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MECHANICS. – **The properties the eigenangle ω of the failure tensor of the anisotropic materials and its systematic classification,** *by Pericles S. Theocaris**, National Academy of Athens.

ABSTRACT

*The spectrally decomposed compliance (**S**), stiffness (**C**) and failure (**H**) tensors for transversely isotropic materials, as they are the fiber reinforced composites, and their respective eigenvalues define in a simple and efficient manner the respective elastic eigenstates of loading of the anisotropic materials [1]. It has been shown [2] that the necessary parameters for an invariant description of the elastic behaviour of the transversely isotropic materials are the four eigenvalues of the compliance or stiffness tensor (from which the two firsts are double) complemented by the eigenangle* ω*, a dimensionless parameter derived from this spectral decomposition. However, thermodynamic restrictions are imposed on the variational bounds of this parameter. It is the purpose of this paper to study the influence of the imposed restrictions along the spectrum of variation of the eigenangle* ω*, on the type and properties of the corresponding elastic bodies and describe the limits imposed by the respective values of* ω *on the properties of the anisotropic materials. It is further shown that this single parameter is sufficient to characterize qualitatively both the elastic properties and the respective toughness of the transversely isotropic bodies.*

1. INTRODUCTION

Decompositions of the fourth-rank tensor representing either stiffness, or compliance, or failure, have been extensively used recently to describe efficiently either the underlying geometry symmetry in crystals, or the mechanical properties of anisotropic materials, because the structures of fourth-order tensor reflect this symmetry and, thus, they can reduce the number of independent components describing this symmetry.

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* ΠΕΡΙΚΛΗΣ ΘΕΟΧΑΡΗΣ.— **Αἱ ἰδιότητες τῆς ἰδιογωνίας ω τοῦ τανυστοῦ ἀστοχίας τῆς ἀνισοτρόπου ὕλης καὶ ἡ συστηματικὴ κατάταξίς της.**

Walpole [3] has presented for the first time a reduction of the algebra of the fourth-rank tensors to irreducible subalgebras, which were simpler than the initial one and, therefore, they facilitated operations between these tensors. In this general form of decomposition Walpole included also the spectral decomposition of the fourth-rank tensor and applied these decompositions to define the properties of various crystalline systems. Later-on Rychlewski [4], [5] has shown the possibility to decompose the elastic stiffness and compliance fourth-rank tensors by using the **spectral decomposition**. The advantage of this decomposition is that the elementary idempotent tensors, to which the fourth-rank tensor is decomposed, present the interesting property to define energy orthogonal stress states, that is states where the stress idempotent tensors are mutually orthogonal and at the same time collinear with their respective strain tensors and therefore they correspond to energy orthogonal stress states [6]. Since the failure tensor is the limiting case for the respective σ_m-tensors, which are eigenstates of the compliance tensor **S**, this tensor also possesses the same remarkable property. On the contrary, the decompositions used by Walpole in his crystallographic applications [1] do not belong to spectral decompositions except for the trivial cases of the isotropic fourth-rank tensor and the tensor corresponding to the cubic system.

While Rychlewski has proved the possibility of a spectral decomposition of the fourth-rank symmetric tensor and has shown that this decomposition is the simplest and the one defining the compliance, **C**, the stiffness **S** and the failure **H** of orthotropic materials, he did not proceed to establish the characteristic eigenvalues of this spectral decomposition. It was Theocaris [7], [8] and Theocaris and Philippides [1], [2] who succeeded to decompose spectrally the compliance, or stiffness tensors for a transversely isotropic material and to evaluate their characteristic values, which defined the respective energy-orthogonal stress-states. Furthermore, bounds of the admissible values for the elastic constants and especially for Poisson's ratios are given, obeying restrictions of thermodynamics.

In this paper, bounds based on thermodynamic principles and the positiviness conditions for the elastic properties, which are previously established [1], [2], were studied all over the spectrum of variation of the basic parameter of the eigenangle ω, which alone characterizes qualitatively both the elastic properties and the toughness of the transversely isotropic materials.

2. THE SPECTRAL DECOMPOSITION OF THE FOURTH-RANK TENSOR BESCRIBING THE TRANSVERSELY ISOTROPIC BODY.

Consider the Cartesian coordinate system, which the stress -and strain- tensor components are referred to, being oriented along the principal material directions, with 3-axis being the axis of infinite symmetry of the material, which is normal to the isotropic (transverse) plane. Using engineering constants with subscript (T) to denote elastic properties on the isotropic plane, and subscript (L) the corresponding ones on the normal, (longitudinal) plane, components of the stiffness **C** -or compliance **S**- tensor, associated with the adopted Cartesian system are given in ref. [2]. The eigenvalues of the associated square matrix of the rank six to tensor **S** are expressed by [2]:

$$\lambda_1 = \frac{(1+\nu_T)}{E_T} = \frac{1}{2G_T} \tag{1.1}$$

$$\lambda_2 = \frac{1}{2G_L} \tag{1.2}$$

$$\left.\begin{matrix}\lambda_3 \\ \lambda_4\end{matrix}\right\} = \frac{(1-\nu_T)}{2E_T} + \frac{1}{2E_L} \pm \left\{\left[\frac{(1-\nu_T)}{2E_T} - \frac{1}{2E_L}\right]^2 + \frac{2\nu_L^2}{E_L^2}\right\}^{1/2} \tag{1.3, 1.4}$$

with the two first eigenvalues λ_1 and λ_2 beig of multiplicity two. Furthermore, along the isotropic plane (T) it is valid that:

$$G_T = \frac{E_T}{2(1+\nu_T)} \tag{2}$$

The minimum polynomial of tensor **S** is a quartic and has as roots the eigenvalues λ_1, λ_2, λ_3, and λ_4. The associated form idempotent tensors $\mathbf{E}_m$ (m=1÷4) of the spectral decomposition of **S** are given by:

$$\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 = \mathbf{I} \tag{3}$$

with **I** the spherical unit tensor.

Furthermore, the eigenangle ω, defined in [2], is experessed by:

$$\tan 2\omega = \frac{-2\sqrt{2}\,\nu_L}{\left[(1-\nu_T)\frac{E_L}{E_T} - 1\right]} \tag{4}$$

If the stress states σ_m constitute the eigenstates of tensor **S**, they satisfy the relationship:

$$\sigma_m = \mathbf{E}_m \cdot \sigma \tag{5}$$

with the index m varying between 1 and 4, where σ is the contracted stress tensor.

Relations (5) imply, through a series of calculations, that:

$$\sigma_1 = \left[\frac{1}{2}(\sigma_1 - \sigma_2),\ -\frac{1}{2}(\sigma_1 - \sigma_2),\ 0,\ 0,\ 0,\ \sigma_{12}\right]^T \tag{6.1}$$

$$\sigma_2 = [0,\ 0,\ 0,\ \sigma_{23},\ \sigma_{31},\ 0]^T \tag{6.2}$$

$$\sigma_3 = \left(\frac{1}{\sqrt{2}}\cos\omega(\sigma_1+\sigma_2) + \sin\omega\sigma_3\right)\left[\frac{1}{\sqrt{2}}\cos\omega,\ \frac{1}{\sqrt{2}}\cos\omega,\ \sin\omega,\ 0,\ 0,\ 0\right]^T \tag{6.3}$$

$$\sigma_4 = \left(\frac{1}{\sqrt{2}}\sin\omega(\sigma_1+\sigma_2) - \cos\omega\sigma_3\right)\left[\frac{1}{\sqrt{2}}\sin\omega,\ \frac{1}{\sqrt{2}}\sin\omega,\ -\cos\omega,\ 0,\ 0,\ 0\right]^T \tag{6.4}$$

Relations (6) imply that the stress eigenstates corresponding to spectral decomposition of the compliance tensor **S** for the transversely isotropic materials brake down the stress tensor σ into four elements. The states σ_1 and σ_2 are shears, simple together with pure, and pure shear respectively, which they are independent of the eigenangle ω. The σ_3 -state represents an equilateral stress state in the isotropic plane together with a superposed tension along the infinite symmetry axis (σ_3) of the material, whereas the σ_4 – state replaces the uniaxial tension by uniaxial compression along the symmetry axis σ_3. To these four stress–tensors σ_1 to σ_4 are associated the four respective strain tensors ε_1, ε_2, ε_3 and ε_4, from which the two first strain tensors correspond to a pure distortion of the solid without any volume change.

From relation (4) it can be deduced that the angle ω takes values in the interval (0°, 180°), whereas for the isotropic solid it was shown that ω equals 125.26°. As it can be seen by this relation, there are two possible values of angle ω corresponding to the isotropic medium, that is ω=35.26° and 125.26°. The former value of ω corresponds to a **negative** value of Poisson's ratio, ν, whereas the latter to a **positive** one. Although a negative value for ν is thermodynamically admissible, only special types of composites are possessing such a negative value [9]. For these materials the zone of variation of the eigenangle ω lies beyond the interval of variation included between ω=90° and ω=180°. It is anticipated that these materials with negative Poisson's ratio ν_L lie inside the interval ω=0° and ω=90°.

3. CLASSIFICATION OF VARIETIES OF TRANSVERSELY ISOTROPIC MATERIALS

The failure theory introduced during recent years and based on failure tensor polynomials was mainly developed for establishing failure conditions for compression strong (C-strong) materials and composites, as they are all compacted and thick-packed materials of condensed matter, which naturally present a high resistance in compression. However, it was shown that the same theory is equally valid for tension strong (T-strong) materials [10].

