ΜΑΘΗΜΑΤΙΚΑ.—**Remarks on Heegaard splittings and the Poincaré conjecture,** *by George M. Rassias**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

1. One of the most important unsolved problems in Geometry-Topology is the classification problem for closed manifolds of any dimension. This asks to list all the topologically distinct kinds of closed n-dimensional manifolds for each integer n. If $n = 1,2$ the problem has been solved by Riemann, but remains unsolved for higher dimensions (only partial results are known). In 1928, A. A. Markov proved that it is impossible to find a computable algorithm for distinguishing n-dimensional manifolds, $n \geqslant 4$. Poincaré made the following conjecture known as Poincaré conjecture (i. e., that every closed simply-connected 3-dimensional manifold is diffeomorphic to the 3-dimensional sphere) which remains unsolved since the early days of 1900. Poincaré conjecture is the most important, interesting and difficult unsolved problem in differential topology. In spite of the enormous efforts by many outstanding mathematicians since the time of Poincaré, this conjecture remains unconquered. Until the Poincaré conjecture is settled, we cannot expect a solution to the classification problem for closed 3-dimensional manifolds. Surprisingly, the higher dimensional analog of the Poincaré conjecture was answered affirmatively by S. Smale [**7**, **8**] and independently by J. Stallings [**11**], in 1960.

2. A closed, orientable 3-dimensional manifold M is the union of two handlebodies H, H′ of the same genus $p (\geqslant 0)$ with their boundaries identified by a homeomorphism, i. e., $M = H \cup H'$, $H \cap H' = \partial H = \partial H'$, where $H \cap H'$ is a closed, orientable 2-dimensional manifold of genus p.

Thus, Poincaré conjecture is true, if and only if, given two handlebodies of the same genus $p (\geqslant 0)$ and identifying their boundaries through a homeomorphism such that the resulting 3-dimensional manifold is simply-connected, then it is homeomorphic to S^3, for each positive integer p.

By M. Dehn [**2**], we know that the above statement is true if $p = 1$. Thus, the first main problem is to examine what is the case if $p = 2$.

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Let $M = H_2 \cup H_2'$, $H_2 \cap H_2' = \partial H_2 = \partial H_2'$ be a Heegaard splitting of genus 2, of M. Then, the homeomorphisms h of ∂H_2 modulo isotopies, are in a $1-1$ correspondence with the automorphisms ψ of $\pi_1(\partial H_2)$ modulo inner automorphisms. Each such homeomorphism h results to a 3-dimensional manifold M_h (after identifying ∂H_2 and $\partial H_2'$ by h). Denote by N the $H_2 \cap H_2' = \partial H_2 = \partial H_2'$.

By Van Kampen's theorem we can find $\pi_1(M_h)$ i.e., the fundamental group of M_h for each h.

By Papakyriakopoulos [**4**], we consider a class of homeomorphisms h of N, which result to 3-dimensional manifolds M_h such that $\pi_1(M_h) = 0$.

The next problem is to check whether these M_h are actually homeomorphic to S^3.

D e f i n i t i o n. Let $M = H_2 \cup H_2'$, $H_2 \cap H_2' = \partial H_2 = \partial H_2' = N$, be a Heegaard splitting of genus 2.

A homeomorphism $h : N \to N$ is said to be a *Papakyriakopoulos homeomorphism* if and only if, $A_i \simeq 0$ in H_2 imply that $h(A_i) \simeq 0$ in H_2', $i = 1,2$, where A_i, B_i is a fundamental system of loops based at 0 (the base point for $\pi_1(N)$). See, Papakyriakopoulos [**3, 4**].

