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ΜΑΘΗΜΑΤΙΚΑ.— **Practical Procedures and Methods for the Numerical Solution of Cauchy-Type Singular Integral Equation,** *by* ***P. S. Theocaris,*** *Fellow of the Academy**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Περικλῆ Θεοχάρη.

ABSTRACT

A large class of problems, of Mathematical Physics can be reduced to the solution of certain systems of Cauchy singular integral equations. Hence, it is of interest to solve numerically these systems of integral equations of the respective boundary value problem, instead of the problem itself. The most effective method of numerical solution of Cauchy-type singular integral equations is the *direct method*, consisting of reducing such an equation (or a system of equations) to a system of linear algebraic equations, by using an appropriate (generally *Gaussian*) numerical integration rule on a properly selected set of collocation points.

The collocation points are defined as roots of a special *transcendental* equation and they are systematically tabulated in tables I to VIII. The collocation points for the *Gauss - Legendre* (G - L), *Gauss - Chebyshev* (G - CH) and *Gauss - Jacobi* (G - J) numerical integration rules are used when simple crack problems are concerned under various loading conditions. In these cases the length of the crack must be finite, while it is allowed to approach an interface. The collocation points for the *Lobatto - Legendre* (LO - L), the *Lobatto - Chebyshev* (LO - CH), the *Lobatto - Jacobi* (LO - J) the *Radau - Legendre* (R - L), the *Radau - Chebyshev* (R - CH) numerical integration

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rules are used in cases where special quantities must be calculated on the crack tips, which in this case may be located close enough to an interface of the body.

In the special case where the crack length is semi-infinite the collocation points for the *Guass - Hermite* (G - H), *Gauss - Laguerre* (G - LAG) and *Radau - Laguerre* (R - LAG) numerical integration rules are introduced.

The preparation of the tables contained in this paper was executed at the Department of Theoretical and Applied Mechanics of the National Technical University of Athens by using its CDC-Cyber computer. Whereas all the algorithms and computer programmes are included in the Computer Library of the Department, practical procedures are presented in this paper, how the optimum approach of numerical solution of problems in Physics, expressed by singular integral equations, may be defined.

1. INTRODUCTION

Several plane and antiplane elasticity problems, not possessing closed-form solutions, can be effectively solved by reduction to a *Cauchy* singular integral equation (or a system of such equations) along their boundary.

The peculiarity of these equations is that the integral, in which the unknown function appears, contains, besides a regular part, one more part not defined in the ordinary sense, but in the *Cauchy* principal-value sense. Naturally, several of these equations can be reduced to equivalent *Fredholm* integral equations of the second kind, for which there exist efficient numerical algorithms. Unfortunately, the above reduction is not, in general, an easy task. Moreover, an equally large class of Cauchy-type singular integral equations is not reducible to an equivalent *Fredholm* integral equation.

For these reasons, the *direct method* of numerical solution of Cauchy-type singular integral equations has received much attention in recent years. The aim of the present paper is to assist the researcher to select the appropriate procedure for the numerical solution of each particular problem and to proceed in its solution by providing the respective values of the collocation points, which have to be used during the numerical solution. These points are generally roots of transcendental functions, and not of algebraic polynomials. The use of these points will permit the evaluation, with a sufficient accuracy, in general, of all quantities required, like stress —and displacement— components, stress intensity factors etc., whose determination constitutes the main goal of the numerical techniques when the solution of a specific boundary-value problem is concerned.

A large number of plane and antiplane isotropic and anisotropic elasticity problems are recently solved [**1**] by using the numerical technique described in this paper and making use of collocation points selected as proposed here (Tables I to VIII). Among them we can mention the following typical problems [**1**] to [**3**]: Simple straight- or curvilinear-cracks, or systems of such cracks, in plane isotropic elasticity, for a finite or infinite medium, arrays of straight or curvilinear cracks in an infinite isotropic elastic medium, branched cracks, kinked cracks, star-shaped cracks, cruciform cracks inside an infinite isotropic medium, cracks along the interfaces of two plane isotropic media, either straight or curvilinear, cracks normal to a bimaterial interface, cracks interacting with a misfitting inclusion, problems of V-notches, contact problems etc.

Finally, the interesting problem of generalizing these techniques to three-dimensional elasticity problems is also successfully faced [**3**].

2. CAUCHY-TYPE INTEGRALS AND NUMERICAL INTEGRATION RULES

Let us consider the following imporper integral :

$$2\pi i\Phi(z) = \int_L \frac{w(t)\,\varphi(t)}{t-z}\,dt\,, \tag{1}$$

where L denotes the interval [a, b] of the real axis, w(t) is a given weight function, defined for every $t \in [a, b]$, and φ(t) is an *analytic* function of t in any plane domain S, containing the interval L. Then, it may be proved [**1**] that the complex *Cauchy* integral Φ(z) is a *sectionally analytic* function of z in the whole plane except L.

If the point z does not belong to L, the *Cauchy-type* integral behaves like a regular integral, provided that the weight function w(t) does not present any strong singularities, but only weak singularities at a finite number of points in the interval L.

On the other hand, in the case where the point z belongs to L the Cauchy-type integral Φ(z) diverges and it is usually defined in the sense of the *Cauchy principal value* (c.p.v.) as follows :

$$2\pi i\Phi(t_0) \equiv \lim_{\varepsilon\to 0} \int_{L-2} \frac{w(t)\,\varphi(t)\,dt}{t-t_0}\,, \tag{2}$$

where $z \equiv t_0 \in L$.

The integral on the right-hand side of Eq. (2) can be defined on the whole interval L by neglecting a small portion $l \equiv 2\varepsilon$ (where ε is a positive real number), which is cut off by a disk of radius ε, centered at the point $z \equiv t_0$, and then proceeding to the limit as $\varepsilon \to 0$. A sufficient condition for the existence of the integral $\Phi(t_0)$ is that its density function w(t) $\varphi(t)$ be a *Hölder*-continuous function in the interval L, except at the neighbourhoods of its ends, where it may present weak singularities [**2**].

In order to calculate numerically the Cauchy-type integral (2), we have to take into account the pole of the integrand at the point $t \equiv z$ [**3**] as following :

$$2\pi i \Phi(z) \simeq \sum_{k=1}^{n} A_k \frac{\varphi(t_k)}{t_k - z} - 2\varphi(z) \frac{q_n(z)}{\sigma_n(z)} , \tag{3}$$

where the functions $\sigma_n(z)$ and $q_n(z)$ are defined as :

$$\sigma_n(z) = \prod_{k=1}^{n} (z - t_k) , \tag{4}$$

$$q_n(z) = -\frac{1}{2} \int_L \frac{w(t)\, \sigma_n(t)}{t - z}\, dt . \tag{5}$$

In the special case, where the complex pole z of the *Cauchy* integral (1) coincides with a point t of L (different from its end-points) we can properly extend the rules of numerical integration, in order to be applicable in the case of Cauchy principal values of the form (2). Then, the following quadrature is valid :

$$\Phi(t_0) \simeq \sum_{k=1}^{n} A_k \frac{\varphi(t_k)}{t_k - t_0} - 2\varphi(t_0) K_n(t_0) , \tag{6}$$

$$t \neq t_k \; (k = 1, 2, \ldots, n)$$

and :

$$\Phi(t_0) \simeq \sum_{\substack{k=1 \\ k \neq m}}^{n} A_k \frac{\varphi(t_k)}{t_k - t_0} + A_m \varphi'(t_0) - 2\varphi(t_0) \Lambda_n(t_0) , \quad (t \equiv t_m, m = 1, \ldots, n) \tag{7}$$

where :

$$K_n(t) = q_n(t) / \sigma_n(t) , \quad t \neq t_k \tag{8}$$

and :

$$\Lambda_n(t) = \frac{1}{\sigma_n'(t)} \{ q_n'(t) + 1/4 A_m \sigma_n''(t) \} , \tag{9}$$

$$(t \equiv t_m \, (m = 1, \ldots, n)) .$$

It should be noted that in all the above considerations to is not permitted to coincide with the end-points a or b of the interval L. In the special case where the variable t_0 coincides with either a or b, the improper integral (2) must be understood in the *finite-part sense* [**4**].

Next, we will determine the functions K_n and Λ_n for several integration rules, which are frequently encountered in elasticity problems :

(i) *The Gauss Legendre rule* [**5**], [**6**] :

This rule has as weight-function the unity and as an integration interval $L \equiv]-1,1[$. The functions $\sigma_n(z)$ and $q_n(z)$ for this rule are :

$$\sigma_n(z) = P_n(z), \qquad q_n(z) = Q_n(z), \tag{10}$$

where $P_n(z)$ and $Q_n(z)$ are the *Legendre* polynomial of degree n and the *Legendre* function of the second kind and order n, respectively. Then, by substituting relation (10) into the Eq. (8) it is obtained that :

$$K_n(t) = Q_n(t)/P_n(t). \tag{11}$$

On the basis of the above general results and particularly Eqs. (6) and (7), it is clear that, in the case where $\bar{t}_0$ is selected as a root of the function $Q_n(\bar{t}_0) \equiv 0$, $P_n(\bar{t}_0) \not\equiv 0$ then $K_n(\bar{t}_0) \equiv 0$ and the Cauchy principal value (6) can be approximated as a regular integral, that is :

$$\Phi(\bar{t}_0) \simeq \sum_{k=1}^{n} A_k \frac{\varphi(t_k)}{t_k - \bar{t}_0}. \tag{12}$$

In fact, the roots $\{\bar{t}_{0i}\}$ of the functions $q_n(t) \equiv Q_n(t)$ are the only abscissae along the integration interval (a, b) for which Eq. (12) for regular integrals can also be used for *Cauchy*-type principal-value integrals, as is clear by a comparison of Eqs. (6) and (12).