Similar materials presenting this phenomenon of being tension strong in failure are detected up to now, besides the foamy materials and ceramics of high porosity, various types of paper sheets, as well as the oriented polypropylene, and materials which present again a high porosity in their structure. For these materials, however, the critical condition defining the type of the anisotropic material, which is expresed by [11]:

$$\mathrm{tr}\mathbf{h} \leq 0 \tag{7}$$

yields always negative values, whereas for the C-strong materials this quantity should be always positive. In this relation the 2nd rank tensor **h** is the strength differential effect (SDE) tensor **h**, intervening into the failure tensor polynomial expressed by [11]:

$$f(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \,.\, \mathbf{H} \,.\, \boldsymbol{\sigma} + \mathbf{h} \,.\, \boldsymbol{\sigma} - 1 = 0 \tag{8}$$

where the normal components of the failure tensor **H** and the (SDE)-tensor **h** are given by:

$$H_{ii} = \frac{1}{\sigma_{Ti}\sigma_{ci}}, \qquad (i \leq 3)$$
$$h_i = \frac{1}{\sigma_{Ti}} - \frac{1}{\sigma_{ci}} = (\sigma_{ci} - \sigma_{Ti})H_{ii} \tag{9}$$

whereas for shear components are given by:

$$H_{ii} = \frac{1}{\sigma_{si}^{+}\sigma_{si}^{-}} \qquad (i > 3)$$
$$h_i = \frac{1}{\sigma_{si}^{+}} - \frac{1}{\sigma_{si}^{-}} = (\sigma_{si}^{-} - \sigma_{si}^{+})H_{ii} \tag{10}$$

The distance y_0 between the symmetry axis of the (EPFS) surface, representing the failure criterion (8), and the hydrostatic axis is given by relation [11]:

$$y_0 = \frac{\sqrt{6}}{9}\frac{(h_1 - h_3)}{H_{33}} \tag{11}$$

and takes always negative values for T-strong materials. The negative value for y_0 indicates again that the symmetry axis of the (EPFS) is beneath the hydrostatic axis

relatively to the positive σ_3 -direction and this is another indication that the material behaves like a T-strong material.

Most of the orthotropic materials are C-strong materials since they are capable to sustain large compressive stresses without failure. However, there is a restricted category of materials which behave like T-strong ones. In these substances the solid material is distributed in little columns or beams forming the cell edges in cellular open-cell foam-like solids, where clusters of microvoids intervene between molecules or crystals, thus creating structures which may be either isotropic or more frequently orthotropic. Most polymers can be readily foamed and techniques exist for doing the same thing with ceramics and glasses even with metals in modern technologies.

Foam-like substances permit the simultaneous optimization of stiffness, strength and lightness of a structure. The mechanical properties of foamed materials reflect to some extent the mode of distribution of the solid material, which depends on the relative density of the solid phase constituting the foam-like material. Thus, for high void fractions of the total volume structures, a lower than linear decrease in flow strength with a decrease of density was exhibited, indicating that bending stresses within the foam structure are progressively a less important feature of the collapse load. Then the structure behaves like a T-strong material presenting a higher strength in the tension-tension-tension octant of the principal stress space until a collapse by internal buckling of the structure takes place.

Thus, for example, oriented polypropylene has a geometric conformation of the individual polymer molecules in the unit cell of a polymer crystallite depending strongly on the repulsion of the methyl groups in the planar zig zag conformation. The molecules assume positions at 120 degrees out of the plane of the chain and are therefore forming helical conformations within the unit-cell of their crystallites. Therefore low-density oriented polypropylene influenced by the spring-like cells presents a T-strong behavior [12]. Similarly, paper sheets consist of networks of fibers placed at different random arrangements. The properties of the individual fibers and the nature and frequency of bonds between them influence the properties of the paper sheet. Moreover, fibers are actually filament wound composite systems, whose cell walls are composed of a number of different layers, the fibrils, which are aggregates of cellulose molecules with cellobiose as the basic repeating unit, arranged either in a orderly, or in a random fashion. The fibrils themselves are arranged in a regular fashion differing within the various layers of the cell-wall and they are held together by the hemicellulose and lignin matrix material. Such conglomerates contain a great number of microvoid clusters distributed randomly inside the structure and this reduces considerably the specific density of the material. The bonded fibers act like springs largely interconnected. It was found that low density paper sheets behave like T-strong anisotropic materials, as it is clearly indicated in [13].

Then, for classical fiber reinforced materials, whose σ_3-axis of symmetry is the strong axis, as well as for all compacted and close-packed C-strong materials, it has been established [2] that their corresponding eigenangle ω varies at the interval $\omega=135^\circ$ and $\omega=180^\circ$ with the more anisotropic materials lying closer the neighbourhood of $\omega=180^\circ$. As the anisotropy of the composites or any C-strong material is reduced, tending to become an isotropic material, the value of the respective eigenangle diminishes from $\omega=180^\circ$ tending to values approaching $\omega_i=125.26^\circ$ for the isotropic materials. Indeed, it has been established that all strong-matrix materials, like those with the metallic or pyrolytic graphite matrices, lie at the transition zone of angle ω, where their ω-angles are much smaller than those of the composites, whose angles ω lie close to $\omega=180^\circ$[2].

Furthermore, a special category of transversely isotropic materials exists, whose mechanical properties are conveniently selected, so that two last eigenstate components of the S-tensor, given by relations (1.4) and (1.3) may identified to contribute, the first, a dilatational type of strain energy, whereas the second, creating a pure distortional one. In this way the four eigenstates (1) are clearly separated into two distinct groups, creating either distortional, or dilatational types of strain energy. It was shown that this family of materials behave like the isotropic body, in spite of their differences in the elastic constants along their principal axes of anisotropy [14].

Since with all fiber-composites it is possible to arrange their mechanical properties by selecting the appropriate ratios between matrix and inclusions, according to their properties, this possibility of selecting in advance the properties of the composite is feasible. Taking into consideration that quasi-isotropic materials develop the smallest stress concentrations in the structures, the development and selection of such composites with quasi-elastic properties becomes very important [15]. Indeed relation (4), expressing the tangent of the double of the eigenangle ω, suggests a particular relationship between E_L, E_T and ν_L, ν_T in order to yield an angle ω_i approaching the value $\omega_i=125.26^\circ$. This relationship is expressed by:

$$\frac{(1-\nu_L)}{E_L} \approx \frac{(1-\nu_T)}{E_T} \tag{12}$$

Introducing this relation into Eq. (4) we obtain:

$$\tan 2\omega = -2\sqrt{2} \tag{13}$$

and therefore, angle ω takes either of the following values:

$$\omega = \omega_i = 125.26^\circ,\ 35.26^\circ \tag{14}$$

These are the only values for angle ω, which yield quasi-isotropic materials with positive or negative values for Poisson's ratios.

Introducing relation (12) into the eigenvalues given by relations (1) we have:

$$\lambda_1 = \frac{1}{2G_T} \quad , \quad \lambda_2 = \frac{1}{2G_L}$$
$$\lambda_3 = \frac{1}{2G_L} \quad \& \quad \lambda_4 = \frac{(1-2\nu_L)}{E_L} \tag{15}$$

These relations indicate that only the first eigenvalue λ_1 depends on the transverse shear modulus, whereas the next two eigenvalues depend exclusively on the longitudinal shear modulus G_L and the fourth λ_4-eigenvalue depends only on the longitudinal bulk modulus.

Moreover, the eigentensors σ_m of the symmetric stress tensor σ are given by:

$$\sigma_1 = \left[\frac{\sigma_1 - \sigma_2}{2}, -\frac{\sigma_1 - \sigma_2}{2}, 0, 0, 0\ \sigma_{12}\right]^T \tag{16.1}$$

$$\sigma_2 = [0, 0, 0, \sigma_{23}, \sigma_{13}, 0]^T \tag{16.2}$$

$$\sigma_3 = \frac{1}{6}(\sigma_1+\sigma_2-2\sigma_3)\ [1, 1, -2, 0, 0, 0]^T \tag{16.3}$$

$$\sigma_4 = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)\ [1, 1, 1, 0, 0, 0]^T \tag{16.4}$$

Relations (15) and (16) indicate that the characteristic states of stress corresponding to the spectral decomposition of the compliance tensor **S** for the transversely isotropic material satisfying also the relationship (12) decompose the generic stress tensor σ into four distinct states, from which the three first are shears, whereas the fourth state (σ_4) is a hydrostatic stress producing a dilatational strain energy, whereas the strain energies of all the other three components are distortional. Thus, this class of transversely isotropic materials maintains the property of isotropic substances to yield the possibility under a certain orientation of loading to admit a clear separation of the total strain energy into two distinct components a distortional and a dilatational one. Then, all the advantages already mentioned [15], valid for the isotropic materials, are extended to this class of transversely isotropic substances in addition to the particular properties deriving from their anisotropy.

From the above concise description of the mechanical properties of this category of quasi-isotropic fiber-reinforced composite it derives the conclusion that this type of materials present values of their eigenangle ω lying in the vicinity of the characteristic eigenangle ω_i=125.26° for the isotropic material and these values are spread in both sides of this angle ω_i.

