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ΜΑΘΗΜΑΤΙΚΑ — **On m - barrelled algebras,** *by Anastasios Mallios* *.
᾿Ανεκοινώθη ὑπὸ τοῦ ᾿Ακαδημαϊκοῦ κ. Φίλ. Βασιλείου.

Our objective in this paper is to derive further information for the class of the topological algebras in title, which have been considered in Ref. [**12**], [**13**].

For completeness sake, we recall the formal definition : Thus, by an *m - barrelled algebra* we mean a topological algebra E, over the complex number field $\mathbb{C}$, in which the ring multiplication is separately continuous (unless otherwise specified), and in such a way that every *m - barrel* in E is a neighborhood of zero.

In this respect, a subset A of E is called an *m - barrel* if it is a barrel, with respect to the topological vector space E, i. e. a closed, balanced, convex and absorbing set, which is moreover idempotent, with respect to the ring structure of E, in the sense that $A . A \subseteq A$.

Thus, the topological algebras we are dealt with in the sequel are such that the respective topological vector spaces are not necessarily locally convex and they comprise the Fréchet (: complete metrisable) topological algebras, and a fortiori the corresponding locally convex and locally m-convex ones [**14**]. On the other hand, they reveal in many respects the situation which one actually has by dealing with these latter classes of algebras, in analogy with the situation one encounters in the

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case of Fréchet (locally convex) topological vector spaces and the barrelled ones.

In particular, we are concerned with topological algebras whose spectra are k-spaces [**6**; p. 248, Definition 9.2]. Thus, m-barrelled topological algebras having this property are of a particular significance in deriving further topological information concerning their spectra. The results obtained herewith have a special bearing on previous recent work of A. G. Dors, concerning Fréchet locally m-convex algebras [**14**] whose spectra are not compactly generated [**5**].

Furthermore, as another application of the class of the topological algebras we are concerned with in this paper, we finally consider the spectra of finitely-generated m-barrelled algebras, extending recent results of R. M. Brooks for finitely-generated Fréchet algebras [**2**].

1. Given a topological space X, the *compactly generated topology* on it, denoted by kX, is the weak topology on X defined by the totality of its compact subsets, i.e. a subset A of X is open in the topology kX if, and only if, $A \cap K$ is open in K, for every compact subset K of X (cf. also [**6**; p. 131]).

In this concern, a topological space X is said to be a *k-space,* if the respective topology kX coincides with the given topology of X (ibid., p. 248, Definition 9.2).

Lemma 1.1. Let X be a topological space and let $(S_\alpha)_{\alpha\in I}$ be a family of compact subsets of X in such a way that the following condition is satisfied:

(1.1) For every compact subset K of X, there exists an index $\alpha \in I$ with $K \subseteq S_\alpha$.

Then, the compactly generated topology kX of X coincides with the weak topology induced on it by the family $(S_\alpha)_{\alpha\in I}$.

Proof: If τ is the weak topology on X determined by the family $(S_\alpha)_{\alpha\in I}$, since S_α, $\alpha \in I$, is a compact subset of X, one obviously has $kX \leqslant \tau$. On the other hand, let B be a τ-closed subset of X, and let K be a compact subset of X, so that K is τ-closed as well. Now, by hypothesis, there exists an index $\alpha \in I$, with $K \subseteq S_\alpha$, so that $B \cap K$ is a τ-closed and hence a compact subset of S_α, so that it is a closed subset of X and hense a closed subset of K as well, that is B is a kX-closed subset of X, and this finishes the proof. ∎

Given a topological space X, we denote by $C(X)$ the set of all complex-valued continuous functions on X, and by $C_c(X)$ the preceding set endowed with the topology of compact convergence in X. Thus, we now have the following.

Theorem 1.1. Let X be a completely regular space. Then, the following assertions are equivalent:

1) X is a k-space.

2) The space $C_c(X)$ is complete and the topology kX of the given space is also completely regular.

3) The space X has the weak topology determined on it by a family $(S_\alpha)_{\alpha \in I}$ of compact subsets of it, satisfying the condition (1.1) above.