The set of the points $\{\bar{z}_{0i}\}$ is called here and in the sequence as *set of collocation points of the numerical integration rule under consideration.* For the *Gauss - Legendre* rule these points are calculated numerically for various natural numbers n (Table I). Furthermore, in the case where $t \equiv t_m$ inside $L \equiv]-1,1[$ the function $\Lambda_n(t)$ of Eq. (9) may be calculated as :

$$\Lambda_n(t) = \frac{Q_{n-1}(t)}{P_{n-1}(t)} + \frac{n+1}{2} A_m \frac{1}{1-t^2}. \tag{13}$$

(ii) *The Lobatto - Legendre rule* [**7**] :

This rule has no weight function like the *Gauss - Legendre* rule, and contains among the abscissae t_k the end points —1,1 of the integration interval L. The functions (4) and (5) are of the form :

$$\sigma_n(z) = P_n(z) - P_{n-z}(z) = \frac{2n-1}{n(n-1)}(z^2-1)P'_{n-1}(z) \tag{14}$$

and :

$$q_n(z) = Q_n(z) - Q_{n-2}(z) = \frac{2n-1}{n(n-1)}(z^2-1)Q'_{n-1}(z)\,. \tag{15}$$

The set of collocation points $\{t_{0i}\}$ coincide with the roots of the function (15). These roots can be calculated numerically and are tabulated for various values of the natural number n (Table II). Furthermore, by taking into account the following relations :

$$\left.\begin{aligned} &\sigma''_n(t)/\sigma'(t) = 0\,, \\ &q'_n(t)/\sigma'_n(t) = Q_{n-1}(t)/P_{n-1}(t)\,, \\ &t = t_m\,(m = 2, 3, \ldots, n-1)\,, \\ &t \neq t_1,\ t_n\ \ (t_1 = 1,\ t_n = -1)\,, \end{aligned}\right\} \tag{16}$$

it can be derived that :

$$\Lambda_n(t) = Q'_{n-1}(t)/P'_{n-1}(t)\,. \tag{17}$$

(iii) *The Gauss-Chebyshev rule* [**1**] :

This rule has as weight function w(t) the function $(1-t)^{\pm 1/2}$ $(1+t)^{\pm 1/2}$, while the integration interval is again the interval]—1,1[. We will examine the case where $w(t) = (1-t^2)^{-1/2}$. The functions $\sigma_n(z)$ and $q_n(z)$ for this case are given by :

$$\sigma_n(z) = T_n(z)\,, \quad q_n(z) = -\frac{\pi}{2}U_{n-1}(z)\,, \quad z \in\,]-1,1[\,, \tag{18}$$

where $T_n(z)$ and $U_n(z)$ denote the *Chebyshev* polynomials of the first and second kind and degree n, respectively, easily expressible in terms of trigonometric functions [**8**]. Then, it is valid that :

$$K_n(t) = -\pi U_{n-1}(t)/nT_n(t). \tag{19}$$

It is obvious that the set of the collocation points $\{t_{0i}\}$ coincides with the roots of the *Chebyshev* polynomials of the second kind :

$$U_{n-1}(t_{0i}) = 0\,, \qquad t_{0i} = \cos\frac{\pi i}{n}\,. \tag{20}$$

$$(k = 1,2,\ldots, n-1)\,.$$

In an analogous manner we can find that :

$$\Lambda_n(t) = -\frac{\pi}{2}\,\frac{U_{n-2}(t)}{T_{n-1}(t)} + \frac{2n-1}{4}\,A_m\,\frac{t}{1-t^2} \tag{21}$$

in the case where $t \equiv t_m$ $(m = 1, \ldots, n)$. It should be noticed that Eq. (19) is valid only for the abscissae t_m lying inside the integration interval $]-1,1[$, while Eq. (21) remains valid only for the abscissas t_m used.

(iv) *The Radau-Legendre numerical integration rule* [**7**] :

This rule has as weight function w(t) the function $1/\sqrt{t}$ while the integration interval L coincides with [0,1]. For this case the functions $\sigma_n(z)$ and $q_n(z)$ can be defined as (the point $t = 1$ is used among the abscissae) :

$$\sigma_n(z) = \frac{(4n-1)}{2n\,(2n-1)}\,(1-z)\,P'_{2n-1}(\sqrt{z}) \tag{22}$$

and :

$$q_n(z) = \frac{1}{\sqrt{z}}\,\frac{4n-1}{2n(2n-1)}\,(1-z)\,Q'_{2n-1}(\sqrt{z})\,, \tag{23}$$

where P and Q are the *Legendre* polynomials of the first and the second kind and of degree $2n-1$ respectively. Then, by taking into account Eq. (8) we can define the collocation points as the roots of the function (23). These points can be calculated numerically and are given in *Table III* for various values of n.

(v) *The Gauss Numerical Integration Rule with a Logarithmic Singularity* [**9**] :

The numerical techniques already presented in the previous paragraphs can be properly extended in order to include Cauchy-type integrals of the form :

$$I(x) = \int_0^1 ln\frac{1}{t}\,\frac{\varphi(t)}{t-x}\,dt\,. \tag{24}$$

For $w(t) = ln(1/t)$ the orthogonal polynomials are given by the recurrence formula :

$$P_0(z) = 4\,,\quad P_1(z) = z - b_1\,,$$

$$P_n(z) = (z - b_n)\,P_{n-1}(z) - c_n P_{n-2}(z)\,,\quad n \geqq 2\,. \tag{25}$$

In ref. [**9**], pp. 90 - 91, the coefficients b_n and c_n are determined. Thus, the functions $q_n(z)$ for the case under consideration are given by :

$$q_n(z) = -\frac{1}{2}\int_0^1 ln\frac{1}{t}\,\frac{P_n(t)}{t-z}\,dt\,. \tag{26}$$

Furthermore, for any value of x other than 0,1 and t_k $(k = 1, 2, \ldots, n)$ denoting the roots of the polynomials $\sigma_n(z)$:

$$\sigma_n(z) = P_u(z) + dnP_{n-1}(z)\,,\quad n \geqq 1\,, \tag{27}$$

where :

$$dn = -\frac{P_n(\tau)}{P_{n-1}(\tau)}\,.\quad \tau = 0 \text{ or } 1\,, \tag{28}$$

the integral (24) can be evaluated as :

$$I(x) \simeq \sum_{k=1}^{n} A_k\,\Phi(t_k) - 2\sum_{k=1}^{m}\frac{e_k\,q_n(z_k)}{\sigma_n(z_k)}\,, \tag{29}$$

with :

$$\Phi(t) = \varphi(t)/t - x\,. \tag{30}$$

In this expression z_k are the simple poles of $\Phi(z)$, lying in the interior of a simple-closed curve C surrounding the integration interval [0,1] and ρ_k

are the residues of $\Phi(t)$ at these poles. Therefore, the roots of the function $q_n(z)$ can be selected as the proper set of collocation points, so that only terms corresponding to poles outside the integration interval remain in the second sum of its right-hand side of Eq. (29). Although these terms are ignored at present, their influence on the resulting values of $\varphi(t_k)$ becomes considerable for those t_k lying in the vicinity of such poles. The collocation points are calculated numerically for various values of the parameter n and are included in *Table IV*.

(vi) *The Gauss-Jacobi Numerical Integration rule* [**1**] :

This rule has as weight function w(t) :

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}, \quad (\alpha, \beta > -1), \tag{31}$$

and as integration interval the interval $L \equiv [-1,1]$. Here, the set of orthogonal polynomials $P_n^{(\alpha,\beta)}(t)$ and $Q_n^{(\alpha,\beta)}(t)$ are the *Jacobi* polynomials of the first and the second kind respectively [**8**]. The collocation points $\{t_{0i}\}$ are defined as roots of the following transcendental equation :

$$q_n^{(\alpha,\beta)}(z) = (z-1)^{\alpha}(z+1)^{\beta} Q_n^{(\alpha,\beta)}(z) \equiv 0 \tag{32}$$

and they are systematically tabulated in *Table V* for various values of the parameters α, β, n, together with the abscissas and weights of the numerical integration rule under discussion.

(vii) *The Lobatto-Jacobi Numerical Integration Rule* [**3**] :

In this case, the collocation points $\{t_{0i}\}$ are the roots of the following transcendental equation :

$$q_n^{(\alpha,\beta)}(z) = (z-1)^{\alpha}(z+1)^{\beta} Q_n^{(\alpha,\beta)}(z) \equiv 0, \tag{33}$$

and are systematically tabulated in *Table VI* by using the same symbols as before (Table V).

(viii) *The Generalized Gauss-Laguerre Numerical Integration Rule :*

This numerical rule has as weight function the function $w(t) = t^{\alpha} e^{-t}$ and as an integration interval $L \equiv [0, \infty)$. The set of orthogonal polynomials are the *Laguerre* polynomials [**8**]. On the other hand, the collocation points are defined as roots of the functions $q^n(z)$, which are estimated by the following reccurence relations :

$$\left\{\begin{array}{l} nq^n(z) = (2n - 1 - z)\, q_{n-1}(z) - (n - 1)\, q_{n-z}(z)\,, \\ q_0(z) = \frac{1}{2}\, e^{-z}\, E_I(z)\,, \\ q_1(z) = (1 - z)\, q_0(z)\,, \\ E_I(z) = \int_{-\infty}^{z} e^t/t dt \end{array}\right\} \qquad (34)$$

The roots of these functions are tabulated in *Table VII* for various values of the parameters α, n.

(ix) *The Generalized Radau-Laguerre Numerical Integration Rule :*

This numerical rule has as weight function the expression $w(t) = t^a \exp(-t)$ and as an integration interval $L \equiv [0, \infty)$. The collocation points are roots of the *Laguerre* polynomials $L_{2n-1}(\sqrt{t})$. These roots are systematically tabulated in *Table VIII* for $a = -0{,}9$ and $n = 4\,(1)\,10$. Of course, between the collocation points is the point $t = 0$ (that is the left edge of the interval L).

Furthermore, it can be clearly seen that the numerical integration rules associated with the *Legendre* and the *Chebyshev* polynomials are special cases of the corresponding rules for the *Jacobi* polynomials (with $\alpha = \beta = 0$ and $|\alpha| = |\beta| = 1/2$ respectively). Similarly, the numerical integration rules associated with the *Hermite* and *Laguerre* polynomials are special cases of the corresponding rules for the generalized *Laguerre polynomials* (with $\alpha = -1/2$ and $\alpha = 0$ respectively).

Since the *Gauss-Chebychev* and *Lobatto-Chebyshev* numerical integration rules are special cases of the *Gauss-Jacobi* and the *Lobatto-Jacobi* numerical integration rules respectively, tables of collocation points are provided only for the latter pair of numerical integration rules.

The same remark is valid for the case of the generalized *Gauss-Laguerre* and *Radau-Laguerre* numerical integration rules. A similar remark holds also true for the *Gauss-Hermite* numerical integration rule, which can be readily reduced either to the generalized *Gauss-Laguerre* rule, when n is an even number, or to the generalized *Radau-Laguerre*, when n is an odd number and $\alpha = -0.5$.

Finally, for the *Gauss-Legendre*, the *Radau-Legendre* and the *Lobatto-Legendre* numerical integration rules, as well as for the *Gauss*-numerical integration rule associated with a logarithmic singularity, no tables for the abscissae and weights are presented here, but only tables for the collocation points compatible with the corresponding tables or abscissae and weights included in the classical book by *Stroud and Secrest* [**10**].

3. CAUCHY-TYPE SINGULAR INTEGRAL EQUATIONS

Let us consider the Cauchy-type singular integral equation :

$$A(x)\,\varphi(x) + B(x)\int_L \frac{\varphi(t)}{t-x}\,dt + \int_L K(t,x)\,\varphi(t)\,dt = f(x)\,, \quad x \in L\,, \tag{35}$$

where $A(x)$, $B(x)$ and $f(x)$ are known functions, $K(t, x)$ the Fredholm kernel, $\varphi(x)$ the unknown function and L the integration interval, which may be a closed contour, a curvilinear arc, or simply a part of the real axis.

The direct method for numerical solution of the *Cauchy* integral equation (35) consists in the direct application of the numerical integration rules (i) to (viii) of the previous paragraph for the approximation of the integrals in Eq. (35). Thus, if the points x_k are selected as roots of the following general transcendental equation :

$$A(x_k)\,w(x_k) - 2B(x_k)\,K_n(x_k) \equiv 0\,, \quad (k = 1, 2, \ldots, m)\,, \tag{36}$$

then Eq. (35) reduces to the following system of linear equations :

$$\sum_{i=1}^{n} A_i \left[\frac{B(x_k)}{t_i - x_k} + K(t_i, x_k)\right] g(x_k) = f(x_k)\,, \quad (k = 1, 2, \ldots, m)\,. \tag{37}$$

These roots of the transcendental equation (36), for various numerical rules, are systematically tabulated in *Tables I to VIII* of the previous paragraph.

