ΜΑΘΗΜΑΤΙΚΑ.— **Plateau's problem and its importance to physics,** *by Themistocles M. Rassias**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φίλωνος Βασιλείου.

1. Since its formulation by Plateau in the 19^{th} century, little has been known about the number of simply connected minimal surfaces spanning a simple closed curve Γ in R^3. One of the most natural questions to ask is whether the solutions to the Plateau problem are unique. In general they are not, R. Courent [**1**], T. Rassias [**8**]. It was generally conjectured that when the Jordan curve is extremal i. e. it lies, on the boundary of its convex hull, then the Douglass solution to the Plateau problem is embedded. The first progress in this direction was made after the discovery of the following very interesting theorems due to T. Rado and J. C. C. Nitsche.

Theorem (T. Rado) 1. *A Jordan curve Γ whose orthogonal projection on some plane is a simply covered convex curve bounds a unique area minimizing surface.*

Theorem (J. C. C. Nitsche) 2. *An analytic simple closed curve Γ with total curvature of Γ less than 4π bounds a unique minimal surface which is free of branch points.*

New proofs of the above two theorems have been given in the spirit of the Morse - Palais - Smale theory [**6**], [**8**], [**10**], on Hilbert manifolds.

2. Consider the space of parametrizations:

$$P = \{\gamma : S^1 \to R^3 \text{ where } \gamma \varepsilon\ C^{2,a},\ S^1 \text{ is the circle } R/^{2\pi}Z \text{ and a is a fixed positive number such that } 0 < a < \frac{1}{2}\},$$

endowed with the C^2 norm, i. e., for $\gamma \varepsilon$ P:

$$\|\gamma\| = \max\{\|\gamma\|_\infty, \|\gamma'\|_\infty, \|\gamma''\|_\infty\}.$$

Then the following theorem is true:

* ΘΕΜΙΣΤΟΚΛΗ Μ. ΡΑΣΣΙΑ, **Τὸ πρόβλημα τοῦ Plateau καὶ ἡ σημασία του εἰς τὴν Φυσικήν.**

Theorem 3. *Almost every* $\gamma \varepsilon$ P *bounds a unique area minimizing surface.*

Following techniques from Morse theory one can also prove the following theorems:

Theorem 4. *A smooth Jordan curve* Γ *of total curvature at most* 6π *bounds only a finite number of minimal surfaces of the type of the disk.*

Theorem 5. *Let* Γ *be an arbitrary smooth simple closed curve lying in the smooth boundary of a uniformly convex subset of* R^3. *Then* Γ *bounds a smoothly embedded minimal disk of least area among all embedded disks having* Γ *as boundary.*

Theorem 6. *In Euclidean space of three dimansions, let* Γ_1, Γ_2, *be any two Jordan curves not intersecting one another. If the minimal surfaces* M_1 *and* M_2 *determined by* Γ_1 *and* Γ_2 *taken separately have in common a point* Q *that is regular for both of them, then there exists a doubly-connected minimal surface* M *bounding by* Γ_1, Γ_2.

Remark. Theorem 6 solves Plateau's problem for two contours in R^3.

3. Research Problems. a. *Does there exist a complete, closed nonorientable minimal surface in* R^3? If we assume the surfaces to be embedded then of cource such surfaces do not exist because by the *Alexander Duality Theorem; A closed codimension one submanifold of* R^n *disconnects* R^n *into two pieces and so a non-orientable manifold* M^{n-1} *of dimension* n—1 *never embeds as a closed submanifold of* R^n. Thus a negative answer to the above problem exists if we assume the complete non-orientable surfaces to be embedded. *The question thus remains open for immersed surfaces in* R^3. This will answer problem number 45 posed by J. C. C. Nitsche [**5**, p. 260].

b. Let M be a complete C^∞ - Riemannian manifold modelled on a separable Hilbert space H. Give necessary and sufficient conditions on M so that any two points P, Q ε M can be joined by a geodesic segment whose length is equal to the Riemannian distance between P and Q.

4. Minimal Surfaces in Physics. It has been proved that in Riemannian manifolds minimal surfaces and their generalizations to higher dimensions have been of use in Physics and other technological sciences. The analogous surfaces in spacetimes in particular manifolds with Lorentz metric, are maximal surfaces. Maximal surfaces have been very essential for questions concerning the understanding of the n - body problem in a gravitational field as well as of the bynamics of the gravitational field. It is not difficult to prove that there are closed spacetimes without any maximal surfaces.

Maximal surfaces in closed spacetimes like the Friedman models (see for example C. W. Misner, K. Thorne and J. A. Wheeler [**3**] occur rarely, unless the universe is static. In fact we can say that the spacetime either represents an ever expanding universe without any maximal spacelike surface, as for example the *«open»* Friedman models or the spacetime possesses exactly one maximal surface as it happens in the *«closed»* Friedman models.