Proof: Admitting 1), we have that X is a completely regular k-space, so that the assertion 1) $\Longrightarrow$ 2) follows by [**10**; p. 231, Theorem 12]. On the other hand, the implication 2) $\Longrightarrow$ 1) is contained in Lemma 2 of [**19**; 268]. Besides, one obviously has that 1) $\Longrightarrow$ 3), and finally 3) $\Longrightarrow$ 1) is an immediate consequence of the preceding Lemma 1.1 and this finishes the proof. ∎

We now specialize to the case the topological space X as above is the spectrum of a topological algebra E (cf. also [**13**; p. 470]. Thus, if $M(E)$ is the spectrum of E and $(U_\alpha)_{\alpha \in I}$ is a local basis of E, then one obtains the following, useful for the sequel relation:

$$M(E) = \bigcup_{\alpha \in I} (M(E) \cap U_\alpha^{\circ}), \tag{1.2}$$

where U_α°, $\alpha \in I$, denotes the polar set of U_α in E', the topological dual of the topological vector space E.

Thus, by considering the particular class of the topological algebras in title, we now have the following.

Lemma 1.2. Let E be an m-barrelled topological algebra whose spectrum is $M(E)$. Moreover, let $(U_\alpha)_{\alpha \in I}$ be a local basis of E. Then, for every compact subset K of $M(\mathrm{E})$ there exists an index $\alpha \in \mathrm{I}$ such that $\mathrm{K} \subseteq M(\mathrm{E}) \cap \mathrm{U}_\alpha^{\mathrm{O}}$, so that

$$\mathfrak{S} \equiv \{M(\mathrm{E}) \cap \mathrm{U}_\alpha^{\mathrm{O}} \equiv M(\mathrm{E}_\alpha) : \alpha \in \mathrm{I}\} \tag{1.3}$$

is a *k-covering family* of the spectrum of the given algebra E, i. e. $\mathfrak{S}$ is a covering of $M(\mathrm{E})$ which also satisfies the condition (1. 1) of Lemma 1. 1 above.

Proof: If $\mathrm{K} \subseteq M(\mathrm{E})$ is compact, then it is a fortiori weakly bounded, so that K^{O}, the polar set of K in E, is a closed, balanced, convex and absorbing subset of E (cf. also [**9**; p. 190, Proposition 1, and p. 198, Proposition 3]), which is also idempotent, so that K^{O} is actually an m-barrel in E, and hence by hypothesis a neighborhood of zero in E. Thus, there exists an index $\alpha \in \mathrm{I}$, with $\mathrm{U}_\alpha \subseteq \mathrm{K}^{\mathrm{O}}$, so that $\mathrm{K} \subseteq \mathrm{K}^{\mathrm{OO}} \subseteq \mathrm{U}_\alpha^{\mathrm{O}}$, and hence $\mathrm{K} \subseteq M(\mathrm{E}) \cap \mathrm{U}_\alpha^{\mathrm{O}}$, which proves the first part of the assertion. On the other hand, since $\mathrm{U}_\alpha^{\mathrm{O}}$ is a closed weakly bounded subset of E′ (ibid.), the same is true for the set $M(\mathrm{E}) \cap \mathrm{U}_\alpha^{\mathrm{O}}$, for every $\alpha \in \mathrm{I}$, so that each of these sets is a compact subset of the spectrum of E by [**13**; p. 470, Theorem 2. 1] [1], and the proof is completed. ∎

Now, by specializing to the case the algebra E has a denumerable local basis, one obtains by the following theorem a strengthening of a previous result of E. A. Michael for Fréchet locally m-convex topological algebras (cf. [**14**; p. 25, Lemma 6. 2]). Thus, we have.