As regards the number m of equations (37), it is generally equal to $(n-1)$, n, or $(n+1)$ [**3**]. In the first case Eq. (37) should be generally supplemented by one more equation, derived from a physical condition of the particular problem treated, which may have the form :

$$\int_L \varphi(t)\,dt = C, \tag{38}$$

where C is a known constant.

On the other hand, in most engineering problems, the Cauchy-type singular integral equations encountered are of the first kind [2] and therefore, for the selection of the collocation points x_k, Eq. (36) takes the simplified form :

$$q_n(x_k) \equiv 0. \tag{39}$$

Next, we can remark that in a numerical integration rule of the form (37) the abscissae t_i are the roots of the polynomial $\sigma_n(x)$ (Eq. 4), whereas the collocation points x_k are the roots of the function $q_n(x)$ (Eq. 5). The roots t_i of the polynomials $\sigma_n(x)$ have been investigated in detail in ref. [**10**], while no attempt has ever been made for the tabulation of the roots x_k of the functions $q_n(x)$ of the second kind, because of the fact that their usefulness during the numerical solution of Cauchy type integral equations is not known.

During the preparation of the Tables I to VIII several special functions have been used (gamma-psi-bilogarithm-hypergeometric-confluent hypergeometric functions), while the numerical algorithms for their approximate evaluation have been taken into account. A detailed study for the construction of the algorithms, whose use yields the values of the roots of the transcendental equation (36) is included in ref. [**11**]. The related computer programmes are available from the computer library of the Laboratory for Testing Materials of the University.

4. THE PRECTICAL IMPORTANCE OF THE TABLES OF COLLOCATION POINTS FOR THE NUMERICAL SOLUTION OF SYSTEMS OF CAUCHY-TYPE SINGULAR INTEGRAL EQUATIONS

Let us consider now that a particular boundary-value problem leads to a Cauchy singular integral equation of the general form of Eq. (35). The numerical solution of such a boundary-value problem follows necessarily the subsequent steps :

i) Formulation of the singular integral equation (or a system of such equations), which solves the boundary-value problem under consideration.

ii) Selection of the set of collocation points (Tables I to VIII) x_k ($x_k = 1, 2, \ldots, m$) for the reduction of Eq. (35) to a system of linear equations.

iii) Selection of the number m of the linear equations (37), the system of which has to be solved numerically.

iv) The numerical solution of the linear system of equations is accomplished by using a standard computer routine for the determination of the values of the unknown function $g(x_k)$ at the collocation points x_k. Next, the function $g(x)$ can be determined along the whole integration interval L, either by using some interpolation rule, or by using the following relationship :

$$\left. \begin{aligned} A(x)\,w(x)\,g(x) + \sum_{i=1}^{n} A_i \left[\frac{B(x_k)}{t_i - x_k} + k(t_i, x_k) \right] g(t_i) - \\ - 2B(x)\,K_n(x)\,g(x) = f(x)\,, \end{aligned} \right\} \tag{40}$$

as a natural interpolation formula (similar to that, which is used in the case of the Fredholm integral equations of the second kind).

From the above discussion it becomes clear that the first step needs the support of the theoretical methods already described in the classical monograph of *Muskhelishvili* [**2**], while the fourth step depends clearly upon the capability of the researcher in using computer routines and other fundamental procedures in numerical analysis.

On the other hand, for the decision concerning the second or the third step, the following points are of particular importance:

i) The selection of the set of collocation points depends clearly upon the nature of the physical problem under consideration.

Here we wish just to remind that the most useful numerical integration rules of *Gaussian* type may be classified as follows according to the categories of boundary-value problems, which can be solved by using them:

a) The *Gauss-Legendre* (G-L) with integration interval $[-1,1]$ and weight function $w(t) = 1$. It can be used for the solution of simple straight crack problems in plane isotropic elasticity, for an infinite medium. The cases of the first and the second fundamental problems of simple crack geometries inside an anisotropic elastic body can be also considered by this rule.

b) The *Radau-Legendre* (R-L) numerical integration rule with the same integration interval and weight function as before. The (R-L) is a Gaussian numerical integration rule of a semiclosed type and can be used in the case of simple-elasticity problems, concerning cracks normal to a bimaterial interface of two plane or antiplane isotropic elastic media, or a free boundary.

c) The *Lobatto-Legendre* numerical integration rule with the same integration interval and weight function as before (cases a, b). The (LO-L)-rule is a Gaussian-numerical integration rule of closed type and can be widely used in cases where special data concerning the crack tips (like SIFs and others) are needed. That is so, because of the fact that the end points —1,1 of the integration interval L are also abscissas of the (LO-L)-rule.

Generally speaking the cases a, b, c concern simple elasticity problems, that is problems where simple load configurations are under discussion. For the most complicated cases, which are frequently encountered in applications the need of a weight function different from unity comes on stage.

d) The *Gauss - Chebyshev* (G - CH) or *Lobatto - Chebyshev* (LO - CH) numerical integration rules have as integration interval the interval [—1,1] and as weight function the function $w(t) = (1 - t^2)^{\pm 1/2}$. They are Gaussian rules of open or closed type respectively and can be used for the solution of more complicated crack problems, like crack geometries along the interface of two plane isotropic elastic media, either straight, or curvilinear, cracks interacting with a misfitting inclusion in an infinite isotropic elastic medium, finite-plane isotropic elastic-media problems etc.

e) A generalization of the above case is the *Gauss-Jacobi* (G-J) or the *Lobatto-Jacobi* (LO-J) numerical integration rules, concerning the interval [— 1,1] and a weight function $w(t) = (1 - t)^{\alpha} (1 + t)^{\beta}$ $(\alpha, \beta > -1)$. By applying the same logic, these Gaussian methods can be used to branched crack problems, kinked cracks, star-shaped or cruciform cracks inside an infinite isotropic solid under various loading conditions.

In order to solve problems concerning semi-infinite crack geometries we introduce the following rules :

f) The *Gauss-Hermitte* (G-H) numerical integration rule ($L \equiv (\infty, \infty)$, $w(t) = \exp(-t)$), or the *Gauss - Laguerre* (G-LAG) numerical integration rule ($L \equiv [0, \infty)$, $w(t) = \exp(-t)$) is used for the solution semi-infinite crack problems associated with plane isotropic elastic media.

Furthermore, it is worthwhile mentioning that the generalized *Gauss - Laguerre* numerical integration rule may be used ($L \equiv [0, \infty)$, $w(t) = t^a \cdot \exp(-t)$) in boundary-value problems concerning semi-infinite cracks under more complicated loading conditions.

Analogous applications to the above-described may be attached to the semi-closed Gaussian integration rule of the *Radau - Laguerre* (R-LAG)-type (i.e. a semi infinite crack problem near a bimaterial interface etc.).

The names of these Gaussian numerical integration rules denote both, if they are of open (*Gauss*)-, semi-closed (*Radau*)- or closed (*Lobatto*)-type and which is the corresponding set of collocation points (see tables I to VIII).

As regards the number m of equations (37) (step 3), it is generally taken equal to (n—1), n, or (n+1). In the first case Eq. (37) should generally be supplemented by a physical condition of the form :

$$\sum_{i=1}^{n} A_i\, g(t_i) = C\,, \tag{41}$$

and this will be the nth linear equation required. Moreover, in the case when $m = n+1$ (or generally greater than m) one can either use only n of the m possible collocation points x_k, or use the whole set of the m-collocation points in the least-square sense to compute the values $g(t_i)$ of the unknown function $g(x)$ at the abscissae t_i used.

Finally, theoretical results of general validity, assuring the existence of at least one collocation point x_k in each subinterval (t_i, t_{i+1}) $(i = 1, 2, n-1)$ of the integration interval L, have been developed in refs. [**1**] and [**3**].

5. CONCLUSIONS

The reasons for which the present method is considered to be superior from all the others, see for example ref. [**12**], for the numerical solution of systems of singular integral equations are fully explained in ref. [**1**]. On the other hand, a large variety of applications for two or three-dimensional boundary-value problems have already mentioned in ref. [**3**].

From the above-described analysis it is clear that we have at our disposition a mathematical devise, which simplifies considerably the numerical calculations for solving various mathematical problems. The complicated problems of the modern research effort necessitate the use of approximate solutions, which have recourse to the digital computer for their solution and the confirmation of theoretical results. The weak point of such a use of computers for solving actual problems lies on the fact that errors introduced during the numerical calculations, due to averaging processes in the calculations, increase considerably from the existence of transcendental functions, which oscillate about infinity in regions of interest.

The actual important problems in two-dimensional and three-dimensional elasticity constitute a typical case, where difficulties are sometimes prohibitive.

The invention of collocation points in the numerical solution of singular integral equations helps considerably in the reduction of the influence of the transcendental components intervening in typical boundary-value problems described by Eq. (36). Then, the introduction of this technique in solving directly singular integral equations, describing various problems in physical sciences, constitutes some kind of optimazitation process for the approximate solution of the general problem of two-and three-dimensional elasticity.

Furthermore, it may be useful to remind that, in the case where the method of collocation points is applied, the following remarks are valid :

i) The (G - L), (G - CH) and (G - J)-rules are used when simple crack problems are concerned under various loading conditions, with the *Jacobi* rules appropriate for more complicated problems and the *Legendre* rules for the simpler ones. The length of the crack is finite and the material of the body may be indifferently either isotropic or anisotropic.

ii) The (LO - L), (LO - CH), (LO - J), (R - L) and (R - CH)-rules are used when special data must be derived near the crack tips, which are located close enough to a given interface. The length of the crack may be finite and the material of the body either isotropic or anisotropic.

iii) The (G - H), (G - LAG) and (R - LAG)-rules are suitable to investigate analogous problems to those of (i) and (ii) groups of problems, but in the special case, where the crack length is semi-infinite.

iv) Finally, special problems concerning logarithmic singularities may be investigated separately.

ΠΕΡΙΛΗΨΙΣ

Μεγάλη κατηγορία συνοριακῶν προβλημάτων τῆς θεωρίας ἐλαστικότητος, ἀλλὰ καὶ γενικώτερον τῆς Μαθηματικῆς Φυσικῆς, καταλήγει νὰ διατυποῦται διὰ συστημάτων ὁλοκληρωτικῶν ἐξισώσεων, ἰδιαιτέρως εἰς τὰς περιπτώσεις ὅπου περιγράφονται φυσικὰ φαινόμενα ἐξαρτώμενα ἐκ διαφόρων παραγόντων. Διὰ περιωρισμένον ἀριθμὸν συνοριακῶν προβλημάτων, τὰ ὁποῖα συνήθως παρουσιάζουν συμμετρίας εἰς τὸ σχῆμα τῶν συνόρων των ἢ εἰς τὸ εἶδος τῆς καταπονήσεώς των, τὸ προαναφερθὲν σύστημα ὁλοκληρωτικῶν ἐξισώσεων δέχεται κλειστὴν λύσιν ἢ κλειστὰς λύσεις. Κατὰ συνέπειαν, διὰ μεγάλην πλειονότητα φυσικῶν προβλημάτων τὰ συστήματα τῶν ὁλοκληρωτικῶν ἐξισώσεων εἶναι ἀνάγκη νὰ λυθοῦν δι' ἐφαρμογῆς ἀριθμητικῶν μεθόδων, αἱ ὁποῖαι ὑλοποιοῦνται μὲ τὴν βοήθειαν τοῦ Ἠλεκτρονικοῦ Ὑπολογιστοῦ.