Finally, it remains one last category of composite materials to be examined and this is the group of weak-axis woven fabric composites. These relatively newly introduced woven fabric composites have shown that they provide more balanced properties on the weave plane than unidirectional fiber composites and therefore they have already gained an increasing importance in technological applications. This was due to the fact that the two-directional reinforcement inside the lamina behaves like a quasi-isotropic substance and resists better in any biaxial loading and especially in impact [16]. All woven composites are considered as laminated bodies, each layer of which is a woven fabric formed by interlacing two sets of threads, the warp and the fill or weft. Warp and weft may be either identical or different threads and the various types of weaves can be identified by the patterns of repeats in the warp and weft directions denoted by the weave parameter n_t meaning the number of threads in the pattern which is equal for warp and weft for non-hybrid fabrics. For equally spaced warps and wefts the fabric is termed as ***plain weave*** and constitutes the simplest and more homogeneous fabric [17].

The failure behavior of woven fabric composites in the form of plain weave fiber unidirectional laminae obeys the same laws as the strong-axis fiber reinforced composites. However, since the transverse weave plane is the strog and isotropic plane of the composite, while the normal to it direction is the weak one, the material is approximated as a weak-axis transversely isotropic composite. Then, the elliptic paraboloid failure surface (EPFS) criterion, continues to describe satisfactorily this type of materials. It was shown [18] that such weak-axis transversely isotropic composites correspond to tension strong composites and their failure surfaces consist of a single-sheet convex surface open to the tension-tension-tension octant of the principal stress space. The main characteristic of such surfaces is that they are oblate along the normal direction to the isotropic plane, in contrast to the typical (EPFS)-criterion for fiber composites, which are prolate along the same direction. While the intersection of this (EPFS)-criterion by the (σ_1, σ_3)-principal stress plane (σ_3 is the weak axis) resembles closely the respective intersection for the unidirectional fiber composites, the (σ_1, σ_2)-isotropic plane intersection, which coincides with the weaving strong plane, approaches very closely a circle, thus indicating that along this isotropic plane the failure stress is hydrostatic and independent of its orientation inside this plane. This property constitutes a significant and promising property, which makes this type of woven composites very attractive in applications.

Indeed, it has been shown [18] that the ellipticity, λ_x, of the paraboloid representing the failure surface for woven fabric weak-axis composites is given by:

$$\lambda_x = \left\{ \frac{H_{33}}{(4H_{11}-H_{33})} \right\}^{1/2} \tag{17}$$

Then, for values $H_{33} \approx 2H_{11}$ the factor λ_x, tends to unity. For the composite studied in ref. [18], which is a T-300 carbon-epoxy woven fabric composite, this value was found to be $\lambda_x = 0.97$. For several other similar materials the same approximate equality is valid. Then, the intersection of the (EPFS) for such materials by the (σ_1, σ_2)-plane tends to become a circle. The critical relationship for such a property to be valid is the relationship:

$$(H_{33} - H_{11} - H_{22}) \approx 0 \qquad (18)$$

From the existing experimental evidence with different woven fabric composites it can be derived that this ratio λ_x remains always close to unity, then it may be concluded the extremely interesting property for this type of materials, that along their weave plane, which is their srong plane, failure is invariant and the stresses at failure, at any combination of the σ_1 -and σ_2- principal stresses, is an invariant characteristic of the material. This is a unique property very advantageous, exclusive for this type of woven-fabric composites, which makes these materials very attractive in applications.

Furthermore, the invariance of the failure strength in the strong weave plane creates a further advantage in the use of these materials as laminates, since it justifies and facilitates the application of a **simple lamination theory** for the creation of multiple-ply laminates, where the contribution of each lamina may be accepted as being invariant of its orientation of the fibers with respect to its neighbors. If one adds to these incontestable advantages of this type of materials the fact, which was derived from this application of the (EPFS)-failure theory, that the woven-fabric composites reinforce the laminates toward the tension-tension quadrant of their failure locus, while they do not create abrupt and high differences in failure limits along the different stress-directions, one may anticipate that these materials should find an exclusive and broad use in crucial modern technological applications.

This last property indicates that the respective values for the eigenangle ω of these composites lies always in the interval around the critical angle $\omega_i = 125.26^\circ$.

4. THE EVALUATION OF THE LONGITUDINAL G_L-SHEAR MODULUS

The spectral decomposition of the elastic stiffness or compliance tensors in elementary fourth-rank tensors serves as a means for the energy orthogonal decomposition of the energy function. The advantage of this decomposition is that the elementary idempotent tensors, to which the fourth-rank tensors are decomposed, have the interesting property of defining energy orthogonal stress states. That is, the stress-idempotent tensors are mutually orthogonal and at the same time collinear with their respective strain tensors, and therefore they correspond to energy-orthogonal stress states, which consequently are

independent to each other. Since the failure tensor is the limiting case for the respective elastic tensors, which are eigenstates of the compliance tensor **S**, this tensor also possesses the same remarkable property. Moreover, it was conclusively proved that the four eigenvalues of the compliance, or stiffness, or failure tensors for a transversely isotropic body, together with the value of the eigenangle ω, constitute the five necessary and most simple parameters, which invariantly describe either the elastic, or the failure behavior of the body. The expressions for the stress-vector thus established represent an ellipsoid centered at the origin of the Cartesian frame, whose principal axes are the directions of the idempotent strain vectors. This ellipsoid is a generalization of the Beltrami ellipsoid for the isotropic materials, introduced in ref. [19].

Furthermore, in combination with extensive experimental evidence, this theory indicates that the eigenangle ω alone characterizes monoparametrically the degree of anisotropy for each transversely isotropic material. Thus, while the angle ω for isotropic materials is always equal to ω_i-125.26°, or 35.26°, the angle $|\omega|$ increases progressively in both sides within the interval 0°-180°, as the anisotropy of the material is increased. Then, the possibilities of modern technology to create new composite materials by acting upon the selection of the particular elastic moduli of the reinforcements and the matrix and choosing the appropriate density of inclusions, allowed to form a series of new composites with selected mechanical and other physical properties convenient for special uses in industry.

For this procedure the definition of a particular eigenangle ω of the composite plays an important role, since it constitutes the criterion for the proper selection of the composite. It has been shown that composites with eigenangles lying at the vicinity of the critical value for the eigenangle, ω_i, for the isotropic materials [9] behave like **quasi-isotropic** materials with equilibrated properties along the principal axes of anisotropy, fact which has a direct influence on an increase of the adhesion between phases and therefore an increase of their toughness.

Furthermore, it has been proved in a recent study [18] that the possibility of disposing orthotropic plates with particular properties interconnecting their mechanical properties improves considerably their mechanical behavior, by reducing drastically the stress concentration factors in discontinuities (holes of cracks) eventually existing in the structures. These stress-concentration factors are reduced considerably, when the composite has properties approaching the respective values for isotropic materials [15]. Then, it is worthwhile to seek convenient types of composites, which not only are highly reinforced by their anisotropy, but also they are favorably designed to develop the lowest possible stress concentration factors at eventual discontinuities of structures made of these materials. It was further indicated that the optimum of such concentration factors can be achieved by the respective isotropic material.

Since for fiber-reinforced composites a vast possibility exists to arrange conveniently the selection of the quantities and the properties of the constituent phases, it becomes feasible to produce convenient composites approaching this simple condition (12). Then, for quasi-isotropic composites it is possible to split the total elastic strain energy into a distortional and a dilatational component, a property which has beneficial repercussions in designing optimal structures. Indeed, it has been shown previously [15] that such materials, whose properties approach those of an isotropic material, present very reduced stress-concentration and stress-intensity factors. Thus, a structure made of such materials presents a superiority over a similar one made by a strong anisotropic composite, since for the same mode and intensity of loading the first type of structure can hold much larger loadings. This phenomenon has been already detected intuitively by the designers of advanced research centers based on their long experience in practical applications. Thus, they have established a long practice and by trial and error procedures, that it is advantageous to introduce in structures strong-matrix composites of woven-fabric composites, whose elastic constants approach better relation (12).

It is worthwhile remarking that the expressions for eigenstates λ_1, λ_3 and λ_4 given by relations (1.1), (1.3) and (1.4), as well as the expression for the eigenangle ω by relation (4) do not depend on the value of the longitudinal shear modulus G_L. Only the λ_2-eigenstate depends exclusively on the G_L-value. It is well known that the evaluation of the longitudinal shear modulus G_L, derived by measuring the shears between the σ_3-axis and the isotropic $(0\sigma_1\sigma_2)$-plane, is a difficult and error-affected experimental procedure. The values for this elastic constant for anisotropic materials present always a large scattering and a high degree of uncertainty. The evaluation of the eigenangle ω of each material, which does not necessitate the knowledge of this elastic constant, as it can be readily checked by relation (4), yields a means for evaluating accurately this constant from the remaining elastic constants of the anisotropic material, provided that some relationship can be established between the angle ω and the shear modulus G_L.

Moreover, it can be readily proved by means of the classical anisotropic elasticity theory [20] that the stress concentration factor, K_T, in the presence of a elliptic crack in a transversely isotropic plate, loaded in tension along the strongest material direction, is given by:

$$K_T = 1 + \left[2\left(\frac{E_L}{E_T}\right)^{1/2} + 2\left(\frac{E_L}{2G_L} - \nu_L\right) \right]^{1/2} \frac{b}{a} \qquad (19)$$

where b/a denotes the ratio of the elliptic crack semi-axes. This relationship indicates that the value of the shear modulus, G_L, is also contributing to the characterization of the fracture toughness of the material and an eventual phenomenological correlation of it with the value of eigenangle, ω, would extend its usefulness as a simple parameter characterizing both elasticity and toughness of the transversely isotropic medium.