Theorem 1. 2. Let E be an m-barrelled topological algebra whose spectrum is $M(\mathrm{E})$. Then, there exists a k-covering family of $M(\mathrm{E})$ (Lemma 1. 2), so that if, in particular, E is also metrisable, the preceding family is denumerable, that is the spectrum of E is a hemicompact topological space. ∎

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1. I take this opportunity to correct an inadvertence which has overpassed the proof-reading of my paper [**13**], concerning Theorem 2 1 in p. 470 by reading «weakly relatively compact» instead of «weakly compact» as it is stated therein. It can besides be spotted by Corollary 2. 1 to that Theorem in the same paper, and it has been also kindly marked out by Professor S. Warner in his review of that paper in Math. Reviews 42 (1971) ⋕ 5047.

The preceding result it also yields a well-known result of S. Warner (cf. [**19**; p. 269, Theorem 4]), in the sense that, in connection with Theorem 3.1 in Ref. [**13**; p. 474], one has the following.

Corollary 1.1. A full Fréchet locally convex algebra E is necessarily locally m-convex, its topology being actually that of the topological algebra $C_c(M(E))$, where its spectrum $M(E)$ is a hemicompact k-space. ∎

On the other hand, by combining Lemma 1.2 as well as Theorem 1.1, one gets the following.

Theorem 1.3. Let E be an m-barrelled topological algebra whose spectrum is $M(E)$. Then, the following two assertions are equivalent:

1) $M(E)$ is a k-space.

2) $M(E)$ has the weak topology determined on it by the family $\mathfrak{S}$ of Lemma 1.3 above.

Consequently, for an m-barrelled topological algebra E, one has the relation

$$M(E) = \varinjlim M(E_\alpha), \tag{1.4}$$

within a homeomorphism, if, and only if, the spectrum of the given algebra E is a k-space (cf. rel. (1.3)). ∎

Scholium. Consider a topological algebra E in which the ring multiplication is (jointly, i.e. in both variables) continuous, with spectrum $M(E)$, and let $\hat{E}$ be the completion of E with spectrum $M(\hat{E})$. Now, the following relation

$$M(\hat{E}) = M(E) \tag{1.5}$$

holds true, *within a continuous bijection,* which is not in general a homeomorphism, even for locally m-convex algebras (cf. [**7**; p. 99], and [**3**; p. 210]). Now, if E is, in particular, an m-barrelled algebra and $(U_\alpha)_{\alpha \in I}$ is a local basis of E, one has the relation

$$M(\hat{E}) \cap U_\alpha^\circ = M(E) \cap U_\alpha^\circ , \tag{1.6}$$

within a homeomorphism, for every $\alpha \in I$: Indeed, this is a consequence of the hypothesis for E and the fact that on an equicontinuous subset of

$M(\mathrm{E})$ (cf. also [**13**; p. 470, Corollary 2.1]) the two members of (1.5) induce the same topology. Therefore, *if* $M(\mathrm{E})$ *is a k-space,* i.e. it has, for instance, the weak topology defined on it by the family (1.6) (cf. also Theorem 1.3), then *the preceding relation* (1.5) *holds true within a homeomorphism.*

Thus, one gets, by the preceding conclusion, a still weaker condition than local equicontinuity for the spectrum of the given algebra E in order that (1.5) to hold true within a homeomorphism (cf. also [**13**; p. 479, Scholium]), since under the hypothesis that E is an m-barrelled algebra the spectrum of E is locally equicontinuous if, and only if, it is locally compact (cf. Theorem 3.1 below). On the other hand, there do exist even Fréchet locally m-convex algebras whose spectra are not k-spaces [**5**]. In this concern, it would be of interest to have necessary conditions which the hypothesis that (1.5) is a homeomorphism would imply for the topological space $M(\mathrm{E})$ and / or the topological algebra E.