Ἡ ἀναζήτησις καὶ εὕρεσις ἀριθμητικῆς μεθόδου, ἡ ὁποία νὰ δύναται εἰς σχετικῶς μικρὸν χρόνον ἀπασχολήσεως τῆς μνήμης τοῦ Ἠλεκτρονικοῦ Ὑπολογιστοῦ νὰ παρέχῃ τὰς ἐπιθυμητὰς λύσεις τοῦ συγκεκριμένου προβλήματος, ἀποτελεῖ ἀντικείμενον μεγάλου ἐνδιαφέροντος τῆς σημερινῆς ἐπιστήμης.

Εἰς ὡρισμένας περιπτώσεις συνοριακῶν προβλημάτων τῆς θεωρίας τῆς Ἐλαστικότητος καὶ συγκεκριμένως εἰς προβλήματα ὅπου μελετᾶται ἡ τασικὴ ἢ ἡ παραμορφωσιακὴ κατάστασις ρηγματωμένων κατασκευῶν, τὰ προαναφερθέντα συστήματα ὁλοκληρωτικῶν ἐξισώσεων παρουσιάζουν ἰδιομορφίας τύπου Cauchy, ὁπότε ἐπεκράτησε νὰ καλοῦνται αἱ ἐξισώσεις ἰδιόμορφοι ὁλοκληρωτικαὶ ἐξισώσεις κατὰ Cauchy. (Cauchy Singular Integral Equations).

Ἡ ἰδιομορφία ἔγκειται εἰς τὸ ὅτι οἱ πυρῆνες εἰς ὡρισμένα ἀπὸ τὰ ὑπεισερχόμενα εἰς τὰς ἐξισώσεις ὁλοκληρώματα, ἀποκλείνουν καὶ τείνουν εἰς τὸ ἄπειρον εἰς σημεῖα τὰ ὁποῖα ἀνήκουν εἰς τὰ χείλη τῶν ρωγμῶν, ὅπου καὶ ἡ ἐντατικὴ κατάστασις παρουσιάζει τὸ μεγαλύτερον ἐνδιαφέρον. Τὰ ὁλοκληρώματα αὐτά, εἰς τὰς συγκεκριμένας περιοχὰς ἀπειρισμοῦ των, λαμβάνουν τότε τὴν κατὰ Cauchy κυρίαν τιμήν των (Cauchy-principal value) καὶ καλοῦνται ἰδιόμορφα ὁλοκληρώματα κατὰ Cauchy.

Μερικὰ ἀπὸ τὰ ἀνωτέρω συστήματα ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων δύνανται νὰ ἀναχθοῦν ἐπὶ τῇ βάσει αὐστηρᾶς θεωρίας εἰς ἰσοδύναμα συστήματα ὁλοκληρωτικῶν ἐξισώσεων τύπου Fredholm δευτέρου εἴδους. Δυστυχῶς, ἡ αναγωγὴ αὐτὴ δὲν εἶναι πάντοτε ἁπλῆ διαδικασία. Ἐπιπροσθέτως, ὡρισμέναι εἰδικαὶ κατηγορίαι ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων δὲν ἀνάγονται εἰς τὰς ἀντι-

στοίχους των, τύπου Fredholm. Κατὰ συνέπειαν, αἱ ἀριθμητικαὶ μέθοδοι δι' ἀπ' εὐθείας ἐπιλύσεως συστημάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ἀποκτοῦν ἰδιαίτερον ἐνδιαφέρον καὶ ἀποτελοῦν ἓν ἐκ τῶν κυρίων ἀντικειμένων μελέτης κατὰ τὰ τελευταῖα ἔτη τῆς περιοχῆς τῆς ἀριθμητικῆς ἀναλύσεως.

Ἡ μελέτη τῆς ἀριθμητικῆς ἐπιλύσεως συστημάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ἀπησχόλησε σειρὰν ἐπιστημόνων ἀπὸ τῶν ἀρχῶν τῆς δεκαετίας τοῦ 1960, τόσον εἰς τὰς Ἡνωμένας Πολιτείας (Williams (1961), Erdogan (1962)), ὅσον εἰς τὴν Εὐρώπην (Krenk, Piessens) καὶ τὴν Σοβιετικὴν Ἕνωσιν (Savruck, Datsychin). Οἱ ἐπιστήμονες αὐτοὶ κατέληξαν εἰς τὴν διατύπωσιν προσεγγιστικῆς μεθόδου, ἡ ὁποία, ἂν καὶ εἰς γενικὰς γραμμὰς ἔδιδε ἱκανοποιητικὰ ἀποτελέσματα κατὰ τὴν ἐπίλυσιν ἁπλῶν προβλημάτων ἀνωμαλιῶν, παρουσίαζε συγκεκριμένας ὑπολογιστικὰς δυσχερείας. Αἱ δυσχέρειαι αὐταὶ ὀφείλοντο κυρίως εἰς τὸν μεγάλον ἀριθμὸν τῶν ἀπαιτουμένων σημείων ὁλοκληρώσεως καὶ εἰς τὴν ἀδυναμίαν τοῦ ἠλεκτρονικοῦ ὑπολογιστοῦ νὰ σταθεροποιῇ τὰ ἀποτελέσματά του, πλησίον τῶν σημείων ἰδιομορφίας (πόλων) τῶν πρὸς ἐπίλυσιν ἐξισώσεων. Ἀπὸ τοῦ 1975 ὁ ὁμιλῶν καὶ οἱ συνεργάται του συνέβαλον, διὰ σειρᾶς ὅλης δημοσιεύσεων εἰς τὸν διεθνῆ ἐπιστημονικὸν τύπον, ἀφ' ἑνὸς μέν, εἰς τὴν διατύπωσιν εὐκάμπτων μεθόδων ἐπιλύσεως προβλημάτων φυσικῶν πεδίων μὲ ἰδιομορφίαν, ἡ ὁποία ἐλάμβανε ὑπ' ὄψιν της τὴν ἰδιαιτερότητα ἑνὸς ἑκάστου προβλήματος, καὶ κατ' αὐτὸν τὸν τρόπον, παρεῖχε εὔκολον καὶ κομψὴν μορφὴν ἐπιλύσεως ἑνὸς ἑκάστου προβλήματος, ἀφ' ἑτέρου δὲ εἰς τὴν εἰς βάθος μελέτην τοῦ τρόπου ἐπιλύσεως καὶ τῶν ἀναγκαίων διαδικασιῶν, ὥστε νὰ παρέχωνται εἰς τὸν ἑκάστοτε ἐρευνητὴν ὅλα τὰ μέσα, ἕτοιμα, διὰ τὴν ταχεῖαν καὶ ἀποδοτικὴν ἀριθμητικὴν ἐπίλυσιν τῶν προβλημάτων αὐτῶν.

Ἡ προταθεῖσα αὐτὴ μέθοδος ἀριθμητικῆς ἐπιλύσεως ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ἔχει ἤδη γίνει ἀποδεκτὴ διεθνῶς ὡς μία ἰσχυρά, πρακτικὴ καὶ εὐπροσάρμοστος ἀριθμητικὴ μέθοδος, ἐκτοπίσασα προοδευτικῶς ὅλας τὰς προηγουμένας τὰς ὁποίας ἐβελτίωσε, χωρὶς νὰ τὰς καταργῇ.

Ἡ μέθοδος αὐτὴ συνίσταται εἰς τὴν ἐφαρμογὴν τῆς προτεινομένης ἰδιομόρφου ὁλοκληρωτικῆς ἐξισώσεως εἰς σύστημα σημείων ταξιθεσίας (collocation points), τὰ ὁποῖα ὁρίζονται ὡς ρίζαι δεδομένης ὑπερβατικῆς ἐξισώσεως, τῆς ὁποίας ἡ μορφὴ ἐξαρτᾶται ἀπὸ τὸ συγκεκριμένον συνοριακὸν πρόβλημα. Κατ' αὐτὸν τὸν τρόπον, διὰ τῆς διαδικασίας αὐτῆς, ἁπλουστεύεται κατὰ πολὺ ἡ μορφὴ τοῦ πρὸς ἐπίλυσιν συστήματος γραμμικῶν ἐξισώσεων, μὲ ἀποτέλεσμα ἡ ἀναμενομένη λύσις νὰ προκύπτει καὶ ταχύτερον, δηλαδὴ μὲ μικρότερον χρόνον ἀπασχολήσεως τῆς μνήμης τοῦ ἠλεκτρονικοῦ ὑπολογιστοῦ, καὶ ἀκριβέστερον.

Ἀπὸ θεωρητικῆς πλευρᾶς, ἡ δυσχέρεια τοῦ ἠλεκτρονικοῦ ὑπολογιστοῦ διὰ τὴν ἐπίλυσιν ἀριθμητικῶς ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ὀφείλεται εἰς δύο κυρίως λόγους :

α) Ὁ πρῶτος εἶναι τὸ ἀποτέλεσμα τοῦ ἀπειρισμοῦ τῶν πυρήνων τῶν ὑπαρχόντων ὁλοκληρωμάτων εἰς τὴν ἐξίσωσιν. Ἡ δυσκολία αὐτὴ ἀντεμετωπίσθη ἀπὸ τὸν Muskhelishvili τὸ ἔτος 1953 δι' εἰσαγωγῆς τῆς θεωρίας τῶν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων, ὅπου διατυπώνονται μέθοδοι καὶ διαδικασίαι παρακάμψεως τῶν δυσκολιῶν αὐτῶν. Ἡ ἐπιρροὴ τῶν πόλων κατὰ τὴν ἀριθμητικὴν ὁλοκλήρωσιν ἐξετιμήθη, κατέστη συγκεκριμένη καὶ ἐξεφράσθη ὡς ὑπερβατικὴ συνάρτησις διαφόρων ἀλγεβρικῶν πολυωνύμων.

β) Διὰ τῆς εἰσαγωγῆς τῶν ὑπερβατικῶν συναρτήσεων ἡ ἀκρίβεια τῶν ἀριθμητικῶν ὑπολογισμῶν διεταράχθη ἀπὸ τὴν ἰσχυρῶς ταλαντουμένην συμπεριφορὰν τῆς ὑπερβατικῆς συνιστώσης τῶν προαναφερθεισῶν ἀλγεβρικῶν ἐκφράσεων εἰς τὰς περιοχὰς πλησίον τῶν πόλων (δηλαδὴ κυρίως εἰς τὰ χείλη τῶν ρωγμῶν, ὅπου καὶ τὰ ἀποτελέσματα παρουσιάζουν ἐνδιαφέρον).