Let $\psi : \pi_1(N, 0) \to \pi_1(N, 0)$ be the automorphism induced by h, where

$$\pi_1(N, 0) = (a_1, b_1, a_2, b_2 ; \prod_{i=1}^{2} [a_i, b_i] = 1)$$

and $[a_i, b_i]$ is the commutator of a_i, b_i, i.e., $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$. Note, that a_i, b_i, $(i = 1,2)$ are the elements of $\pi_1(N, 0)$ corresponding to A_i, B_i, $(i = 1,2)$ respectively. Let $A = \langle a_1, a_2 \rangle$ be the normal closure of a_1, a_2 i.e., the smallest normal subgroup of $\pi_1(N, 0)$ containing a_1, a_2. We have a diagram of fundamental groups, where the homomorphisms i_1, i_2 are induced from the inclusions. Then, $\pi_1(M)$ is isomorphic to the quotient of $\pi_1(N)$ by $(\ker i_1) \cdot (\ker i_2)$ because i_1, i_2 are surjective homomorphisms. By Papakyriakopoulos [**3, 4**] we have that

$$\pi_1(M, 0) \cong \pi_1(N, 0) / \langle A, \psi(A) \rangle$$

where

$$\langle A, \psi(A) \rangle = \langle a_1, a_2, \psi(a_1), \psi(a_2) \rangle$$

the normal closure of a_1, a_2, $\psi(a_1)$, $\psi(a_2)$. Then, M is simply-connected, if and only if,

$$\pi_1(N,0) = \langle a_1, a_2, \psi(a_1), \psi(a_2)\rangle, \quad \text{where} \quad \psi \in \mathrm{Aut}\big(\pi_1(N,0)\big)$$

induced by $h : N \to N$ homomorphism.

Now, by Birman [**1**], the manifold $M_h = H_2 \underset{h}{\cup} H_2'$ is homeomorphic to S^3, if and only if, $[\psi] \in AB$, where

$$[\psi] \in H(N) = \mathrm{Aut}\big(\pi_1(N)\big) / \mathrm{Inn}\big(\pi_1(N)\big)$$

the homeotopy group of N, and

$$\begin{aligned} A &= \{[\psi] \in H(N) : (\langle a_1, a_2\rangle)\,\psi \subseteq \langle a_1, a_2\rangle\} \\ B &= \{[\psi] \in H(N) : (\langle b_1, b_2\rangle)\,\psi \subseteq \langle b_1, b_2\rangle\} \\ AB &= \{[\psi_a] \cdot [\psi_\beta] \ : [\psi_a] \in A \quad \text{and} \quad [\psi_\beta] \in B\} \end{aligned}$$

Thus, in order that the following is true, namely, every closed simply-connected 3-dimensional manifold M obtained as a Heegaard splitting of genus 2, $M = H_2 \cup H_2'$, $H_2 \cap H_2' = \partial H_2 = \partial H_2' = N$, by identifying the boundaries of the handlebodies H_2, H_2' by a Papakyriakopoulos homomorphism, is homeomorphic to S^3 (3-dimensional sphere) it is sufficient that the set

$$W = \big\{[\psi] \in H(N) : \pi_1(N) = \langle a_1, a_2, \psi(a_1), \psi(a_2)\rangle \ \text{where} \ \psi \in \mathrm{Aut}\big(\pi_1(N)\big)\big\}$$

is a subset of AB. Let $Sp(4, \mathbb{Z})$ denote the group of 4×4 symplectic matrices with integer entries: matrices of the form

$$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

where y_i, $i = 1, 2, 3, 4$ are 2×2 matrices such that

$$\begin{gathered} y_1 y_2^t = y_2 y_1^t, \quad y_2^t y_4 = y_4^t y_2, \quad y_1^t y_3 = y_3^t y_1, \quad y_3 y_4^t = y_4 y_3^t, \\ y_1^t y_4 - y_3^t y_2 = I, \quad y_1 y_4^t - y_2 y_3^t = I \end{gathered}$$

where by y^t we mean the transpose of y.

Generally, there exists a homomorphism from $H(N)$ onto $Sp(4, \mathbb{Z})$ defined as follows: Suppose

$$\psi(a_i) = F_i(a_1, a_2, b_1, b_2) = \prod_{\lambda=1}^{2} a_\lambda^{\psi_{i,\lambda}} \cdot b_\lambda^{\psi_{i,2+\lambda}} \bmod [\pi_1(N), \pi_1(N)]$$

$$\psi(b_i) = G_i(a_1, a_2, b_1, b_2) = \prod_{\lambda=1}^{2} a_\lambda^{\psi_{2+i,\lambda}} \cdot b_\lambda^{\psi_{2+i,2+\lambda}} \bmod [\pi_1(N), \pi_1(N)]$$

where ψ is a representative of an element $[\psi] \in H(N)$, $i = 1, 2$. Then, the homomorphism $\sigma : H(N) \to Sp(4, \mathbb{Z})$ is defined by $([\psi])\,\sigma = (\psi_{mn})$ which is the matrix having as its entries the exponents of a_λ, b_λ appearing in the above equalities.