Now, assuming that E is a complete locally m-convex algebra, if $(\mathrm{U}_\alpha)_{\alpha\in\mathrm{I}}$ is a local basis of E consisting of m-barrels, one obtains a projective system $(\mathrm{E}_\alpha)_{\alpha\in\mathrm{I}}$ of Banach algebras in such a way that $\mathrm{E} = \varprojlim \mathrm{E}_\alpha$, within a topological algebraic isomorphism [**14**], and hence the spectrum of E is given by the relation

$$M(\mathrm{E}) = \varinjlim M(\mathrm{E}_\alpha), \tag{1.7}$$

within a continuous bijection of the second member onto the first, this being not always a homeomorphism (cf., for instance, [**8**; p. 161, Remark 1.4]). Now, as an implication of Theorem 1.3 above, one has the following strengthening of an analogous result of A. G. Dors [**5**], obtained for commutative Fréchet locally m-convex algebras with identity elements. That is, we have:

Theorem 1.4. Let E be a complete, m-barrelled locally m-convex algebra whose spectrum is $M(\mathrm{E})$. Then, one has the preceding relation (1.7), i.e.

$$M(\mathrm{E}) = \varinjlim M(\mathrm{E}_\alpha), \tag{1.8}$$

within a homeomorphism if, and only if, $M(\mathrm{E})$ is a k-space. ∎

We finally remark that the preceding result provides supplementary information concerning the topological structure of the spectrum of certain (locally convex) topological algebras admitting functional representations (cf. [**13**; p. 474, Theorem 3. 1, and p. 475, Corollary 3. 1]). On the other hand, it also supplements, for the case under consideration, the relevant result in Ref. [**8**; p. 160, Proposition 1. 2, and p. 161, Remark 1. 4]).

2. Suppose X is a completely regular (Hausdorff topological) space, so that by considering the evaluation map, we may regard X as a topological subspace of the weak topological dual $(C_c(\mathrm{X}))_s'$ of the space $C_c(\mathrm{X})$ (cf. also the preceding section).

Now, we say that a (completely regular) topological space X is a *Nachbin - Shirota space* if the weakly bounded sets and the weakly relatively compact sets in X coincide, when the space X is identified as above, i. e. X has the «Heine - Borel property», when it is considered as a topological subspace of $(C(\mathrm{X}))_s'$.

Thus, by the classical result of L. Nachbin (cf. [**16**; p. 471, Theorem 1]) and T. Shirota (cf. [**17**; p. 294, Theorem]), *the* (topological vector) *space* $C(\mathrm{X})$ *is barrelled if, and only if,* (the completely regular space) X *is a Nachbin - Shirota space.*

The following result characterizes the Nachbin-Shirota spaces as those which actually appear as spectra of m - barrelled algebras. That is, more precisely, we have.

Theorem 2. 1. A Hausdorff topological space X is a Nachbin - Shirota space if, and only if, it is within a homeomorphism, the spectrum of an m - barrelled topological algebra.

Proof: If X is a Nachbin - Shirota space then it is homeomorphic to the spectrum of the algebra $C_c(\mathrm{X})$ which by the preceding is a barrelled space and hence a fortiori an m - barrelled algebra. Conversely, the spectrum of an m - barrelled algebra is a Nachbin - Shirota space by [**13**; p. 470, Corollary 2. 1], and this finishes the proof. ∎

Now, the question concerning the coincidence of the three classes of subsets of the spectrum $M(\mathrm{E})$ of a given topological algebra E, which appear in Corollary 2.1 in Ref. [**13**; p. 470], relative to the property of E of being an m-barrelled algebra remains still unsettled; (in this respect, cf. also [**12**; p. 306]). We have, of course, a trivial exception in case the algebra E admits a functional representation, where the situation is actually explained by the Nachbin-Shirota theorem [**16**], [**17**] (cf. also Theorem 2.2 below).

In particular, we comment in the sequel on certain implications, which the following condition implies, referred to the spectrum $M(\mathrm{E})$ of a given topological algebra E. That is, consider the following statement.

(2.1) A subset of $M(\mathrm{E})$ is weakly bounded if, and only if, it is equicontinuous.

In this concern, we thus obtain first the folowing.

Lemma 2.1. Let E be a topological algebra the spectrum $M(\mathrm{E})$ of which satisfies the condition (2.1). Then, one has the following:

1) The respective Gel'fand map $g: \mathrm{E} \rightarrow C_c(M(\mathrm{E}))$ is continuous.

2) $M(\mathrm{E})$ is a Nachbin-Shirota space and hence the range of the map g above is a barrelled (and hence a fortiori an m-barrelled) algebra.