Διὰ τῆς εἰσαγωγῆς τῆς ἰδέας διὰ τὰ σημεῖα ταξιθεσίας, νὰ ἀποτελοῦν τὰς ρίζας τῆς ὑπερβατικῆς αὐτῆς ἐκφράσεως καὶ κατὰ συνέπειαν νὰ μηδενίζουν τὴν ἐπιρροήν της, παρεκάμφθη ἡ δυσκολία αὐτὴ καὶ ηὐξήθη ἡ ἀκρίβεια τῶν ἀποτελεσμάτων τὰ ὁποῖα προκύπτουν ἀπὸ τὸν ἠλεκτρονικὸν ὑπολογιστήν.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν πινακοποιοῦνται συστηματικῶς τὰ σημεῖα ταξιθεσίας κατὰ περίπτωσιν, συμφώνως μὲ τὸν κανόνα ἀριθμητικῆς ὁλοκληρώσεως ποὺ ἐφαρμόζεται εἰς ἕκαστον πρόβλημα, καὶ δίδονται ὁδηγίαι διὰ πρακτικοὺς τρόπους ἐπιλογῆς των, ἀναλόγως πρὸς τὸ ἐπίλυσιν πρόβλημα.

Τοιουτοτρόπως, εἰς τοὺς κυρίους τύπους κανόνων ὁλοκληρώσεως κατὰ Gauss, Radau καὶ Lobatto κατανέμονται αἱ ἀκόλουθοι κατηγορίαι ὀρθογωνίων πολυωνύμων : Legendre, Chebyshev, Jacobi, Laguerre καὶ Hermite.

Δι' ἕκαστον ἐπιλεγέντα συνδυασμὸν κανόνος ὁλοκληρώσεως καὶ ὀρθογωνίου πολυωνύμου εἶναι φανερὸν ὅτι προκύπτει ἀφ' ἑνὸς μὲν συγκεκριμένος τύπος ἀριθμητικῆς ὁλοκληρώσεως, ὁ ὁποῖος εἶναι κατάλληλος διὰ τὴν ἐπίλυσιν συγκεκριμένης γεωμετρίας συνοριακῶν προβλημάτων, καὶ ἔχει κατάλληλον συνάρτησιν βάρους διὰ τὴν ἀντιμετώπισιν συγκεκριμένων ἰδιομορφιῶν φορτίσεως, ἀφ' ἑτέρου δέ, ἰδιαιτέραν κατηγορίαν σημείων ταξιθεσίας τὰ ὁποῖα καὶ πινακοποιοῦνται.

Τὰ ὀρθογώνια πολυώνυμα τύπων Legendre, Chebyshev καὶ Jacobi χρησιμοποιοῦνται εἰς προβλήματα ρωγμῶν πεπερασμένου μήκους, ἐνῶ τὰ πολυώνυμα Laguerre καὶ Hermite εἰς τὴν περίπτωσιν ἡμιαπείρων ρωγμῶν.

Ἡ νέα ἄμεσος μέθοδος ἀριθμητικῆς ἐπιλύσεως ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων συμβάλλει τόσον ἀπὸ θεωρητικῆς, ὅσον καὶ ἀπὸ καθαρῶς ἀριθμητικῆς πλευρᾶς εἰς τὴν προώθησιν τῆς ἐρεύνης εἰς τὸν τομέα τῆς προσεγγιστικῆς ἐπιλύσεως προβλημάτων ρωγμῶν τῆς διδιαστάτου ἐλαστικότητος. Εἰς τὸν θεωρητικὸν τομέα ἡ συμβολή της συνίσταται εἰς τὴν ταξινόμησιν τῶν ἤδη ὑπαρχουσῶν μεθόδων διαμορφώσεως τῶν ὁλοκληρωτικῶν ἐξισώσεων ἀναλόγως τοῦ τιθεμένου προβλήματος, καθὼς ἐπίσης καὶ ἡ ἀναπαραγωγὴ τῶν ἐξισώσεων ποὺ ἀφοροῦν τὰς ρωγμὰς ἀπὸ τὰς ἀντιστοίχους ποὺ ἀφοροῦν ἐγκλείσματα.

Εἰς τὸν καθαρῶς ἀριθμητικὸν τομέα ἡ συμβολὴ τῆς μεθόδου ὑπῆρξε σημαντική, πρᾶγμα ποὺ πιστοποιεῖται καὶ ἀπὸ τὴ διεθνῆ ἀποδοχὴν καὶ χρησιμοποίησιν τῆς μεθόδου κατὰ τὰ τελευταῖα ἔτη. Ἡ γενικὴ ἀποδοχὴ τῆς μεθόδου ὀφείλεται κυρίως εἰς τὸ γεγονὸς ὅτι, διὰ τῆς χρησιμοποιήσεως καταλλήλων σημείων ταξιθεσίας, ἁπλουστεύονται κατὰ πολὺ οἱ ἀριθμητικοὶ ὑπολογισμοὶ καὶ προκύπτουν ἀκριβῆ ἀριθμητικὰ ἀποτελέσματα ἀκόμη καὶ εἰς περιπτώσεις πολυπλόκων συνοριακῶν προβλημάτων. Ὁ τρόπος ἐκλογῆς τῶν σημείων αὐτῶν ταξιθεσίας, ἀναλόγως τῆς γεωμετρίας τοῦ τιθεμένου προβλήματος καὶ τοῦ πολυπλόκου τῆς ἐφαρμοζομένης φορτίσεως, ἐξηγεῖται πλήρως εἰς τὴν παροῦσαν ἀνακοίνωσιν. Κατὰ συνέπειαν, ὁ ἐρευνητὴς ἀπαλλάσσεται ἀπὸ τὸν κόπον νὰ ἀνατρέξῃ εἰς ἀλγορίθμους εὑρέσεως τῶν ριζῶν τῶν ὑπερβατικῶν ἐξισώσεων κάθε φορὰ ποὺ ἔχει πρὸς ἐπίλυσιν συγκεκριμένον πρόβλημα φυσικῆς, ἐνῶ παραλλήλως ἔχει πρὸ ὀφθαλμῶν τὰ κριτήρια διὰ τῶν ὁποίων ἐπιλύεται, ἀναλόγως τοῦ τιθεμένου συνοριακοῦ προβλήματος, ἡ πλέον κατάλληλος μέθοδος ἀριθμητικῆς ὁλοκληρώσεως.

Τέλος, ἀναφέρεται ἐπὶ πλέον ὅτι ἡ ἔρευνα τῶν τελευταίων ἐτῶν εἰς τὸ Ἐργαστήριον Ἀντοχῆς Ὑλικῶν τοῦ Ε.Μ.Π. ὡδήγησε εἰς τὴν ἀπ' εὐθείας ἐπέκτασιν καὶ γενίκευσιν τῶν ἀνωτέρω μεθόδων καὶ εἰς τὴν τρισδιάστατον θεωρίαν ἐλαστικότητος, διὰ τὴν ἀντιμετώπισιν τοῦ δυσχεροῦς προβλήματος τῆς τρισδιαστάτου ρωγμῆς εὑρισκομένης ἐντὸς ἐλαστικῶς παραμορφωμένου χωρίου.

Γενίκευσις τῆς παρούσης ἐργασίας θὰ ἀποτελέσῃ μονογραφίαν ὑπὸ δημοσίευσιν εἰς ξένον ἐκδοτικὸν οἶκον, ἡ ὁποία ἐμπεριέχει τὸ θεωρητικὸν ὑπόβαθρον, τὴν κατασκευὴν τῶν ἀλγορίθμων, καθὼς ἐπίσης καὶ τὰ προγράμματα μὲ τὰ ὁποῖα ἀναπαράγονται οἱ πλήρεις πίνακες τῶν σημείων ταξιθεσίας, τμήματα τῶν ὁποίων περιλαμβάνονται εἰς τὴν παροῦσαν πρόδρομον ἀνακοίνωσιν.

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TABLE I

Collocation Points for the Gauss - Legendre Numerical Integration Rule $(w(x) = 1,\ n = 2(1)14)$.

	N = 3	
0.93761	21436	53014
0.00000	00000	00000
	N = 4	
0.96780	21190	01004
0.42970	74773	72613
	N = 5	
0.98042	91140	58448
0.63900	33629	48181
0.00000	00000	00000
	N = 6	
0.98686	69984	07231
0.75275	58283	49230
0.28059	49441	13258
	N = 7	
0.99058	39159	31232
0.82064	31284	00257
0.46337	42424	95580
0.00000	00000	00000
	N = 8	
0.99292	12537	10609
0.86417	21023	21523
0.58650	08788	18819
0.20745	53246	99112
	N = 9	
0.99448	56229	51440
0.89367	01684	35670
0.67254	79446	90800
0.36062	31751	75360
0.00000	00000	00000
	N = 10	
0.99558	36211	14547
0.91454	95165	94484
0.73471	95809	67262
0.47529	42697	12834
0.16436	83590	13911
	N = 11	
0.99638	36912	49525
0.92985	48461	72357
0.78095	72438	68050
0.56268	67666	34290
0.29442	31929	26610
0.00000	00000	00000
	N = 12	
0.99698	45893	48534
0.94140	07597	33994
0.81620	79610	90278
0.63049	61576	06385
0.39802	67583	20281
0.13603	86168	62008
	N = 13	
0.99744	73127	49755
0.95032	14395	65408
0.84366	11861	39043
0.68400	35619	80290
0.48137	03639	41131
0.24839	18471	65121
0.00000	00000	00000
	N = 14	
0.99781	11829	90069
0.95735	45464	10232
0.86543	89260	25803
0.72687	83295	94467
0.54913	39955	90481
0.34178	65335	45709
0.11601	35899	34070
	N = 15	
0.99810	24706	56799
0.96299	62805	73431
0.88299	32889	30273
0.76171	20602	95879
0.60481	61440	82986
0.41964	04176	09980
0.21484	30645	83170
0.00000	00000	00000

TABLE II

Collocation Points for the Lobatto - Lagendre Numerical Integration Rule $\left(w(x) = 1,\ n = 3(1)15\right)$.

	N = 4	
0.62317	47097	82724
	N = 5	
0.80540	45239	69493
0.00000	00000	00000
	N = 6	
0.88169	88950	45163
0.34837	00516	19560
	N = 7	
0.92060	89339	31333
0.54733	94959	60627
0.00000	00000	00000
	N = 8	
0.94307	63694	74857
0.66899	39708	96066
0.24144	26610	09340
	N = 9	
0.95720	55975	69440
0.74808	07780	19740
0.40944	93384	58940
0.00000	00000	00000
	N = 10	
0.96666	25550	46562
0.80214	02845	23118
0.52916	36689	95431
0.18470	40223	21681
	N = 11	
0.97330	03429	80677
0.84062	70078	26961
0.61676	26809	79460
0.32604	92142	82648
0.00000	00000	00000
	N = 12	
0.97813	71222	97429
0.86895	44995	25128
0.68248	33444	27771
0.43535	96722	92250
0.14954	94914	85576
	N = 13	
0.98176	98563	38032
0.89038	78341	20706
0.73290	50215	90650
0.52105	69855	83719
0.27056	20592	56464
0.00000	00000	00000
	N = 14	
0.98456	72757	51948
0.90698	51131	06143
0.77235	91115	38042
0.58919	53702	76787
0.36900	82037	78448
0.12563	41511	25230
	N = 15	
0.98676	70858	55028
0.92009	32818	10647
0.80377	03252	97449
0.64410	4475	05833
0.44972	26512	92276
0.23109	05217	24748
0.00000	00000	00000
	N = 16	
0.98852	80994	86472
0.93062	26953	19608
0.82916	20460	06358
0.68892	51366	95172
0.51647	32889	75503
0.31987	11325	75480
0.10831	19149	62625

TABLE III

Collocation Points for the Radau - Legendre Numerical Integration Rule $(w(x) = 1, \; n = 2(1)11)$.