In order to investigate a possible monoparametric correlation between the value of the angle ω and that of G_L, and since there does not exist an exact and simple expression for G_L in terms of the other elastic properties, we should make recourse to experimental evidence. Thus, by plotting the values of the ratio $E_L/2G_L$ for all known transversely isotropic materials, fiber composites and inorganic crystals of the hexagonal lattice arrangement, which are strongly orthotropic, versus their respective eigenangles ω, a universal curve of dependence of the $E_L/2G_L$ -ratio versus angle ω can be derived, which constitutes a sure and accurate relationship between all elastic constants of orthotropic materials. These experimental points were already plotted in refs [2], [14] and [22] with coordinates the values ω and $E_L/2G_L$. Most materials represented by these experimental points were also conveniently tabulated (Table 1 of refs [2] and [14]). The continuous curve plotted passing through these points represents the mean characteristic behaviour of all these experimental points, which are ordered in a very prescribed manner. It is evident that the value of the ratio $E_L/2G_L$ diminishes as the angle ω approaches the value of 125.26°, whereas in the broad vicinity of this value, the ratio $E_L/2G_L$ changes mildly, thus retaining values near those of isotropic materials.

The plotting of this figure is repeated here in Fig. 1, where, besides the points corresponding to different materials as they have derived from experience, the curve $|\cot 2\omega| = f(\omega)$ was also plotted. It can be immediately derived that all values $E_L/2G_L = f(\omega)$ lie on this curve with some scatter, due to inherent experimental error in measuring all these elastic constants, that is E_L, E_T, ν_T. Then, it is legal to accept that the variation of the $E_L/2G_L$ -ratio with angle ω follows a curve defined by:

$$\frac{E_L}{2G_L} = a[1+|\cot 2\omega|] \tag{20}$$

Introducing the value for $\cot 2\omega$ from Eq.(4) we have:

$$\frac{E_L}{2G_L} = a\left[1 + \left|\frac{\frac{1}{2E_L} - \frac{1-\nu_T}{2E_T}}{\sqrt{2}\frac{\nu_L}{E_L}}\right|\right] \tag{21}$$

where the a-factor is evaluated from Fig.3 to be $a = \sqrt{2}$.

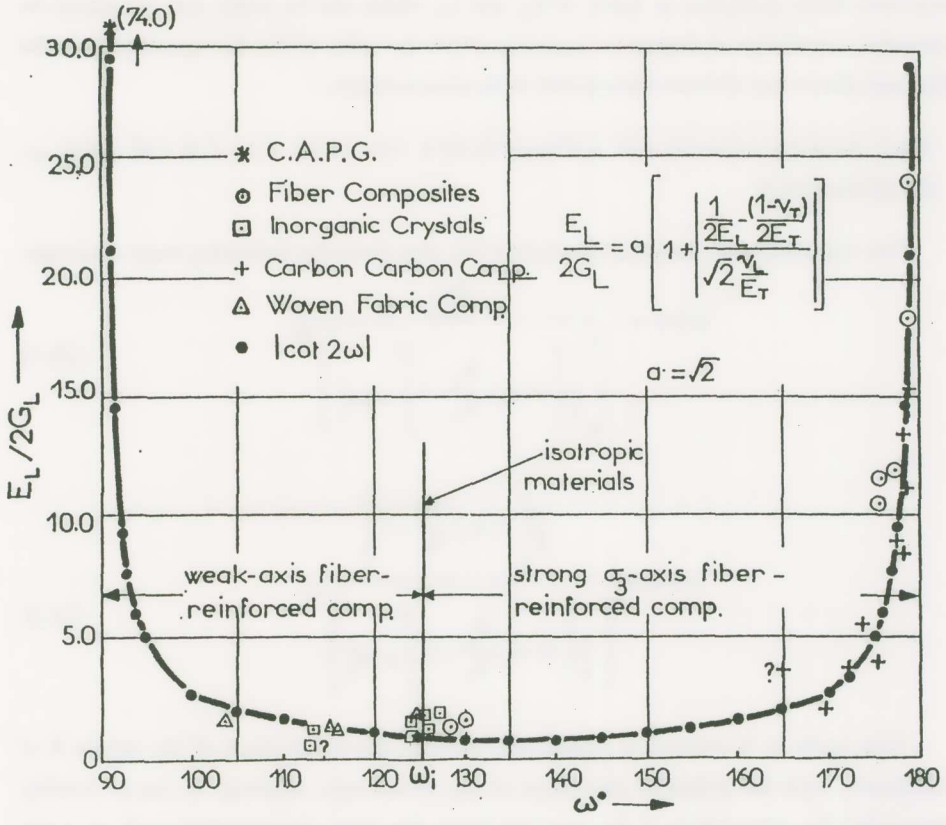

Fig. 1. The universal curve of variation of the $E_L/2G_L$-ratio versus the eingenangle ω, expressed by the relation$E_L/2G_L$=[1+/cot2ω/]and the comparison with the values derived from experimental results for a series of materials used in applications.

Then, relation (21), when solved for G_L, yields:

$$G_L = \frac{\nu_L}{\left|\frac{1+2\sqrt{2}\,\nu_L}{E_L} - \frac{1-\nu_T}{E_T}\right|} \qquad (22)$$

Relation (22) yields the value of the longitudinal shear modulus G_L of the composite in terms of the elastic moduli E_L and E_T and the respective Poisson ratios ν_L and ν_T. Since both G_L and ν_L elastic constants are difficult to be measured and they demand sofisticated loading devices and measuring systems, the importance of relation (22), which intercon-

nects both these quantities in terms of E_T and ν_T, which can be easily and accurately be evaluated, constitutes an important means to check the value of the one quantity from the other and detect any obvious discrepancy in its measurement.

5. THE DEPENDENCE OF ANISOTROPY ON THE VALUE OF THE ω-EIGENANGLE

The eigenangle ω, defined by relation (4), also gives the following basic relations:

$$\sin 2\omega = -\frac{\sqrt{2}\,\nu_L}{\left[\frac{1}{4}\left((1-\nu_T)\frac{E_L}{E_T}-1\right)^2+2\nu_L^2\right]^{1/2}} \tag{23.1}$$

$$\cos 2\omega = \frac{\frac{1}{2}\left[(1-\nu_T)\frac{E_L}{E_T}-1\right]}{\left[\frac{1}{4}\left((1-\nu_T)\frac{E_L}{E_T}-1\right)^2+2\nu_L^2\right]^{1/2}} \tag{23.2}$$

This angle ω is intimately connected, through the eigenvalues of the tensor **S** of compliance, with the principal directions of the orthotropic material. It can be readily derived that the orientations of the principal strain directions depend exclusively on angle ω. This is valid for every transversely isotropic material, as well as for the isotropic body. It can be readily shown that for the isotropic body, the vector $\mathbf{e}_4$ has the positive direction of the hydrostatic axis, whereas the $\mathbf{e}_3$-vector lies on the deviatoric plane. Both of these vectors remain always on the principal diagonal plane $(\sigma_3\delta_{12})$ for the transversely isotropic bodies. From relations (4) and (7) it can be readily shown that thermodynamically accepted values for the eigenangle ω lie in-between 0° and 180°, whereas for the isotropic body ω_i=125.26° or 35.26°.

For the typical fiber-reinforced composite materials, whose axis of symmetry of the parallel fibers constitutes the strong axis of the material, angle ω takes values lying inside the interval between 125.26° and 180°. Furthermore, it should be mentioned that relation (4), or relations (7), yield two values for the isotropic material ω=125.26° and ω=35.26°. While the first value corresponds to a positive Poisson's ratio ν_L, the second value refers to a negative Poisson's ratio ν_L. However, it has been established that the limits of variation of the isotropic Poisson's ratio ν_i are defined by the inequalities:

$$-1.0 \le \nu_i < 0.50 \tag{24}$$

whereas for the transversely isotropic material it is valid that the longitudinal, ν_L, and the transverse, ν_T, Poisson's ratios must satisfy the relations [2]:

$$|\nu_T| \le \left[(1-\nu_T)\frac{E_L}{2E_T} \right]^{1/2} \quad \text{and} \quad |\nu_T| \le 1 \tag{25}$$

Then, it is of interest to study the influence of the value of the eigenangle ω on the type of anisotropy holding in the material. This study will be limited to only the transversely isotropic materials which are of great interest in the constructions.

The general representation of relations (4) and (23) in the trigonometric circle of radius:

$$r_t = \left[\frac{1}{4}\left\{ (1-\nu_T)\frac{E_L}{E_T} - 1 \right\}^2 + 2\nu_L^2 \right]^{1/2}$$

yields Fig. 2, where, for the angle A0C=2ω, the argument 0B=cos2ω and the argument BC=sin2ω.