Proof: Since every (weakly) compact subset of $M(\mathrm{E})$ is (weakly) bounded, so that by (2.1) equicontinuous, the assertion 1) follows by [**12**; p. 305, Theorem 3.1]. Now, the assertion 2) is a consequence of the Nachbin-Shirota and the Alaoglu-Bourbaki theorems, and this completes the proof of the lemma. ∎

On the other hand, as a corollary to the preceding and by applying the argumentation in the proof of Theorem 3.1 in Ref. [**13**; p. 474], one obtains the folowing strengthened (cf. [**12**; p. 306]) form of that result. That is, we have.

Theorem 2.2. Let E be a full, Pták locally convex algebra whose spectrum $M(\mathrm{E})$ satisfies the condition (2.1) above. Then, E is a Michael algebra, for which the respective Gel'fand map is a topological (algebraic) isomorphism and $M(\mathrm{E})$ is a k-space. In particular, E is a barrelled (locally m-convex) algebra. ∎

3. We are in the sequel concerned with the continuity of the

Gel'fand map of a given topological algebra supplementing the relevant results obtained in Ref. [**12**].

Thus, we first have the following lemma, the proof of which has been motivated by that of an analogous argument in Ref. [**4**; p. 294, Lemma 3. 2]. That is, one has.

Lemma 3. 1. Let E be a topological algebra whose spectrum $M(E)$ is locally equicontinuous (i. e. every point of $M(E)$ has an equicontinuous neighborhood). Then, $M(E)$ is a locally compact (Hausdorff) space.

Proof: If U is an open equicontinuous neighborhood of an element f in $M(E)$, then it is also weakly relatively compact in E_s' (Alaoglu-Bourbaki), so that $\bar{U} \cap M(E) = \bar{U} - \{0\}$ is a locally compact subset of $M(E)$, and hence U as well, so that there exists a compact neighborhood of $f \in U$, and this proves the assertion. ∎

We are now in a position to state the following.

Theorem 3. 1. Let E be a topological algebra whose spectrum is $M(E)$. Moreover, consider the following two statements:

1) $M(E)$ is locally equicontinuous.

2) $M(E)$ is a locally compact (Hausdorff) space.

Then (Lemma 3. 1), 1) implies 2). On the other hand, if the respective Gel'fand map of the algebra E is continuous, then the two statements are equivalent.

Proof: We actually have to prove that 2) implies 1) as well, under the supplementary supposition for the Gel'fand map of E. But then, every compact subset of the spectrum of E is also equicontinuous by [**12**; p. 305, Theorem 3. 1], and this finishes the proof. ∎

The preceding result constitutes a strengthening of a similar result in Ref. [**12**; p. 302, Theorem 2. 1], since *every m-barrelled topological algebra has the respective Gel'fand map continuous* (ibid. p. 306, Corollary 3. 1; the respective proof of the cited result is obviously valid for the present «non-locally convex case»), while the converse is not in general true (ibid. p. 306).

On the other hand, we also have the following.

Theorem 3. 2. Let E be a topological algebra whose spectrum $M(E)$ is locally equicontinuous. Then, the respective Gel'fand map of the algebra is continuous.

Proof: By hypothesis and the fact that «a local uniform limit of continuous functions gives a continuous function», one can prove by standard argumentation that the map

$$(x, f) \to f(x) : E \times M(E) \to \mathbb{C}$$

is continuous (cf. also [**1**; p. 24, Corollaire 3]), so that the assertion now follows by Théorème 3 of the same Ref. [**1**; p. 45], and this completes the proof. ∎

The preceding two results can finally be combined into the form of the following.

Theorem 3. 3. Let E be a topological algebra whose spectrum is $M(E)$. Then, the following two statements are equivalent:

1) $M(E)$ is locally equicontinuous.