Consider an automorphism ψ of $\pi_1(N)$ such that $[\psi] \in W$ and let it be defined by

$$\psi(a_1) = a_1^{\lambda_1''} \cdot a_2^{\lambda_2''} \cdot b_1^{\lambda_3''} \cdot b_2^{\lambda_4''}, \qquad \psi(a_2) = a_1^{\nu_1''} \cdot a_2^{\nu_2''} \cdot b_1^{\nu_3''} \cdot b_2^{\nu_4''}$$

$$\psi(b_1) = a_1^{m_1''} \cdot a_2^{m_2''} \cdot b_1^{m_3''} \cdot b_2^{m_4''}, \quad \psi(b_2) = a_1^{k_1''} \cdot a_2^{k_2''} \cdot b_1^{k_3''} \cdot b_2^{k_4''}.$$

The image of $[\psi]$ by σ, is the symplectic matrix

$$\sigma([\psi]) = \begin{bmatrix} \begin{pmatrix} \lambda_1'' & \lambda_2'' \\ \nu_1'' & \nu_2'' \end{pmatrix} & \begin{pmatrix} \lambda_3'' & \lambda_4'' \\ \nu_3'' & \nu_4'' \end{pmatrix} \\ \begin{pmatrix} m_1'' & m_2'' \\ k_1'' & k_2'' \end{pmatrix} & \begin{pmatrix} m_3'' & m_4'' \\ k_3'' & k_4'' \end{pmatrix} \end{bmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}.$$

Now, suppose $\sigma([\psi])$ can be written as

$$\sigma([\psi]) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ M_3 & M_4 \end{pmatrix} \cdot \begin{pmatrix} N_1 & N_2 \\ 0 & N_4 \end{pmatrix}.$$

In order that $\psi \in \mathrm{Aut}(\pi_1(N))$, it has to be true that

$$\prod_{j=1}^{2} [\psi(a_j),\ \psi(b_j)] = 1.$$

Also, since the first homology group of $H_2 \underset{\psi}{\cup} H_2'$ is trivial, if and only if, the matrix P_1 has determinant ± 1, it must be true that $\det(P_1) = \det(M_1 N_1) = \pm 1$ so that the fundamental group of $H_2 \underset{\psi}{\cup} H_2'$ is trivial. Thus, $\det(M_1) = \lambda_1 \nu_2 - \lambda_2 \nu_1 = \pm 1$, $\det(N_1) = \lambda_1' \nu_2' - \lambda_2' \nu_1' = \mp 1$. If the previous conditions are satisfied then we construct automorphisms ψ_1, ψ_2 of $\pi_1(N)$ such that $[\psi_1] \in A$, $[\psi_2] \in B$, and

$$\sigma([\psi_1]) = \begin{pmatrix} M_1 & 0 \\ M_3 & M_4 \end{pmatrix} \quad \text{and} \quad \sigma([\psi_2]) = \begin{pmatrix} N_1 & N_2 \\ 0 & N_4 \end{pmatrix}.$$

Namely,

$$\left\{ \begin{array}{ll} \psi_1(a_1) = a_1^{\lambda_1} \cdot a_2^{\lambda_2}, & \psi_1(b_1) = a_1^{m_1} \cdot a_2^{m_2} \cdot b_1^{m_3} \cdot b_2^{m_4} \\ \psi_1(a_2) = a_1^{\nu_1} \cdot a_2^{\nu_2}, & \psi_1(b_2) = a_1^{k_1} \cdot a_2^{k_2} \cdot b_1^{k_3} \cdot b_2^{k_4} \end{array} \right\} \text{ and}$$

$$\left\{\begin{array}{ll} \psi_2(a_1) = a_1^{\lambda_1'} \cdot a_2^{\lambda_2'} \cdot b_1^{\lambda_3'} \cdot b_2^{\lambda_4'}, & \psi_2(b_1) = b_1^{m_3'} \cdot b_2^{m_4'} \\ \psi_2(a_2) = a_1^{\nu_1'} \cdot a_2^{\nu_2'} \cdot b_1^{\nu_3'} \cdot b_2^{\nu_4'}, & \psi_2(b_2) = b_1^{k_3'} \cdot b_2^{k_4'} \end{array}\right\}$$