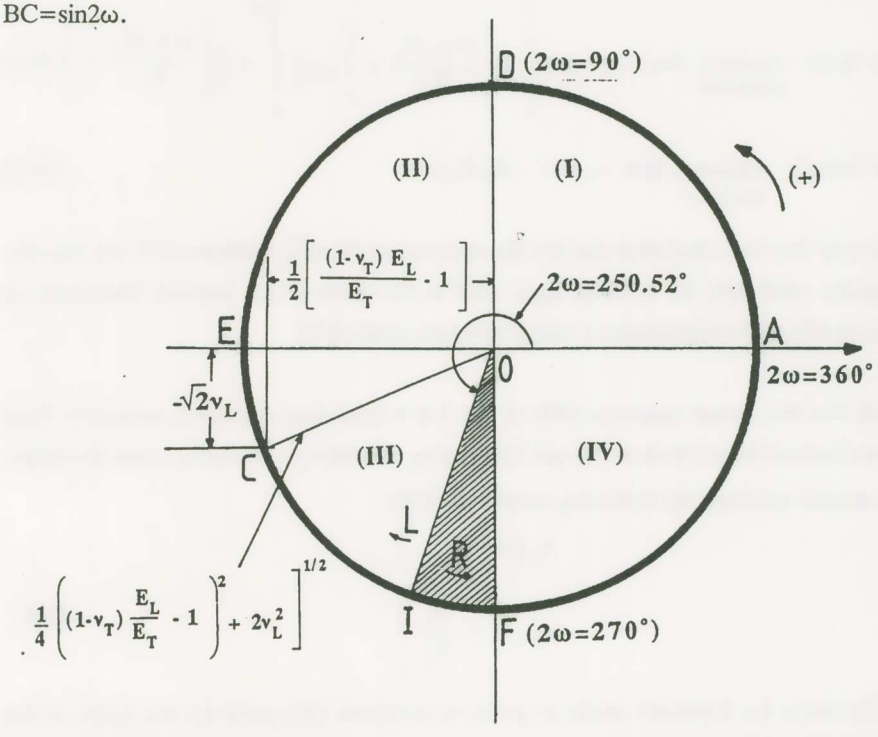

Fig. 2. The trigonometric circle with radius $r=\{1/4\,[1-\nu_T)E_L/E_L-1]^2+2\nu_L^2\}^{1/2}$ with the four different regions of variation of eigenangle ω.

(i) Then, when angle 2ω lies in the first quadrant A0D in Fig. 4 it is valid that $\cos 2\omega$, $\sin 2\omega \geq 0$. Furthermore, since from relations (4) and (23) $\tan 2\omega$ should be positive it may be derived that it is valid that:

$$\nu_L \leq 0 \quad \text{and} \quad \left(\frac{(1-\nu_T)}{E_T} E_L - 1 \right) \geq 0 \tag{26}$$

Relations (26) yield the following conditions valid for the first quadrant (I):

$$\nu_L \leq 0$$

$$\nu_T \leq \left(1 - \frac{E_T}{E_L} \right) \tag{27}$$

Introducing these conditions into the expressions for the limits of Poisson's ratios for the anisotropic materials, expressed by relations (25), we find for the limiting value of $\nu_T = \{1-(E_T/E_L)\}$ and for $2\omega = \pi/2$, that $\nu_L \leq 1/\sqrt{2}$.

Then, for the limits of this first quadrant we have:

$$\text{(i) For } 2\omega=0 \text{ : } \begin{matrix}\cos 2\omega=1, \text{ then } \nu_L=0 \text{ and} \\ \sin 2\omega=0\end{matrix} \left[\frac{1}{4} \left(\frac{(1-\nu_T)E_L}{E_T} - 1 \right)^2 + 2\nu_L^2 \right]^{1/2} = \frac{1}{2} \left[\frac{(1-\nu_T)E_L}{E_T} - 1 \right] \tag{28.1}$$

$$\text{(ii) For } 2\omega=\pi/2 \text{ : } \begin{matrix}\cos 2\omega=0, \text{ then } \nu_T = \{1 - (E_T/E_L)\} \\ \sin 2\omega=1\end{matrix} \tag{28.2}$$

It may then be concluded that for the interval $\omega \in [0, \pi/2]$ relations (27) are thermodynamically valid with the equality signs valid at the limits of the interval. Obviously, in this first quadrant the eigenangle ω varies between $\omega \in [0, \pi/4]$.

(ii) For the second quadrant D0E of Fig. 2 it is valid that: $\cos 2\omega < 0$, $\sin 2\omega \geq 0$. Then it may be derived from relations (4) and (23) that $\nu_L \leq 0$ and $\nu_T > (1-E_T/E_L)$ and, therefore, for the second quadrant the following conditions hold:

$$\nu_L \leq 0$$

$$\nu_T > \left(1 - \frac{E_T}{E_L} \right) \tag{29}$$

The limits for Poisson's ratios as given by relations (25) yield for the limits of the second quadrant that:

(a) For $2\omega=\pi/2$, $\cos2\omega = 0$, $\sin2\omega = 1$ as previously, and
(b) for $2\omega=\pi$, $\cos2\omega = -1$, $\sin2\omega = 0$ and therefore $\nu_L=0$.

Therefore, for a thermodynamically acceptable case in the interval $2\omega\in [\pi/2, \pi]$ relations (29) hold with the sign of equality for the limits as indicated. Obviously, angle ω runs in-between $\pi/4$ and $\pi/2$ ($\omega\in [\pi/4, \pi/2]$).

(iii) For the third quadrant (E0F), where $2\omega\in [\pi, 3\pi/2]$, we have $\sin2\omega<0$ and $\cos2\omega\leq 0$, which, because of relations (23), yield:

$$\nu_L > 0 \quad \text{and} \quad \nu_T \geq \left(1 - \frac{E_T}{E_L} \right) \tag{30}$$

Therefore, for the third quadrant the following relations are valid:

$$\nu_L > 0 \quad , \quad \nu_T \geq \left(1 - \frac{E_T}{E_L} \right) \tag{31}$$

For these limits Poisson's ratios check well, yielding: $|\nu_T| \leq 1$ and $|\nu_L| \leq 1/\sqrt{2}$.

At the limits of this interval we have:

(a) for $2\omega=\pi$, $\cos2\omega = -1$, $\sin2\omega = 0$,
and (b) for $2\omega=3\pi/2$, $\cos2\omega = 0$, $\sin2\omega = -1$, which yield:

$$\nu_T = \left(1 - \frac{E_T}{E_L} \right) \tag{32}$$

Therefore, for a thermodynamically acceptable case in the interval $2\omega\in [\pi, 3\pi/2]$ relations (30) are valid with the signs of equality holding at the limits of the interval. Obviously, the eigenangle ω varies in this interval $\omega\in [\pi/2, 3\pi/4]$.

(iv) For the fourth quadrant (F0A), where $2\omega\in [3\pi/2, 2\pi]$, it is valid that $\sin2\omega\leq 0$ and $\cos2\omega>0$ which yield:

$$\nu_L \geq 0 \quad \text{and} \quad \nu_T < \left(1 - \frac{E_T}{E_L} \right)$$

Therefore, for this interval we have the validity of relations:

$$\nu_L \geq 0 \quad , \quad \nu_T < \left(1 - \frac{E_T}{E_L} \right) \tag{33}$$

Relations (33) hold for the quadrant IV and they give termodynamically acceptable values for Poisson's ratios in this interval $2\omega \in [3\pi/2, 2\pi]$ and therefore for $\omega \in [3\pi/4, \pi]$.

(v) Finally, we examime the situation around the characteristic value ω_i for the isotropic materials ($\omega_i = 125.26°$). For this angle $\cos 2\omega_i = -1/3$, then from relation (23.2) we have:

$$\frac{1}{2}\left[(1-\nu_T)\frac{E_L}{E_T} - 1 \right] > -\frac{1}{3}\left[\frac{1}{4}\left\{ (1-\nu_T)\frac{E_L}{E_T} - 1 \right\}^2 + 2\nu_L^2 \right]^{1/2}$$

which, for $\omega \in [125.26°, 135°]$, yields:

$$\nu_L > \left[1 - (1-\nu_T)\frac{E_L}{E_T} \right] , \quad \nu_T \geq \left[1 - \frac{E_T}{E_L} \right] \tag{34}$$

Similarly, for $\omega \in [90°, 125.26°]$, we find that:

$$\nu_L < \left[1 - (1-\nu_T)\frac{E_L}{E_T} \right] , \quad \nu_T > \left[1 - \frac{E_T}{E_L} \right] \tag{35}$$

From the above-developed thorough study of the termodynamically admissible limits of variation of the two components of Poisson's ratios the following results may be derived:

(I) The interval I of variation of the eigenangle $\omega°$, $\omega° \in [0°, 90°]$, contains all transversely isotropic materials which have negative longitudinal Poisson's ratios ν_L. Such materials are all the so-called **auxetic materials** [22] with very interesting properties described recently in several papers, referred to in [9]. These materials presumably should have a high porosity and correspond to a great number of open-cell foamy materials, based on polymers or metals, with re-entrant cell structures, as well as various types of papers, ultra high molecular weight oriented polypropelyne and polyethylene, as well microporous oriented polytetra- fluoroethylene. In the same category also belong several types of honeycombs, as well as rank-2 composites, which are composite mixtures of two materials of whom the one is a rank-1 composite, consisting of alternating layers of the stiffer and the more flexible phases, then the rank-2 composite is constructed of alternating layers of the stiff material and the rank-1 composite in layers of different orientations. Depending on the average densities of the two phases, the stiff phase and the rank-1 composite and

their relative orientation, it is possible to obtain composites with negative Poisson's ratios approaching the negative unit [23].

However, these newly developed types of materials, very useful in special applications, are as yet not thoroughly studied and their mechanical properties, as well as their failure modes are as yet not well known, in order to allow a definite classification of these materials. Therefore, the domain of variation of the eigenangle ω between 0° and 90° is not clearly defined, the only definite result being that for the characteristic value of ω, $\omega_i=35.26^\circ$ in the interval $\in [0^\circ, 90^\circ]$, where the material becomes again an isotropic one.