2) $M(E)$ is a locally compact (Hansdorff) space, and the respective Gel'fand map of the algebra is continuous. ∎

We conclude this section with the following theorem, which constitutes a strengthened form of a combined result in Ref. [**13**: p. 472, Theorem 2. 2] and [**18**; p. 7, Theorem 6] in case of complete algebras with an identity element. It also reinforces the relevant result in Ref. [**14**; p. 59, Theorem 13. 6]). That is, one has the following.

Theorem 3. 4. Let E be a locally m-convex algebra with an identity element, whose spectrum is $M(E)$. Consider the following two assertions:

1) $\hat{E}$ (· the completion of E) is a Q-algebra.

2) $M(E)$ is a weakly compact subset of E'.

Then, 1) implies 2). Moreover, if E is commutative, with the respective Gel'fand map continuous, then the two preceding assertions are equivalent.

Proof: The assertion 1) $\Rightarrow$ 2) is a consequence of Theorem 2. 2. in Ref. [**13**; p. 472]. Now, if the Gel'fand map of E is continuous with $\mathcal{M}(E) \subseteq E_s'$ compact, then by [**12**; p. 305, Theorem 3. 1], $\mathcal{M}(E)$ is an equicontinuous subset of E', so that by hypothesis and [**11**; p. 174, Lemma 2. 2], $\hat{E}$ is a Q - algebra, and this finishes the proof. ∎

Corollary 3. 1. A commutative, complete locally m - convex algebra with an identity element is a Q - algebra if, and only if, its spectrum is (weakly) compact and the respective Gel'fand map of the algebra continuous. ∎

As another application of the preceding Theorem 3. 4, we also have the following Corollary, which is to be compared with the analogous result in Ref. [**14**; p. 59, Theorem 13.6]. That is, one has.

Corollary 3. 2. Let E be a commutative locally m - convex algebra with an identity element and the respective Gel'fand map continuous. Then, the following assertions are equivalent:

1) $\hat{E}$ (: the completion of E) is a Q - algebra.
2) $\mathcal{M}(E)$ is weakly compact.
3) $\mathcal{M}(E)$ is equicontinuous. ∎

Corollary 3. 3. (M. Bonnard). Let E be a commutative, complete locally m - convex algebra with an identity element. Then, E has a continuous inversion (L. Waelbroeck) if, and only if, its spectrum $\mathcal{M}(E)$ is compact and the respective Gel'fand map of E continuous. ∎ (Cf. M. Bonnard, Sur les applications du calcul fonctionnel holomorphe. Bull. Soc. math. France, Memoire 34 (1973), 5 - 54).

4. In this final section we consider the spectra of certain finitely-generated topological algebras, motivated by and extending as well similar recent results of R. M. Brooks in Ref. [**2**]. Thus, we first have the following.

Lemma 4. 1. Let E be a commutative complete finitely - generated locally m - convex algebra with an identity element, and whose spectrum is $\mathcal{M}(E)$. Moreover, let $(x_1, \ldots, x_n)$ be a generating family for E, and let $\varphi : \mathcal{M}(E) \to \sigma(x_1, \ldots, x_n) \subseteq \mathbb{C}^n$ be the natural map defined by the relation

$$\varphi(h) \underset{\text{Def.}}{:=} (\hat{x}_1(h), \ldots, \hat{x}_n(h)), \tag{4.1}$$

for every $h \in \mathcal{M}(E)$, where $\sigma(x_1, \ldots, x_n)$ denotes the «joint-spectrum» of the family $(x_1, \ldots, x_n)$. Finally, consider the following two statements:

1) There exists a local basis $(U_\alpha)_{\alpha \in I}$ of E in such a way that $(\sigma_\alpha(x_1, \ldots, x_n))_{\alpha \in I}$ is a k-covering family for $\sigma(x_1, \ldots, x_n)$.

2) The map φ defined by (4.1) is a homeomorphism.

Then, under the preceding circumstances, 1) implies 2). Moreover, if the given algebra E is m-barrelled, then the two statements are equivalent.