N = 3		
0.89775	51168	55309
— 0.45550	48176	33487
N = 4		
0.95515	22921	66484
0.23487	55795	28816
— 0.74364	64280	74884
N = 5		
0.97489	24031	91546
0.54444	87458	48756
— 0.21215	02191	96384
— 0.85290	91946	01549
N = 6		
— 0.98396	67214	98314
0.70076	34685	36436
0.14724	83882	18905
— 0.46619	17362	63469
— 0.90499	80605	41710
N = 7		
0.98887	92513	45741
0.78925	52944	84119
0.37709	01931	14811
— 0.13790	68323	22420
— 0.61776	41633	56531
— 0.93369	92714	73874
N = 8		
0.99183	56348	66521
0.84385	31692	50343
0.52830	56093	04982
0.10715	64229	00751
— 0.33621	01079	82790
— 0.71396	77595	87774
— 0.95114	40073	76222
N = 9		
0.99375	21338	25422
0.87979	37595	96243
0.63174	14699	58357
0.28695	14724	98191
— 0.10210	95294	45902
— 0.47621	20348	23708
— 0.77838	98371	31105
— 0.96252	23523	74720
N = 10		
0.99506	50278	39839
0.90466	54768	49131
0.70512	45859	50253
0.42018	91189	89038
0.84206	90573	71125
— 0.26230	05692	51154
— 0.57753	78745	51593
— 0.82347	23274	43095
— 0.97034	88424	52733
N = 11		
0.99600	36079	40324
0.92257	18437	51297
0.75886	75526	63106
0.52065	61220	71895
0.23123	94794	33118
— 0.81055	01790	61701
— 0.38565	79993	91412
— 0.65275	10310	14546
— 0.85618	06730	92247
— 0.97595	98423	97506
N = 12		
0.99669	77688	68087
0.93588	28752	11709
0.79931	16376	28101
0.59783	45520	28219
0.34776	10276	14484
0.69348	15339	96432
— 0.21484	92164	44621
— 0.48180	67953	43563
— 0.70989	56683	91063
— 0.88063	04071	20300
— 0.98011	82020	67793

TABLE IV

Collocation Points for the Gauss Numerical Integration Rule with a logarithmic Singularity $(w(x) = \ln 1/x,\ n = 2(1)16)$.

N = 2			
0.96382	80325	19246	(—2)
0.32761	17872	10051	
0.85363	83149	71065	
N = 3			
0.57380	82969	83158	(—2)
0.19011	93754	14739	
0.57255	16718	87686	
0.91862	34371	29737	
N = 4			
0.38363	26889	40582	(—2)
0.12377	19864	12760	
0.39448	00207	60224	
0.71328	75345	04982	
0.94847	10432	33792	
N = 5			
0.27580	23964	04911	(—2)
0.86962	41133	86361	(—1)
0.28475	19556	10073	
0.54580	01533	55082	
0.79658	38762	13621	
0.96451	83602	61045	
N = 6			
0.20844	26070	94728	(—2)
0.64464	28304	96628	(—1)
0.21414	60286	94201	
0.42410	75094	36936	
0.65160	44426	88373	
0.84892	82306	31715	
0.97410	57929	00854	
N = 7			
0.16340	71167	74841	(—2)
0.49715	94969	11838	(—1)
0.16651	75968	65514	
0.33645	80024	70133	
0.53343	80526	29037	
0.72618	71703	58030	
0.88366	69772	86455	
0.98028	05771	42813	
N = 8			
0.13174	02206	98576	(—2)
0.39524	19177	18783	(—1)
0.13303	18039	81994	
0.27233	85657	58531	
0.44076	95323	93504	
0.61740	83581	59500	
0.77997	48923	16254	
0.90779	26658	49610	
0.98448	68978	97472	
N = 9			
0.10858	81915	41415	(—2)
0.32185	79192	95447	(—1)
0.10865	51909	43705	
0.22443	54192	63157	
0.36844	38408	44550	
0.52632	44139	81235	
0.68206	87122	87614	
0.81974	28633	64294	
0.92518	48860	20052	
0.98747	95327	23548	

Table IV (continued)

N = 10			N = 11		
0.91126	95414	45828 (—3)	0.77619	02236	20516 (—3)
0.26725	24689	43416 (—1)	0.22551	22276	37129 (—1)
0.90384	72624	02508 (—1)	0.76350	25712	52554 (—1)
0.18788	49447	46058	0.15944	74030	10408
0.31160	65181	40744	0.26645	01142	11372
0.45143	94905	43335	0.39006	66419	16961
0.59574	51828	54787	0.52170	41390	97259
0.73239	60163	38804	0.65211	68206	00202
0.84984	14382	60409	0.77208	81944	69823
0.93811	59405	88267	0.87310	58505	37353
0.98968	37146	55939	0.94798	13736	14597
			0.99135	36905	41407

TABLE V

Collocation Points, Abscissas and Weights for the Gauss - Jacobi Numerical Integration Rule $w(x) = (1-x)^A (1+x)^B$, A, B = — 0.9 (0.2) — 0.1, n = 4 (1) 8 (2) 10.

X (K) A = — 0.9	T (I)	N (I) B = — 0.9
	N = 4	
0.78213 19357 90350	0.98361 67172 10890	0.46891 39288 16123 (+1)
0.00000 00000 00000	0.43167 61604 13596	0.97240 41994 46645
— 0.78213 19357 90350	— 0.43167 61604 13596	0.97240 41994 46645
	— 0.98361 67172 10890	0.46891 39288 16123 (+1)
	N = 5	
0.86536 40738 98259	0.99001 14602 32392	0.44545 42081 12288 (+1)
0.33929 12734 56154	0.63934 82599 47124	0.86101 81933 53403
— 0.33929 12734 56154	0.00000 00000 00000	0.69196 64262 63185
— 0.86536 40738 98259	— 0.63934 82599 47124	0.86101 81933 53403
	— 0.99001 14602 32392	0.44545 42081 12288 (+1)
	N = 6	
— 0.90873 91446 46430	0.99327 60731 70182	0.42776 88246 58800 (+1)
0.53675 59713 19402	0.75221 59303 85072	0.79874 58494 25756
0.00000 00000 00000	0.27930 02312 05132	0.58510 93915 94116
— 0.53675 59713 19402	— 0.27930 02312 05132	0.58510 93915 94116
— 0.90873 91446 46430	— 0.75221 59303 85072	0.79874 58494 25756
	— 0.99327 60731 70182	0.42776 88246 58800 (+1)

	N = 7	
0.93412 47087 78025	0.99516 58469 90492	0.41366 10218 90333 (+1)
0.65904 31477 84437	0.81974 04595 69053	0.75738 95628 60562
0.23727 13641 18955	0.46116 21362 70113	0.52801 68659 83647
— 0.23727 13641 18955	0.00000 00000 00000	0.47905 36797 20667
— 0.65904 31477 84437	— 0.46116 21362 70113	0.52801 68659 83647
— 0.93412 47087 78025	— 0.81974 04595 69053	0.75738 95628 60562
	— 0.99516 58469 90492	0.41366 10218 90333 (+1)
	N = 8	
0.95023 67437 05471	0.99635 74279 38683	0.40198 13913 13108 (+1)
0.73926 54285 11643	0.86315 85213 75426	0.72701 98317 79291
0.40363 83817 00698	0.58388 29844 82266	0.49193 94977 02281
0.00000 00000 00000	0.20620 86914 08977	0.42274 02449 95222
— 0.40363 83817 00698	— 0.20620 86914 08977	0.42274 02449 95222
— 0.73926 54285 11643	— 0.58388 29844 82266	0.49193 94977 02281
— 0.95023 67437 05471	— 0.86315 85213 75426	0.72704 98317 79291
	— 0.99635 74279 38683	0.40198 13913 13108 (+1)
	N = 10	
0.96875 19236 01204	0.99771 93100 73687	0.38344 54035 80020 (+1)
0.83397 61737 99210	0.91358 91523 64639	0.68397 33526 46738
0.61060 59601 37845	0.73207 17126 27617	0.44775 61745 88683
0.32241 49016 70851	0.47282 64430 82099	0.36351 79478 47948
0.00000 00000 00000	0.16338 99355 77397	0.33184 19767 24311
— 0.32241 49016 70851	0.16338 99355 77397	0.33184 19767 24311
— 0.61060 59601 37845	0.47282 64430 82099	0.36351 79478 47948
— 0.83397 61737 99210	0.73207 17126 27617	0.44775 61745 88683
— 0.96875 19236 01204	0.91358 91523 64639	0.68397 33526 46738
	0.99771 93100 73687	0.38344 54035 80020 (+1)

Table V (continued)

X (K) A = — 0.7	T (I)	N (I) B = — 0.9
	N = 4	
0.72641 12398 89691	0.94971 25852 29672	0.12584 13105 71564 (+1)
— 0.48907 75009 83224 (—1)	0.37238 60596 27058	0.83278 14631 44890
— 0.79470 80265 34269	— 0.46250 74508 71285	0.10302 25926 04337 (+1)
	— 0.98459 11939 85445	0.53436 17639 40964 (+1)
	N = 5	
0.82810 69008 83827	0.96887 42192 98772	0.10836 64227 80914 (+1)
0.29531 32182 05809	0.59486 66656 39782	0.67132 46282 75335
— 0.36625 13821 80037	— 0.37340 22552 04251 (—1)	0.67986 63482 56819
— 0.87142 47796 86413	— 0.65497 86299 69479	0.94278 97297 91524
	— 0.99046 96484 87269	0.50873 93200 18072 (+1)
	N = 6	
0.88229 50012 94642	0.97885 14651 07906	0.96225 93173 17942
0.50059 39700 58036	0.71868 29646 97521	0.57696 12830 74289
— 0.30186 85278 27807 (—1)	0.24452 51413 79405	0.53724 34910 91014
— 0.55274 11377 57197	— 0.30280 33365 16208	0.60843 09304 04658
— 0.91210 16547 54854	— 0.76111 38075 60020	0.88856 21841 82243
	— 0.99352 70424 93220	0.48915 80928 24339 (+1)

N = 7

0.91445	32235	70902	0.98469	92619	79212	0.87175	17755	08478
0.62971	33825	07125	0.79387	01310	48832	0.51303	61459	95386
0.20858	70200	78011	0.43099	20532	36875	0.45681	83591	50015
— 0.25795	64186	42436	— 0.25329	49550	72906 (—1)	0.47348	06889	98689
— 0.66918	07015	80444	— 0.47656	25511	78795	0.56573	52431	35022
— 0.93617	77120	19322	— 0.82525	46936	28497	0.85009	52274	17050
			— 0.99531	79317	56788	0.47341	20694	10890 (+1)