II) It remains to classify systematically the extent of anisotropy of the materials in the range of the ω-eigenangle: $\omega^\circ \in [90^\circ, 180^\circ]$. According to the previous analysis, as well as from the experience gathered in examining different types of composites presenting different values for their respective ω-eigenangle, there are three distinct sub-domains where the properties of the materials present different anisotropic behaviour. These domains are defined as follows. The first domain extents in the interval $\omega \in [90^\circ, 125^\circ]$, the second one in the interval $\omega \in [125^\circ, 135^\circ]$ and the third and last one inside the range $\omega \in [135^\circ, 180^\circ]$.

(a) In the interval $\omega \in [90^\circ, 125^\circ]$ are included most of the weak-axis of symmetry composites with the compression annealed pyrolytic graphite-graphite fiber composites presenting a value of its eigenangle $\omega^\circ \approx 90^\circ$ [2], [14]. Indeed, all woven-fabric composites disposing their σ_3-axis of symmetry with the lowest strength, in comparison with their transverse isotropic plane (σ_1, σ_2) which is reinforced with fibers in a woven fabric arrangement, resemble very much the quasi-isotropic materials and they all lie inside this interval [18], [22]. In the same interval lie also all open-cell foamy materials [12], [24]. All these composites present the property to be tension-strong materials and they obey the condition $\mathrm{tr}\mathbf{h}<0$. Finally, in the same interval lie the majority of inorganic crystals with the property that one of the values of their Poisson's ratios is always negative.

Since the negativeness of Poisson's ratio is an indication of increased porosity of the materials, it seems that there is an interdependence between the property of the materials to be T-strong materials with the property to eventually present negative or very low one of their Poisson's ratios and in this case the longitudinal one, ν_L. This argument is further supported by the fact that oriented polypropylene, as well as oriented polytetrafluoroethylene, are both T-strong materials [25], and they present a texture of their macromolecules resembling the texture of open-cell foams. Similar phenomena and failure behaviour appear with various types of papers, which again are sheets formed from randomly arranged arrays

of cellulosic fibers presenting some kind of parallel ordering, bound together by lignin, a polymeric substance, thus presenting an important amount of porosity [13].

(b) For the second interval of the ω-eigenangle, where $\omega^\circ \in [125.26^\circ, 135^\circ]$ we proceed to a straightforward transform to the two last eigenvalues λ_3 and λ_4 of relations (1) and we derive that:

$$\lambda_3 = (A+B) = \frac{(1-\nu_T)}{2E_T}\left(1+\frac{1}{\cos 2\omega}\right)+\frac{1}{2E_L}\left(1-\frac{1}{\cos 2\omega}\right)$$
$$\lambda_4 = (A'+B') = \frac{1}{2E_L}\left(1+\frac{1}{\cos 2\omega}\right)+\frac{1-\nu_T}{2E_T}\left(1-\frac{1}{\cos 2\omega}\right) \tag{36}$$

where A, B, A′, B′ are functions of the moduli and Poisson's ratios.

Relations (36) may replace the expressions fot the two last eigenvalues λ_3 and λ_4 given by Eqs. (1). These relations, expressed in terms of tangent of the eigenangle ω, take the form:

$$\lambda_3 = \frac{1}{(1-\tan^2\omega)}\left[\frac{1-\nu_T}{E_T}-\frac{1}{E_L}\tan^2\omega\right]$$
$$\lambda_4 = \frac{1}{(1-\tan^2\omega)}\left[\frac{1}{E_L}-\frac{1-\nu_T}{E_T}\tan^2\omega\right] \tag{37}$$

Figure 3 presents the variation of $\tan^2\omega = f(\omega)$ and $(1-\tan^2\omega)^{-1} = f(\omega)$ for $\omega^\circ \in [90^\circ, 180^\circ]$. It is clear from this plotting that both eigenvalues λ_3 and λ_4 become indeterminate for $\omega^\circ = 135^\circ$. Then, relation (4), expressing the tangent of the 2ω-angle becomes negative infinite for $\omega = 135^\circ$ suggesting that, for this value of the eigenangle, the denominator of relation (4) becomes equal to zero. Then, it is valid that:

$$\left\{(1-\nu_T)\frac{E_L}{E_T}-1\right\} = 0 \tag{38}$$

and therefore relations (37) for this angle become:

$$\lambda_3 = \lambda_4 = \frac{1}{E_L} \tag{39}$$

Then, in the interval of variation of the ω-eigenangle, $\omega^\circ \in [125.26^\circ, 135^\circ]$, the compliance, stiffness and failure tensors degenarate to simple quasi-isotropic forms, varying between the bounds:

$$\text{for } \omega_i = 125.26^\circ : \lambda_1 = \frac{1}{2G}, \lambda_2 = \frac{1}{2G}, \lambda_3 = \frac{1}{2G}, \lambda_4 = \frac{1}{3K} \tag{40}$$

and

$$\text{for } \omega = 135^\circ : \lambda_1 = \frac{1}{2G_T}, \lambda_2 = \frac{1}{2G_L}, \lambda_3 = \frac{1}{E_L}, \lambda_4 = \frac{1}{E_L} \tag{41}$$

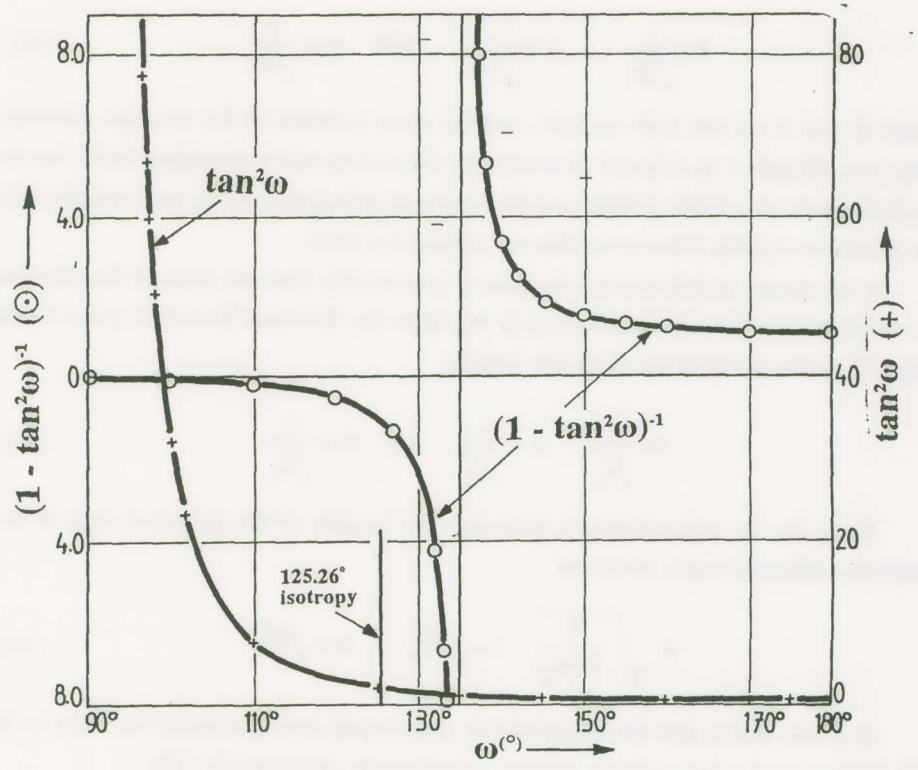

Fig. 3. The variation of the quantities $\tan^2\omega$ and $(1-\tan^2\omega)^{-1}$ versus the eigenangle for the interval $\omega \in [90°, 180°]$.

In-between this interval of ω ($\omega \in [125.26°, 135°]$), as well as outside this interval and in the zones of variation of the ω-eigenangle, where relation (12) for the quasi-isotropic materials is valid, these bounds become:

$$\lambda_1 = \frac{1}{2G_T},\ \lambda_2 = \frac{1}{2G_L},\ \lambda_3 = \frac{1}{2G_L},\ \lambda_4 = \frac{(1-2\nu_L)}{E_L} \tag{42}$$

In a previous publication of the author [19] the Beltrami-Haigh ellipsoid of stresses and deformations for the isotropic materials [26], [27], valid only for isotropic materials, was extended to cover the whole range of transversely isotropic composites and other substances. It was shown that for the isotropic case the Beltrami ellipsoid is an ellipsoid of revolution, whose principal semi-axes have the lengths:

$$\ell=\frac{1}{\sqrt{3K}} \quad , \quad k=\frac{1}{\sqrt{2G}} \quad \text{and} \quad m=\frac{1}{\sqrt{2G}} \tag{43}$$

where K and G are the bulk modulus and the shear modulus of the isotropic material. Thus, this ellipsoid is an ellipsoid of revolution, whose long axis is coinciding with *l*, lies on the hydrostatic axis of the material and the transverse principal plane (k, m) coincides with the deviatoric π-plane. Since k=m, this intersection is a circle.