Proof: It is easy to prove that the map $\varphi : \mathcal{M}(E) \to \mathbb{C}^n$ defined by (4.1) is a continuous bijection. On the other hand, since $\sigma(x_1, \ldots, x_n)$ is a k-space, φ is a homeomorphism if, and only if, it is a proper map (cf. also [**2**; p. 146, Lemma 1.3]). Thus, for every compact $K \subseteq \sigma(x_1, \ldots, x_n)$, there exists by hypothesis $\alpha \in I$ with $K \subseteq \sigma_\alpha(x_1, \ldots, x_n)$, so that $\varphi^{-1}(K) \subseteq \mathcal{M}(E_\alpha) \underset{\text{homeo}}{\cong} \mathcal{M}(\hat{E}_\alpha)$, the latter space being a compact subset of the spectrum of E, which proves the assertion. Now, if E is an m-barrelled algebra, then 2) $\Rightarrow$ 1) by Lemma 1.2 in the preceding, and this finishes the proof of the lemma. ∎

We conclude with the following.

Theorem 4.1. Let E be a commutative, complete, finitely-generated, m-barrelled, locally m-convex algebra with an identity element, such that its spectrum $\mathcal{M}(E)$ is homeomorphic (via the relation (4.1)) to a subset of the complex space $\mathbb{C}^n$. Then, this set, say $S \subseteq \mathbb{C}^n$, is polynomially convex, i.e. for every compact $K \subseteq S$, one has the relation:

$$\mathcal{M}(P(K)) = H(K) \subseteq S, \tag{4.2}$$

within a homeomorphism, where the first member of the preceding relation denotes the spectrum of the uniform closure P(K) of the algebra of polynomials on K.

Proof: Let $(x_1, \ldots, x_n)$ be a generating family for the given algebra E, so that $\mathcal{M}(E) = \sigma(x_1, \ldots, x_n)$ within a homeomorphism defined by the relation (4.1) above. Now, by the preceding Lemma 4.1, there exists a local basis $(U_\alpha)_{\alpha \in I}$ of E such that $\{\sigma_\alpha(x_1, \ldots, x_n)\}_{\alpha \in I}$ is a k-covering family for $\sigma(x_1, \ldots, x_n)$. Therefore, if K is any compact subset of $S \equiv \sigma(x_1, \ldots, x_n)$, there exists $\alpha \in I$, with $K \subseteq \sigma_\alpha(x_1, \ldots, x_n)$. Hence, one obtains, if H(K) denotes the polynomially convex hull of K,

$$H(K) = \mathcal{M}(P(K)) \underset{\text{homeo. into}}{\overrightarrow{\subset}} \mathcal{M}\big(P(\sigma_\alpha(x_1, \ldots, x_n))\big)$$

$$\underset{\text{Def.}}{:=} \mathcal{M}\big(P(\sigma_{\hat{E}_\alpha}(\dot{x}_1, \ldots, \dot{x}_n))\big) \underset{\text{homeo.}}{=} \sigma_{\hat{E}_\alpha}(\dot{x}_1, \ldots, \dot{x}_n),$$

the latter set being a compact subset of S, by its definition, and this finishes the proof. ∎

In connection with the preceding theorem, we remark that *the spectrum* $S = \sigma(x_1, \ldots, x_n)$ *of the m-barrelled algebra* E *under consideration is polynomially convex if, and only if, it is «polynomially convex» with respect to the k-covering family* $\{\sigma_\alpha(x_1, \ldots, x_n)\}$, $\alpha \in I$, in the sense of Ref. [**2**; p. 146. Cf. also ibid., Theorem 1.2].

By referring to the foregoing, we also note that the characterization of polynomially convex subsets of $\mathbb{C}^n$ which might be spectra of finitely-generated m-barrelled algebras, not necessarily Fréchet ones, is still lacking. In the latter case such a characterization has been given by R. M. Brooks in Ref. [**2**; p. 149, Theorem 2.2], and a similar result for algebras of holomorphic functions, actually Stein algebras, has been recently announced by E. R. Heal and M. P. Windham (: Finitely-generated Fréchet algebras with applications to Stein manifolds. Notices Amer. Math. Soc. 20 (1973), A-149).