N = 8

0.93505	86946	63835	0.98841	82675	02286	0.80103	07502	60733
0.71532	69134	04993	0.84271	00810	26993	0.46594	10989	64938
0.37788	78876	56530	0.55821	23303	40660	0.40383	09845	95362
— 0.21816	85942	28829 (—1)	0.18183	41972	40143	0.39853	48275	97479
— 0.41815	92513	34043	— 0.22461	85752	40765	0.43543	47106	57219
— 0.74605	80992	27810	— 0.59441	04035	05167	0.53661	28522	45293
— 0.95157	99976	20658	— 0.86680	05477	14817	0.82057	59128	23291
			— 0.99645	64607	60444	0.46030	76997	16922 (+1)

N = 10

0.95895	29129	59500	0.99270	39457	47853	0.69648	64283	58713
0.81743	79554	87289	0.89999	06645	66444	0.39977	92401	18686
0.59080	33148	43733	0.71346	74950	57680	0.33629	25864	00684
0.30264	21704	50607	0.45265	64017	96813	0.31536	01311	36913
— 0.17077	83365	46740 (—1)	0.14468	62983	39263	0.31722	25610	19555
— 0.33510	99378	76145	— 0.17841	72201	88068	0.33984	06575	27756
— 0.61838	22436	43757	— 0.48305	55462	21711	0.31101	60900	08412
— 0.83743	14939	41425	— 0.73754	78447	48806	0.49813	07868	04465
— 0.96941	69258	37968	— 0.91541	17779	37370	0.77683	07073	17646
			— 0.99776	80685	86013	0.43940	78945	62071 (+1)

Table V (continued)

X (K) A = —0.5	T (I)	N (I) B = —0.9
	N = 4	
0.67415 21136 36083	0.91495 59511 74794	0.64075 43926 69922
— 0.93026 74759 50072 (—1)	0.31816 13208 64163	0.74564 08463 88151
— 0.80589 97381 75955	— 0.49008 52660 46024	0.11020 01474 70647 (+1)
	— 0.98545 62484 17176	0.60928 98542 42177 (+1)
	N = 5	
0.79209 72623 61151	0.94660 74579 04028	0.50297 83815 84596
0.25432 07228 88282	0.55287 03094 44660	0.55099 01757 76998
— 0.39103 25169 29539	— 0.71857 98485 90645 (—1)	0.67867 14515 81389
— 0.87696 05789 62066	— 0.66929 02786 36680	0.10367 09741 65832 (+1)
	— 0.99088 76433 87828	0.58119 45505 58501 (+1)
	N = 6	
0.85626 12801 46188	0.96339 82329 68705	0.41441 68856 11098
0.46619 68712 70208	0.68639 96848 18022	0.44056 09108 86530
— 0.58517 12210 50879 (—1)	0.21170 65660 61302	0.50331 94622 97537
— 0.56763 77853 76424	— 0.32477 13050 41571	0.63763 19306 83811
— 0.91522 43305 66774	— 0.76938 83224 57159	0.99071 66468 90423
	— 0.99375 99506 88921	0.55946 49419 81692 (+1)

N = 7

0.89484 58010 54252	0.97335 76078 65607	0.35255 27697 52008
0.60143 16278 67319	0.76863 55026 41458	0.36840 67533 41767
0.18128 01415 44276	0.40211 69233 09248	0.40444 68025 71172
— 0.27750 95514 43395	— 0.49346 34886 98564 (—1)	0.47293 76927 05087
— 0.67872 40949 18213	— 0.49108 60669 27966	0.60886 04945 40721
— 0.93810 62983 85833	— 0.83043 91013 38792	0.95544 37754 16782
	— 0.99546 07389 01921	0.54186 46967 85878 (+1)

N = 8

0.91979 52698 24058	0.97974 41367 25131	0.30684 28164 89014
0.69201 72586 83049	0.82257 64774 20429	0.31719 66091 91206
0.35312 02722 11144	0.53339 47155 93356	0.33997 25559 15631
— 0.42657 06794 34904 (—1)	0.15848 05569 52370	0.38050 34442 00715
— 0.43197 08085 14852	— 0.24216 24016 87408	0.45136 92249 72088
— 0.75250 19513 28684	— 0.60440 86570 46611	0.58699 31309 17407
— 0.95285 25031 50299	— 0.87025 26590 44366	0.92694 94994 34816
	— 0.99655 02511 04681	0.52714 67975 06544 (+1)

N = 10

0.94898 14739 84474	0.98716 30960 87191	0.24375 23380 15988
0.80112 41169 80070	0.88643 39809 19640	0.24888 71620 09494
0.57149 29609 68224	0.69523 89327 03914	0.25977 92472 16751
0.28348 49253 00921	0.43305 86277 64797	0.27788 12125 17069
— 0.33554 97876 89565 (—1)	0.12660 64331 34835	0.30603 78664 35499
— 0.34732 33268 83369	— 0.19289 32419 84231	0.34989 01797 95298
— 0.62584 87551 29047	— 0.49288 59112 65044	0.42181 11043 25628
— 0.84074 49283 24344	— 0.74280 20588 36065	0.55491 66706 51148
— 0.97005 41796 16655	— 0.91715 88350 49337	0.88267 96111 69919
	— 0.99781 47859 16560	0.50356 59864 04952 (+1)

Table V (continued)

X (K) A = — 0.3			T (I)			N (I) B = — 0.9		
			N = 4					
0.62492	21409	49030	0.87990	73775	26273	0.40452	98918	05573
— 0.13306	84247	92933	0.26829	20450	74982	0.69065	88661	36282
— 0.81592	54301	74598	— 0.51491	12670	72257	0.11881	23138	78215 (+1)
			— 0.98622	93319	99586	0.69506	73879	87279 (+1)
			N = 5					
0.75722	58181	53028	0.92358	58137	42759	0.29095	86333	28646
0.21598	30491	13499	0.51310	31235	92929	0.47068	78056	50856
— 0.41389	70948	37534	— 0.10388	17265	78180	0.68662	46725	86003
— 0.88203	71542	12147	— 0.68244	57899	85485	0.11441	80208	15771 (+1)
			— 0.99127	05116	71114	0.66415	34456	87358 (+1)
			N = 6					
0.83061	36036	32828	0.94716	91022	40041	0.22316	32930	41141
0.43341	27132	28747	0.65526	76169	14347	0.35149	31483	82943
— 0.85168	15625	21958 (—1)	0⁰18066	55525	19877	0.47974	05458	74979
— 0.58155	57609	66197	— 0.34535	55117	15899	0.67283	85377	13326
— 0.91813	20315	28733	— 0.77710	34135	39791	0.11068	48324	94036 (+1)
			— 0.99397	66797	51909	0.63999	01926	64405 (+1)

	N = 7	
0.87530 23829 16605	0.96131 82918 59358	0.17867 72800 07109
0.57412 59657 52040	0.74399 76351 70929	0.27714 33797 18069
0.15524 34742 89118	0.37444 10792 97279	0.36530 47582 81564
— 0.29602 51448 18371	— 0.72156 82972 00641 (—1)	0.47685 74819 05754
— 0.68772 48474 13364	— 0.50480 74572 36885	0.65791 99309 80785
— 0.93992 14720 76348	— 0.83532 26271 14127	0.10751 74553 76065 (+1)
	— 0.99559 50922 56491	0.62029 08391 94287 (+1)
	N = 8	
0.90445 29990 87326	0.97046 45441 29039	0.14753 97659 42170
0.66930 16594 03258	0.80274 29422 22319	0.22664 87454 65630
0.32927 19406 00532	0.50937 83382 89262	0.29257 64240 75825
— 0.62589 20670 02443 (—1)	0.13607 85770 23445	0.36744 11807 42383
— 0.44512 53629 10968	— 0.25890 26809 09240	0.47060 11270 01641
— 0.75862 34776 02081	— 0.61391 74274 86464	0.64376 39663 05810
— 0.95405 97164 24988	— 0.87352 94130 70206	0.10479 64237 18539 (+1)
	— 0.99663 92045 22480	0.60374 50329 87795 (+1)
	N = 10	
0.93884 45071 35495	0.98117 27293 82048	0.10732 09080 50625
0.78502 68486 21065	0.87291 90232 04652	0.16305 33278 91238
0.55265 21483 61470	0.67737 03071 22921	0.20570 01446 71407
0.26491 12503 62519	0.41400 48570 77756	0.24834 26077 58469
— 0.49464 75415 79521 (—1)	0.10911 67572 07241	0.29774 69850 12387
— 0.35904 29805 08887	— 0.20684 93245 83332	0.36202 14591 28037
— 0.63302 39766 64152	— 0.50234 10198 72175	0.45625 95182 55412
— 0.84392 51005 28892	— 0.74784 77340 49162	0.61895 29466 27483
— 0.97066 53882 31511	— 0.91883 49440 46401	0.10033 62466 66453 (+1)
	— 0.99785 95876 13762	0.57712 25452 53721 (+1)

Table V (continued)

X (K) A = —0.1	T (I)	N (I) B = —0.9
	N = 4	
0.57842 23679 14642	0.84494 75992 80885	0.28725 45307 66244
— 0.16960 08083 32198	0.22222 04969 55233	0.65705 99709 31763
— 0.82495 95232 66215	— 0.53738 65810 58362	0.12894 46881 01794 (+1)
	— 0.98692 43723 20613	0.79326 46001 91458 (+1)
	N = 5	
0.72343 79825 78727	0.90008 50953 78178	0.19020 98350 96451
0.18002 51659 59271	0.47536 17125 97567	0.41523 18451 00333
— 0.43506 56035 31009	— 0.13368 70527 96588	0.70267 19540 72426
— 0.88670 96202 78919	— 0.69548 16968 45499	0.12668 47139 86412 (+1)
	— 0.99162 25027 78102	0.75914 46610 49719 (+1)
	N = 6	
0.80535 91667 92133	0.93035 70605 38274	0.13622 50438 96728
0.40211 33365 95500	0.62521 19578 35826	0.29062 51603 24674
— 0.11029 35922 78042	0.15124 79156 77541	0.46414 83043 65640
— 0.59459 02985 20716	— 0.36468 72310 36701	0.71432 33952 27087
— 0.92084 62856 90698	— 0.78431 44804 05640	0.12388 34286 85429 (+1)
	— 0.99417 88589 72937	0.73222 51193 96210 (+1)

				N = 7				
0.85584	46420	89700	0.94871	95071	25131	0.10287	05633	84830
0.54773	62280	08362	0.71993	08854	83178	0.21659	93525	01572
0.13038	23531	32566	0.34788	10584	46881	0.33576	50204	68646
— 0.31358	68638	54596	— 0.93855	06391	47788 (—1)	0.48487	54695	10965
— 0.69622	86015	13437	— 0.51779	29955	42499	0.71353	43120	09663
— 0.94163	29738	27493	— 0.83993	09014	52483	0.12112	73960	29531 (+1)
			— 0.99572	17209	14660	0.71014	88706	45954 (+1)
				N = 8				
0.88905	47644	76471	0.96068	15552	29432	0.80715	64949	03630 (—1)
0.64715	20044	06231	0.78320	34785	63805	0.16855	37330	39112
0.30628	61742	79217	0.48611	79538	31080	0.25670	98035	93985
— 0.81675	01746	72416 (—1)	0.11456	61130	12191	0.35848	54344	21786
— 0.45767	00339	11592	— 0.27489	57247	59346	0.49325	18664	75814
0.76444	65247	90089	— 0.62297	24478	60676	0.70770	48587	85621
0.95520	65456	29669	— 0.87664	39088	97736	0.11856	68613	71647 (+1)
			— 0.99672	36857	85416	0.69153	16425	10737(+1)
				N = 10				
0.92855	94461	44125	0.97479	34041	67311	0.53879	41549	01163 (—1)
0.76914	14091	33598	0.85945	07329	76631	0.11145	14473	36219
0.53426	05676	40133	0.65984	83980	63846	0.16649	66000	44956
0.24689	18917	65338	0.39546	93739	39326	0.22482	08636	13460
0.64837	83224	14076 (—1)	0.92186	39367	41520 (—1)	0.29194	35490	49661
0.37041	46035	63589	— 0.22031	43150	91378	0.37629	90077	38406
0.63992	49210	66233	— 0.51144	24927	70192	0.49476	60687	81105
0.84697	99444	33189	— 0.75269	71825	62534	0.69110	24721	88792
0.97125	21182	11920	— 0.92044	43548	89546	0.11409	85296	02435 (+1)
			— 0.99790	25891	36563	0.66145	72663	46346 (+1)

TABLE VI

Collocation points abscissas and weights for the Lobatto - Jacobi numerical integration rule
$w(x) = (1-x)^A (1+x)^B$, A, B = −0.9 (0.2) −0.1, A ⩾ B, N = 4 (1) 8 (2) 10.