This can be done since the homomorphism σ is surjective. Note, that $\{\psi_i(a_1), \psi_i(a_2), \psi_i(b_1), \psi_i(b_2)\}$, should still generate $\pi_1(N)$ and

$$\prod_{j=1}^{2} [\psi_i(a_j), \psi_i(b_j)] = 1$$

in order that $[\psi_i] \in \mathrm{Aut}(\pi_1(N))$. Thus, in such a case, $\psi \in \mathrm{Aut}(\pi_1(N))$ where $[\psi] \in W$, can be written as a composition of automorphisms $\psi_1, \psi_2 \in \mathrm{Aut}(\pi_1(N))$ where $[\psi_1] \in A$ and $[\psi_2] \in B$. Thus, $H_2 \underset{\psi}{\cup} H_2'$ is homeomorphic to S^3.

E x a m p l e s. Let $M_i \approx H_2 \underset{\psi_i}{\cup} H_2'$ de a Heegaard splitting of genus 2, where $\psi_i \in \mathrm{Aut}(\pi_1(N))$, $N = H_2 \cap H_2' = \partial H_2 = \partial H_2'$ and $i = 1, 2, 3, 4, 5$. Define

$$\{\psi_1(a_1) = a_1 b_1, \quad \psi_1(a_2) = a_2 b_2, \quad \psi_1(b_1) = b_1, \quad \psi_1(b_2) = b_2\}$$
$$\{\psi_2(a_1) = a_1 b_1^{-1}, \quad \psi_2(a_2) = a_2 b_2^{-1}, \quad \psi_2(b_1) = b_1, \quad \psi_2(b_2) = b_2\}$$
$$\{\psi_3(a_1) = a_1, \quad \psi_3(a_2) = a_2, \quad \psi_3(b_1) = b_1 a_1, \quad \psi_3(b_2) = b_2 a_2\}$$
$$\{\psi_4(a_1) = a_1, \quad \psi_4(a_2) = a_2, \quad \psi_4(b_1) = b_1 a_1^{-1}, \quad \psi_4(b_2) = b_2 a_2^{-1}\}$$
$$\{\psi_5(a_1) = a_1 b_1, \quad \psi_5(a_2) = a_2 b_2, \quad \psi_5(b_1) = a_1^{-1}, \quad \psi_5(b_2) = a_2^{-1}\}$$

where

$$\pi_1(N) \cong (a_1, a_2, b_1, b_2; \quad a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdot a_2 \cdot b_2 \cdot a_2^{-1} \cdot b_2^{-1} = 1)$$

Then, the Poincaré conjecture is true for M_i, $i = 1, 2, 3, 4, 5$.

R e m a r k. The automorphisms $\psi_1, \psi_2 \in \mathrm{Aut}(\pi_1(N))$ are inverse of each other. Similarly $\psi_3, \psi_4 \in \mathrm{Aut}(\pi_1(N))$ are inverse of each other.

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ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἐργασία ἀναφέρεται εἰς ὡρισμένας παρατηρήσεις, ὡς καὶ συμπεράσματα, ἐπὶ τῆς περιφήμου εἰκασίας τοῦ Poincaré κατὰ τὴν ὁποίαν «κάθε συμπαγής, τριδιάστατος πολλαπλότης, ἄνευ συνόρου καὶ ἁπλῶς συναφής, εἶναι ὁμοιομορφικὴ πρὸς τὴν τριδιάστατον σφαῖραν».

Αἱ ἐφαρμοζόμεναι διὰ τὴν ἐργασίαν αὐτὴν μέθοδοι εἶναι ἀλγεβρικαί.

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