(Note added in proof). In connection with Theorem 2.1 above, we remark that *the spectrum of a given topological algebra may be a Nachbin-Shirota space without the respective algebra to be an m-barrelled one.* This follows by some recent considerations of A. K. Chilana: The space of bounded sequences with the mixed topology. Pacific J. Math. 48 (1973), 29-33. (Cf. ibid., p. 30, II, and p. 31, VII, as well as § 2 above). So-

mething stronger is actually true (cf. also [12; p. 306]), as it is pointed out in a forthcoming paper by the present author (: On the barrelledness of a topological algebra relative to its spectrum. Remarks).

ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα μελέτη ἀποτελεῖ συνέχειαν ἐπὶ τῆς θεωρίας τῶν m - κυλινδροειδῶν (: m - barrelled) τοπολογικῶν ἀλγεβρῶν, μὴ κατ' ἀνάγκην τοπικῶς κυρτῶν, αἱ ὁποῖαι ἐθεωρήθησαν εἰς προηγουμένας ἐργασίας τοῦ συγγραφέως (πρβλ. [**12**], [**13**]). Μελετῶνται ἰδιαιτέρως τὰ στοιχεῖα τῆς ὡς ἄνω κατηγορίας τῶν ἀλγεβρῶν, τῶν ὁποίων τὰ φάσματα εἶναι k - χῶροι, διὰ τὴν ἀντίστοιχον τοπολογίαν Gel'fand. Διαφαίνεται ὅτι ἡ ἐν λόγῳ κατηγορία τῶν ἀλγεβρῶν ἐνέχει ἐν πολλοῖς ἀνάλογον σημασίαν διὰ τὴν θεωρίαν τῶν τοπολογικῶν ἀλγεβρῶν Fréchet, τοπικῶς m - κυρτῶν (Michael [**14**]), πρὸς ἐκείνην τῶν κυλινδροειδῶν χώρων (barrelled spaces) εἰς τὴν θεωρίαν τῶν (τοπικῶς κυρτῶν) τοπολογικῶν διανυσματικῶν χώρων, ὡς πρὸς τοὺς ἀντιστοίχους χώρους Fréchet.

Ἀφ' ἑτέρου, θεωροῦνται οἱ τοπολογικοὶ χῶροι Nachbin - Schirota, ἐξ ἀφορμῆς τοῦ ὁμωνύμου κλασσικοῦ θεωρήματος, χαρακτηρίζονται δὲ οὗτοι ὡς φάσματα, ὡς πρὸς τὴν τοπολογίαν Gal'fand, τοπολογικῶν ἀλγεβρῶν τῆς ὡς ἄνω κατηγορίας. Συναφῶς παρατηρεῖται ὅτι ἡ συνέχεια τῆς ἀντιστοίχου ἀπεικονίσεως Gal'fand (πρβλ. [**12**: σελ. 305]) διὰ δοθεῖσαν τοπολογικὴν ἄλγεβραν ἀντικαθιστᾷ, εἰς ὡρισμένας ἐνδιαφερούσας περιπτώσεις, τὴν ἰσχυροτέραν ὑπόθεσιν ὅτι ἡ θεωρουμένη ἄλγεβρα εἶναι m - κυλινδροειδής.

Τελικῶς μία περαιτέρω ἐφαρμογὴ τῆς θεωρουμένης κατηγορίας τοπολογικῶν ἀλγεβρῶν δίδεται εἰς τὰ πλαίσια τῆς θεωρίας τῶν τοπολογικῶν ἀλγεβρῶν ὁλομόρφων συναρτήσεων ἐπὶ μιγαδικῶν χώρων, εἰς ὅ,τι ἀφορᾷ εἰς τὸν χαρακτηρισμὸν ἑνὸς «πολυωνυμικῶς κυρτοῦ» ὑποσυνόλου ἑνὸς (ἀριθμητικοῦ) μιγαδικοῦ χώρου, ὡς φάσματος μιᾶς m - κυλινδροειδοῦς ἀλγέβρας, μὴ κατ' ἀνάγκην Fréchet, «πεπερασμένως παραγομένης».

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