In another paper an explanation of the importance of Global Variational Theory and in particular minimal surface theory for the study of some current research problems of Theoretical Physics will be given.

ΠΕΡΙΛΗΨΙΣ

Τὸ πρόβλημα τοῦ Plateau καὶ ἡ σπουδαιότης του εἰς τὴν Φυσικὴν εἶχεν ἀπασχολήσει ὡρισμένους τῶν διασημοτέρων Μαθηματικῶν ἐρευνητῶν ὅλου τοῦ κόσμου ἀπὸ τῆς ἐποχῆς τοῦ 18ου αἰῶνος. Ἰδιαιτέρως ὁ προσδιορισμὸς τοῦ ἀριθμοῦ τῶν ἐλαχίστων ἐπιφανειῶν, αἱ ὁποῖαι φράσονται ἀπὸ μίαν δεδομένην καμπύλην (ἢ σύστημα καμπύλων) τοῦ Jordan εἰς τὸν χῶρον τοῦ Εὐκλείδου παρέμενεν ἕνα ἄλυτον καὶ πολὺ δύσκολον πρόβλημα. Ὁ συγγραφεὺς διὰ τῆς παρούσης ἐργασίας του ἀνευρίσκει μέθοδον βασιζομένην εἰς τὴν Γεωμετρίαν τῶν ἀπειροδιαστάτων πολλαπλοτήτων τύπου Hilbert (ἢ Banach) διὰ τὸν καθορισμὸν τοῦ ζητουμένου ἀριθμοῦ ἐλαχίστων ἐπιφανειῶν. Ἡ πορεία σκέψεώς του ἐπιλύει καὶ τὸ πρόβλημα τοῦ Plateau διὰ δύο καμπύλες τοῦ Jordan (θεώρημα 6) εἰς τὸν χῶρον τοῦ Εὐκλείδου καὶ συνεπάγει τὰ θεωρήματα τῶν T. Rado καὶ J. Nitsche, περὶ τῆς μοναδικότητος λύσεως τοῦ προβλήματος τοῦ Plateau, ὡς ἁπλᾶ πορίσματα. Ἡ παροῦσα ἐργασία τελειώνει μὲ τὴν ἔκφρασιν σχέσεως τοῦ προβλήματος τοῦ Plateau, εἰς τὰ Μαθηματικά, μὲ τὸ πρόβλημα τῶν μεγίστων ἐπιφανειῶν εἰς τὴν Θεωρητικὴν Φυσικήν, πρᾶγμα τὸ ὁποῖον ἀποτελεῖ ἕνα τῶν πλέον ἐνδιαφερόντων ἐρευνητικῶν θεμάτων τῆς Φυσικῆς.

REFERENCES

1. R. Courant, Dirichlet's Principle Conformal Mappings and Minimal Surfaces, Interscience Publ. New York (1950).
2. J. Douglass, The higher topological form of Plateau's problem, Ann. R. Scuola Norm. Super. Pisa, Ser. II 8 (1939), 1 - 24.
3. C. W. Misner - K. Thorne and J. A. Wheeler, Gravitation, W. H. Freeman Co., San Francisco (1973).
4. C. B. Morrey, The Problem of Plateau on a Riemannian manifold, Annals of Math. 2 (49), (1948), 807 - 851.
5. J. C. C. Nitsche, On new results in the theory of minimal surfaces, Bull. Amer. Math. Soc. 71 (1965), 195 - 270.
6. R. Palais, Morse theory on Hilbert manifolds, Topology 2 (1963), 299-340.
7. T. Rado, On Plateau's problem, Ann. Math. 31 (1930), 457 - 469.
8. T. M. Rassias, Morse Theory on Hilbert Manifolds and the Plateau's problem, Proceedings of the National Academy of Athens, Greece. Vol. 52 (1977), 365 - 374.
9. ——, Morse theory and the Plateau's Problem, To Appear in the Einstein Anniversary Volume, edited by Dr. T. M. Rassias and Dr. G. M. Rassias, (1981).
10. S. Smale, Morse theory and a non-linear generalization of the Dirichlet problem, Ann. of Math. 2 (80), (1964), 382 - 396.
11. ——, On the Calculus of Variations, Differentiable Analysis, Bombay Colloquium (1964).
12. F. Tomi, On the local uniqueness of the Problem of least area, Arch. Rat. Mech. Anal. 52 (1973), 312 - 318.
13. A. Tromba, The Morse - Sard - Brown theorem for functionals and the Problem of Plateau, to appear in Amer. J. Math.
14. H. S. Tropp, History of Plateau's Problem. To appear.