises, Zeitsch. ang. Math and Mech. Vol. 6, p. 199 (1926).
9. F. Schleicher, Zeitsch. ang. Math. und Mech., Vol. 5, p. 478 (1925).
10. L. F. Coffin Jr., Jnl. Applied Mech., Vol. 17, No. 3, pp. 223 - 248 (1950).
11. R. C. Grassi and I. Cornet, Jnl. Appl. Mech., Vol. 16, pp. 178 - 182 (1949).
12. G. I. Taylor and H. Quinney, Phil. Trans. Roy. Soc. London, Ser. A, Vol. 230, p. 323 (1931).
13. W. Lode, Zeitsch. der Physik, Vol. 36, p. 913 (1926).
14. F. Stassi d'Alia, Meccanica, Vol. 3, pp. 178 - 195 (1967).
15. N. W. Tschoegl, Polymer Sci. Symposia, Vol. 32, p. 239 (1967).
16. R. Raghava, R. M. Caddell and G. S. Y. Yeh, Jnl. of Mat. Sci., Vol. 8, pp. 225 - 232 (1973).
17. P. S. Theocaris, Intern. Jnl. Mech. Sci., Vol. 18, No. 4, pp. 171 - 178 (1976).
18. E. E. Gdoutos, Jnl. Appl. Polymer Sci., Vol. 27, pp. 879 - 892 (1982).
19. E. E. Theotokoglou, G. Tsamasphyros and P. S. Theocaris, Proc. IUPAC Internant. Symp. on Interrelations Between Processing, Structure and Properties of Polymeric Materials, Athens Greece August 1982.

20. A. Nadai, "Theory of Flow and Fracture of Solids", McGraw - Hill Publ. p. 210 and 220 (1950).

21. S. Sternstein and L. Ongchin, A. C. S. Polymer Rep. Vol. 10, No. 2, p. 1117.

22. J. C. Bauwens, Jnl. Polymer Sci., Ser. A-2, Vol. 8, p. 893 - 901 (1970).

23. W. A. Spitzig and O. Richmond, Polymer Engrg. and Sci., Vol. 19, No. 16, pp. 1129 - 1139 (1979).

24. D. C. Drucker and W. Prager, Quart. Appl. Math., Vol. 10, pp. 157 - 165 (1952).

25. O. Richmond and W. A. Spitzig, "Theoretical and Applied Mechanics", Proc. of the XV Intern. Congr. of Theor. and Appl. Mech., Edited by F. P. J. Rimrott and B. Tabarrok, North Holland Publ., pp. 377 - 386 (1980).

26. W. A. Spitzig, R. J. Sober and O. Richmond, Acta Mettalurgica, Vol. 23, pp. 885 - 893 (1975).

27. F. Mc Clintock, Jnl. Appl. Mech., Trans. ASME, Vol. 35, No. 2, pp. 363 - 371 (1968).

28. A. L. Gurson, Jnl. Engrg. Mat. and Technology, Vol. 99, pp. 2 - 15 (1977).

29. L. M. Brown and J. D. Embury, Proc. Third Intern. Conf. on Strength of Metals and Alloys, Int. of Metals, London, pp. 164 - 169 (1973).

30. V. Tvergaard, Intern. Jnl. Solids and Struct. Vol. 30, No. 4, pp. 265-286 (1982).