It was shown in [19] that for the general transversely isotropic material the ellipsoid of revolution for the isotropic materials becomes the Beltrami-Theocaris generalized ellipsoid, whose principal axes have the lengths:

$$\ell=\frac{1}{\sqrt{\lambda_4}}, \quad k=\frac{1}{\sqrt{\lambda_1}} \quad \text{and} \quad m=\frac{1}{\sqrt{\lambda_3}} \tag{44}$$

Thus, for the quasi-isotropic materials the lengths of the principal axes of the respective ellipsoids take the forms:

$$\ell=\sqrt{\frac{E_L}{(1-2\nu_L)}}, \quad k=\sqrt{2G_T} \ , \quad m=\sqrt{2G_L} \tag{45}$$

It is also shown that the ellipsoids for transversely isotropic materials cease to be ellipsoids of revolution and they present a slenderness, expressed by [19]:

$$s=\sqrt{\frac{\lambda_3}{\lambda_1}} \tag{46}$$

The slenderness, s, for the transversely isotropic materials having an ω-angle lying in the interval $\omega \in [90°, 125.26°]$, is horizontal, that is these ellipsoids seem to be compressed along the vertical σ_3-principal stress axis. Beyond this zone and inside the interval $\omega \in [125.26°, 180°]$ the slenderness of the ellipsoids is increased, as the ω-eigenangle tends to its extreme limits, either $\omega=90°$ or $\omega=180°$. It is of interest to remark that, for the particular value of $\omega=135°$, the ellipsoid of principal stresses or strains becomes again an ellipsoid of revolution, but with its longitudinal axis of symmetry along the σ_1-principal direction, normal to the principal diagonal $\sigma_3\delta_{12}$-plane. This is because for this particular value of the ω-angle it is valid that $\lambda_3=\lambda_4$ (see relation (41)).

Furthermore, while the Beltrami ellipsoid for the isotropic materials has as principal axes, the hydrostatic axis coinciding with σ_4-principal direction and the deviatoric plane for the (σ_1, σ_3)-plane, all the other ellipsoids for the transversely isotropic materials have their

principal axes rotating about the origin with the σ_4-principal direction subtending an angle $(\omega-\pi/2)$ with the (σ_1, σ_2) transverse isotropic plane, the σ_1-direction always normal to the principal diagonal plane (σ_3, δ_{12}) and the σ_3-principal direction subtending again an angle $(\omega-\pi/2)$ with the σ_3-principal stress direction.

Thus, for the strong fiber-axis composites the Beltrami-Theocaris ellipsoids become almost vertical, since for these materials, angle ω tends to 180°. As the composites dispose stronger matrices and therefore the angle ω diminishes, the B-T ellipsoid rotates about the σ_1-principal axis tending to the angle $\omega=35.26°$ for the isotropic materials and the respective ellipsoids tend to become ellipsoids of revolution, approaching the isotropic Beltrami ellipsoid.

For the interval of $\omega \in [90°, 125.26°]$ the Beltrami-Theocaris ellipsoids for the transversely isotropic materials start to become horizontally flattened and their longitudinal σ_4-axis rotates about the σ_1-principal direction, subtending an angle varying in the interval [35.26°, 0°] as the angle ω tends to $\omega\to90°$. As an example, Fig. 4 presents a series of typical Beltrami-Theocaris ellipsoids (a) for a Thornel 75S graphite epoxy composite, for which $\omega=178.57°$ [2], (b) for a Borsic 1100/Aluminum composite with $\omega=134.92°$, (c) for an isotropic material ice-I at -16°C, (d) for a glass-epoxy woven fabric composite with $\omega=114.96°$ and finally (e) for a compression annealed pyrolite graphite material with $\omega=90.84°$. It is of interest to remark the relative positions of these ellipsoids, as well as their principal dimensions [14].

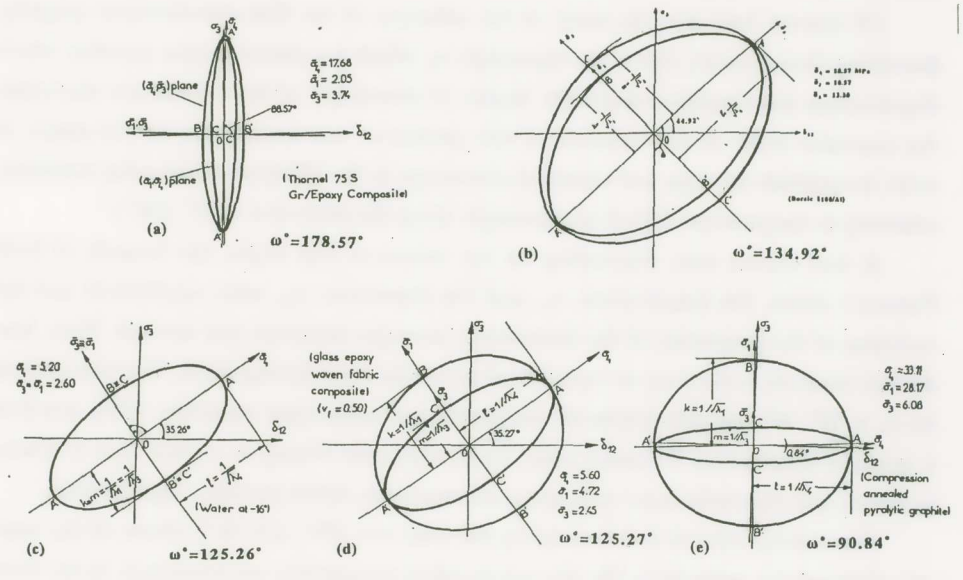

Fig. 4. The normalized to total strain energy 2T, B-T strain-energy ellipsoids for five different transversely isotropic materials with respective eigenangles $\omega°$ varying between $\omega=90°$ and $\omega=180°$.

7. CONCLUSIONS

The spectral decomposition of the fourth-rank tensors of elastic stiffness, compliance and failure of transversely isotropic materials gave the possibility of decomposing in an energy orthogonal mode the stress and the strain tensors in this important category of materials, encompassing almost the totality of engineering materials of the praxis. Then, the spectral decomposition of these tensors allowed the extension of the advantageous properties of linearity and coincidence of the components of principal stresses and strains at a privileged Cartesian frame, beyond their purely elastic domain, to a broader area of the transversely isotropic materials.

This spectral decomposition of the abovementioned tensors gave six eigenvalues λ_i ($i=1\div4$), with the two first double, which allowed the splitting of the strain energy into four components, the two firsts of which representing a distortional type of elastic strain energy, whereas the two last eigenvalues gave stress —and strain— states consisting of a superposition of a equilateral stress or strain in the transverse isotropic plane, superimposed with either a tensile, or a compressive stress along the strong axis of symmetry of the material. In this way the strain-energy function may be splitted into four components in the general transversely isotropic or orthotropic material, none of which is expressing exclusively a dilatational type of strain energy. This decomposition of the strain energy presented in detail in previous papers [19], [1].

Of interest here was the study of the influence of the fifth characteristic quantity, describing these tensors, that is the eigenangle ω, which is a dimensionless quantity which characterizes monoparametrically the degree of anisotropy of the orthotropic materials. An extensive study of the variation of this parameter was undertaken in this paper, in order to establish the type and extent of anisotropy in the different engineering materials, relatively to the position of their ω-eigenangle along the interval $\in \omega\, [0°, 180°]$.

It was shown that, depending on the values of this angle, the bounds of both Poisson's ratios, the longitudinal, ν_L, and the transverse, ν_T, were established and the variation of the properties of the transversely isotropic materials was derived. Thus, four distinct intervals in the zone of variation of the tan2ω were distinguished. The first interval for $\omega \in [0°, 90°]$ should contain all composites and anisotropic materials, which manifest a negative longitudinal Poisson's ratio. These materials belong to a great part to foamy materials and especially those with open-cell structures, which present a high porosity.

The second interval is delineated by the zone $\omega \in [90°, 125.26°]$, where all the weak axis of symmetry composites, like the woven-fabric composites, are contained. In the same zone foamy materials with closed-cell structures are included, as well as a great number of inorganic crystals and different types of papers.

For the limiting values of angle ω equal to $\omega=125.26^\circ$, as well as $\omega=35.26^\circ$ (for the first interval), the materials become isotropic. It was also shown that as the angle ω changed from 90° to 125.26°, the anisotropy of the materials was reduced and in both sides of the neighbourhood of the isotropic angle $\omega_i=125.26^\circ$ the anisotropic materials tend to a quasi-isotropy presenting the remarkable fact to acquire the property, valid only for the isotropic substances, to split their strain energy into dilatational and distortional components, like the isotropic ones.

The third interval is in the zone $\omega \in [125.26^\circ, 135^\circ]$, which is characterized by almost complete isotropy. Finally, in the fourth interval of $\omega \in [135^\circ, 180^\circ]$ lie all strong-axis composites and fiber reinforced materials, with those of a strong matrix concentrating closer to the ω_i-angle. As the anisotropy of the material increased the angle ω_i is receding, approaching the limiting value of $\omega=180^\circ$, where all the weak-matrix, strong-fiber composites are placed.

As a further confirmation of the variation of the anisotropy of the materials with the change of angle ω, the modified Beltrami-Theocaris ellipsoids of elastic strains or stresses were plotted for five different characteristic materials along the intervals studied and the form and orientation of these ellipsoids were established [19].

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REFERENCES

1 Theocaris, P. S. and Philippides, T. P., "Spectral Decomposition of compliance and stiffness fourth-rank Tensors suitable for orthotropic materials", *Zeitsch. angew. Math. & Mech.*, **71**(3), 161-171 (1991).

2. Theocaris, P. S. and Philippides, T. P., "Variational bounds on the eigenvalue ω of tranversely isotropic materials", *Acta Mechanica*, **85**(1), 13-26, (1990).

3. Walpole, L. J., "Fourth-rank tensors of the thirty-two crystal classes: Multiplication tables", *Proced. Roy. Soc. London*, **A391,** 149-179 (1984).

4. Rychlewski, J., "On the Hooke's law", *Prikl. Matem. i Mekh.*, **48**(3), 303-314 (1984).

5. Rychlewski, J., "Elastic Energy decompositions and limit criteria", *Advances in Mechanics (Uspechi Mekhanikii)*, **10**(3), 83-102 (1987).

6. Thomson, W., "Elements of a mathematical theory of elasticity", *Phil. Trans. Royal Soc.*, **A155**, 481-498 (1856).

7. Theocaris, P. S., "The Compliance fourth-rank tensor for the transtropic material and its spectral decomposition", *Proc. National Academy of Athens*, **64**(1), 80-100 (1989).

8. Theocaris, P. S., "The decomposition of the strain-energy density of a single-ply laminate to orthonormal components", *Jnl. Reinf. Plast. and Comp.*, **8**(6), 565-583 (1989).

9. Theocaris, P. S., Stavroulakis, G. E., and Panagiotopoulos P. D., "Negative Poisson's Ratios in Composites with Star-shaped inclusions: A Numerical Homogenization Approach", *Archive Applied Mechanics*, in print (1997).

10. Theocaris, P. S., "The Elliptic Paraboloid Failure Surface for 2D-Transtropic Plates (Fiber Laminates), *Engng. Fracture Mech.*, **33**(3), 185-203 (1989).

11. Theocaris, P. S., "Failure Criteria for Anisotropic Bodies" *Handbook of Fatigue Crack Propagation in Metallic Structures*, A. Carpinteri Editor, Elsevier, Amsterdam, Publ., **1,** 3-45 (1994).

12. Theocaris, P. S., "Failure Modes of Closed Cell Polyurethane Foams", *Intern. Jnl. of Fracture*, **56**(4), 353-375 (1992).

13. Theocaris, P. S., "Failure Behaviour of Paper Sheets", *Jnl. of Reinforced Plastics and Composites*, **8**(6), 601-626 (1989).

14. Theocaris, P. S. "On a Family of Quasi-Isotropic Fiber-Reinforced Composites", *Acta Mechanica*, **96**(2), 163-180, (1993).

15. Theocaris, P. S. and Philippides T. P., "Stress Distribution in Orthotropic Plates with Coupled Elastic Properties", *Acta Mechanica*,. **80**(2), 95-111 (1989).

16. Zweben, C. & Norman, J. C., "Kevlar 49/Thormed 300 Hybrid Fabric Composites for Aerospace Applications, *Soc. Amer. Plast. Engs.*, Quarterly No **2**, 1-8, (1976).

17. Ishikawa, T. & Chou, T. W., "Non-Linear Behaviour of Woven Fabric Composites", *Jnl. Composite Materials*, **18**, 399-413 (1983).

18. Theocaris P. S., "Failure Criteria for Weak-Axis Quasi-Isotropic Woven-Fabric Composites", *Acta Mechanica*, **95**(1), 69-86 (1992).

19. Theocaris P. S., "The Extension of Beltrami's Ellipsoid to Anisotropic Materials" *Archive of Applied Mechanics*, **65**(2), 86-98 (1995).

20. Lekhnitskii S. G., "*Anisotropic Plates*", Gordon & Breach, (1968).

21. Evans K. E., Nkansah M. A. Hutchinson I. J. & Rogers S. G. "Molecular Network Design", *Nature*, **353**, 124, (1991).

22. Theocaris P. S., "Failure Modes of Woven Fabric Composites Loaded in the Transverse Isotropic Plane", *Acta Mechanica*, **103**(1-4), 157-175 (1994).

23. Milton G. W., "On Characterizing the Set of possible Effective Tensors of composites: The Variational Method and the Translation Method", *Comm. Pure and Appl. Math.*, **43**, 63-125, (1990).

24. Theocaris P. S., "The Elliptic Paraboloid Failure Criterion for Cellular Solids and Brittle Foams", *Acta Mechanica*, **89**(2), 93-121, (1991).

25. Theocaris P. S., "Failure Criteria for TranstropicPressure Dependent Materials: The Fiber Composites", *Rheologica Acta*, **27**(5), 451-465 (1988).

26. Beltrami E., "Sulle Conditioni di resistenza dei corpi elastici. Opere Matematiche IV (1920) (see also Rendiconti del R. Institutto Lombardo di Scienze Lettere e Arti, 18, 704-714 (1885).

27. Haigh R. P., "The Strain Energy Function and the Elastic Limit", *Engeneering CX*, 158-160 (1920).

Αἱ ἰδιότητες τῆς ἰδιογωνίας ω τοῦ τανυστοῦ ἀστοχίας τῆς ἀνισοτρόπου ὕλης καὶ ἡ συστηματικὴ κατάταξίς της

Οἱ τανυσταὶ ἐνδόσεως (S), δυσκαμψίας (C) καὶ ἀστοχίας (H) τῶν ἐγκαρσίως ἰσοτρόπων ὑλικῶν, ἀποσυντιθέμενοι φασματικῶς, δίδουν ἓξ ἰδιοτιμὰς ἢ ἰδιοτανυστάς, ἐξ ὧν δύο εἶναι διπλοῖ, μεγέθη τὰ ὁποῖα ἐπιτρέπουν τὸν καθορισμὸν τῶν ἐλαστικῶν ἰδιοκαταστάσεων φορτίσεως τῶν ἀνισοτρόπων ὑλικῶν καὶ κατὰ συνέπειαν τὸν καθορισμὸν τῶν βασικῶν καὶ στοιχειωδῶν ὅρων διαχωρισμοῦ τῆς ἐλαστικῆς ἐνεργείας τοῦ καταπονουμένου σώματος [1]. Ἔχει ἀποδειχθῆ ὅτι αἱ ἀναγκαῖαι παράμετροι διὰ τὴν **ἀναλλοίωτον περιγραφὴν** τῆς ἐλαστικῆς συμπεριφορᾶς τῶν ἐγκαρσίως ἰσοτρόπων ὑλικῶν ἐκφράζονται διὰ τῶν τεσσάρων τιμῶν τῶν ἰδιοτανυστῶν εἰς τοὺς ὁποίους ἀναλύεται ὁ ἑκάστοτε τανυστὴς ἀστοχίας τῶν ὑλικῶν, δεδομένου ὅτι αἱ δύο πρῶται ἰδιοτιμαὶ τοῦ τανυστοῦ εἶναι διπλαῖ [2]. Αἱ τέσσαρες αὐταὶ τιμαὶ τῶν ἰδιοτανυστῶν συμπληροῦνται ὑπὸ τῆς ἰδιογωνίας ω, ἡ ὁποία ἀποτελεῖ ἀδιάστατον παράμετρον προκύπτουσαν ἀπὸ τὴν φασματικὴν ἀποσύνθεσιν τοῦ τανυστοῦ.

Ἐν τούτοις, περιορισμοὶ εἰς τὴν παραμετρικὴν ἔκφρασιν τοῦ μεγέθους τῆς ἰδιογωνίας, ἐπιβαλλόμενοι ἐκ θερμοδυναμικῶν συνθηκῶν, καθορίζουν τὰ ὅρια μεταβολῆς τῆς ἰδιογωνίας ω. Σκοπὸς τοῦ ἄρθρου τούτου εἶναι ἡ ἐξέτασις τῆς ἐπιδράσεως τῶν ἐπιβαλλομένων περιορισμῶν ἐφ' ὁλοκλήρου τοῦ φάσματος μεταβολῆς τῆς ἰδιογωνίας ω, ἐπὶ τοῦ τύπου, τῆς μορφῆς καὶ τῶν ἰδιοτήτων τῶν ἀντιστοίχων ἀνισοτρόπων σωμάτων καὶ ὁ καθορισμὸς καὶ ἡ περιγραφὴ τῶν ὁρίων, ἐκ τῶν ἐπιβαλλομένων εἰς τὰς τιμὰς τῆς ἰδιογωνίας ω, ἐπὶ τῶν ἰδιοτήτων τῶν ἀνισοτρόπων ὑλικῶν.

Ἀποδεικνύεται περαιτέρω εἰς τὸ ἄρθρον αὐτὸ ὅτι ἀρκεῖ ἡ μοναδικὴ αὐτὴ παράμετρος τῆς ἰδιογωνίας νὰ χαρακτηρίση ποιοτικῶς, τόσον τὰς ἐλαστικὰς ἰδιότητας, καθὼς ἐπίσης καὶ τὴν ἀνθεκτικότητα τῶν ἐγκαρσίως ἰσοτρόπων ὑλικῶν.

Δεδομένου ὅτι ἡ φασματικὴ ἀποσύνθεσις τοῦ τανυστοῦ ἀποτελεῖ τὴν ἁπλουστέραν μορφὴν ἀναλύσεως τοῦ συμμετρικοῦ τανυστοῦ τετάρτης τάξεως, ἡ ὁποία ἐπιτυγχάνει νὰ ἀναλύη τοὺς τανυστὰς τῶν τάσεων καὶ τῶν παραμορφώσεων εἰς συγγραμμικὰ ἀνύσματα, ἀνὰ δύο κάθετα ἐπ' ἀλλήλων, ἡ ἀνάλυσις αὐτὴ παρέχει ταχὺ καὶ δραστικὸν μέσον ὁρισμοῦ τῶν ὅρων ἀναλύσεως τῆς εἰδικῆς ἐνεργείας παραμορφώσεων. Τοιουτοτρόπως, τὰ χωριζόμενα μέρη τῆς ἐνεργείας αὐτῆς δύνανται νὰ καθορίσουν τοὺς ὅρους ἀναλύσεως τῆς ἐνεργείας καὶ νὰ ὁρίσουν ἐπακριβῶς τὴν μορφὴν τῶν τόπων ἀστοχίας τῶν ἀνισοτρόπων ὑλικῶν.