X (K) A = −0.9	T (I)	W (I) B = −0.9
	N = 4	
− 0.92258 37367 60431	0.10000 00000 00000 (+1)	0.42890 48096 67263 (+1)
0.00000 00000 00000	0.55901 69943 74947	0.13724 95390 93524 (+1)
− 0.92258 37367 60431	− 0.55901 69943 74947	0.13724 95390 93524 (+1)
	− 0.10000 00000 00000 (+1)	0.42890 48096 67263 (+1)
	N = 5	
0.05914 11472 72313	0.10000 00000 00000 (+1)	0.40209 82590 63059 (+1)
0.41891 27121 03424	0.75955 45253 12750	0.12080 81880 56279 (+1)
− 0.41891 27121 03424	0.00000 00000 00000	0.86495 80328 28981
− 0.95914 11472 72313	− 0.75955 45253 12750	0.12080 81880 56279 (+1)
	− 0.10000 00000 00000 (+1)	0.40209 82590 63059 (+1)
	N = 6	
0.97484 91363 83965	0.10000 00000 00000 (+1)	0.38295 07229 17199 (+1)
0.62766 40119 77570	0.84995 08284 07479	0.11205 92685 02688 (+1)
0.00000 00000 00000	0.33304 19232 21047	0.71144 35734 09003
− 0.62766 40119 77570	− 0.33304 19232 21047	0.71144 35734 09003
− 0.97484 91363 83965	− 0.84995 08284 07479	0.11205 92685 02688 (+1)
	− 0.10000 00000 00000 (+1)	0.38295 07229 17199 (+1)

	N = 7	
0.98298 56262 96197	0.10000 00000 00000 (+1)	0.36822 18489 58846 (+1)
0.74289 93648 91104	0.89777 68107 51827	0.10630 38664 01030 (+1)
0.27579 92651 50251	0.53005 02993 25836	0.63683 83541 72063
— 0.27579 92651 50251	0.00000 00000 00000	0.55889 59596 74111
— 0.74289 93648 91104	— 0.53005 02993 25836	0.63683 83541 72063
— 0.98298 56262 96197	— 0.89777 68107 51827	0.10630 38664 01030 (+1)
	— 0.10000 00000 00000 (+1)	0.36822 18489 58846 (+1)
	N = 8	
0.98773 24395 22079	0.10000 00000 00000 (+1)	0.35634 37247 98883 (+1)
0.81241 30217 13349	0.92599 45968 37978	0.10207 24640 06257 (+1)
0.45698 70034 64534	0.65309 91340 82000	0.59182 68416 59455
0.00000 00000 00000	0.23511 96333 22967	0.48555 47578 97028
— 0.45698 70034 64534	— 0.23511 96333 22967	0.48555 47578 97028
— 0.81241 30217 13349	— 0.65309 91340 82000	0.59182 68416 59455
— 0.98773 24395 22079	— 0.92599 45968 37978	0.10207 24640 06257 (+1)
	— 0.10000 00000 00000 (+1)	0.35634 37247 98883 (+1)
	N = 10	
0.99276 32880 83844	0.10000 00000 00000 (+1)	0.33799 54029 25770 (+1)
0.88794 75328 70409	0.95615 53204 58895	0.96020 55173 98230
0.66625 19659 45384	0.79023 86583 80434	0.53826 23666 60876
0.35665 70944 66964	0.52010 49613 44891	0.41351 62682 80648
0.00000 00000 00000	0.18134 48601 46527	0.36960 53060 10426
— 0.35665 70944 66964	— 0.18134 48601 46527	0.36960 53060 10426
— 0.66625 19659 45384	— 0.52010 49613 44891	0.41351 62682 80648
— 0.88794 75328 70409	— 0.79023 86583 80434	0.53826 23666 60876
— 0.99276 32880 83844	— 0.95615 53204 58895	0.96020 55173 98230
	— 0.10000 00000 00000 (+1)	0.33799 54029 25770 (+1)

Table VI (continued)

X (K) A = —0.7			T (I)			W (I) B = —0.9		
				N = 4				
0.88593	72907	82417	0.10000	00000	00000 (+1)	0.97319	62738	02769
— 0.54788	29597	31353 (—1)	0.49631	10551	23656	0.11192	55933	07034 (+1)
— 0.92789	26466	32908	— 0.58722	01460	32747	0.14811	07877	50276 (+1)
			— 0.10000	00000	00000 (+1)	0.48914	78049	93767 (+1)
				N = 5				
0.93842	29492	46836	0.10000	00000	00000 (+1)	0.80666	00130	45076
0.37167	57876	17060	0.71843	24191	73362	0.87090	16043	21665
— 0.44566	85667	77696	— 0.40721	22775	98835 (—1)	0.85040	46117	37172
— 0.96123	76073	13093	— 0.77146	11914	13479	0.13426	09600	23971 (+1)
			— 0.10000	00000	00000 (+1)	0.45944	62304	96992 (+1)
				N = 6				
0.96161	01320	32523	0.10000	00000	00000 (+1)	0.69921	90387	91715
0.59147	58370	58025	0.82180	28107	03832	0.73483	96134	30557
— 0.32377	39307	99515 (—1)	0.29586	19726	31129	0.64346	92588	03379
— 0.64195	01664	22773	— 0.35694	28765	78861	0.74629	75732	76095
— 0.97587	83395	30605	— 0.85596	00019	94194	0.12606	83179	64136 (+1)
			— 0.10000	00000	00000 (+1)	0.43805	29470	37043 (+1)

	N = 7	
0.97381 71571 46646	0.10000 00000 00000 (+1)	0.62299 39024 87377
0.71524 91510 82469	0.87751 21900 96720	0.64566 05626 97214
0.24535 28702 53561	0.49910 29360 41384	0.53633 48161 69764
— 0.29700 72827 26272	— 0.26862 58324 37562 (—1)	0.55256 28623 50729
— 0.75127 63789 43765	— 0.54470 30827 63374	0.68890 32053 87490
— 0.98356 42734 02356	— 0.90120 33062 84821	0.12034 76359 48335 (+1)
	— 0.10000 00000 00000 (+1)	0.42151 06425 73771 (+1)
	N = 8	
0.98101 49211 58081	0.10000 00000 00000 (+1)	0.56551 52983 88839
0.79088 63648 57994	0.91077 72955 86150	0.58133 16188 27864
0.43036 02853 53694	0.62778 00564 50749	0.46872 97269 60762
— 0.22949 25385 28881 (—1)	0.20942 20095 51675	0.45446 30966 32854
— 0.47121 33845 85791	— 0.25404 15565 77061	0.50264 02952 15752
— 0.81770 46877 29193	— 0.66258 71693 70239	0.65129 12856 36987
— 0.98808 92581 05146	— 0.92812 48291 90020	0.11598 41841 00895 (+1)
	— 0.10000 00000 00000 (+1)	0.40812 24971 64153 (+1)
	N = 10	
0.98871 83540 39894	0.10000 00000 00000 (+1)	0.48357 49485 96193
0.87404 58354 15039	0.94672 15859 16091	0.49280 98656 98875
0.64680 38308 48012	0.77298 23389 19198	0.38527 54431 33828
0.33618 09516 68080	0.49960 87174 67433	0.35212 60833 68700
— 0.17767 57880 44651 (—1)	0.16183 29521 17713	0.35183 69047 34821
— 0.36939 12006 12767	— 0.19678 21063 70969	0.37973 39307 45938
— 0.67334 03775 13171	— 0.52998 38847 24397	0.44769 27836 57487
— 0.89042 92370 88845	— 0.79480 26951 67053	0.60287 29761 71132
— 0.99272 64607 32865	— 0.95713 78837 67772	0.10954 22710 22938 (+1)
	— 0.10000 00000 00000 (+1)	0.38736 92487 97718 (+1)

Table VI (continued)

X (K) A = —0.5	T (I)	W (I) B = —0.9
	N = 4	
0.84786 95306 85164	0.10000 00000 00000 (+1)	0.42356 39325 16889
— 0.10379 89327 11212	0.43809 33603 69183	0.96526 64436 07812
— 0.93252 14830 09853	— 0.61200 64038 47443	0.16112 65313 88150 (+1)
	— 0.10000 00000 00000 (+1)	0.55811 99566 18012 (+1)
	N = 5	
0.91611 88137 70959	0.10000 00000 00000 (+1)	0.31263 05216 19608
0.32745 77698 06016	0.67860 35151 05762	0.67039 13071 11740
— 0.47003 24615 93685	— 0.78183 20912 69483 (—1)	0.85040 43938 14800
— 0.96312 96584 98600	— 0.78223 84877 96996	0.14967 40409 54597 (+1)
	— 0.10000 00000 00000 (+1)	0.52511 28624 09420 (+1)
	N = 6	
0.94706 72223 54541	0.10000 00000 00000 (+1)	0.24768 79494 95728
0.55671 65478 36151	0.79388 68808 26899	0.51732 74249 52389
— 0.62668 63746 72998 (—1)	0.26072 73846 47528	0.59535 70282 57893
— 0.65516 88462 19583	— 0.37915 59938 26313	0.78879 99870 79064
— 0.97682 66804 23892	— 0.86150 47832 76021	0.14204 41252 73764 (+1)
	— 0.10000 00000 00000 (+1)	0.50116 81613 66360 (+1)

				N = 7				
0.96361	46468	46760	0.10000	00000	00000 (+1)	0.20506	63218	22762
0.68825	30293	88472	0.85711	10184	46803	0.42244	66741	09406
0.21635	93245	77750	0.46935	00306	20605	0.46381	96762	33367
— 0.31697	93375	79236	— 0.52277	42494	61679 (—1)	0.55239	01000	18514
— 0.75912	04855	59247	— 0.55845	58809	85587	0.74825	59037	09739
— 0.98410	48756	41285	— 0.90440	69884	18672	0.13636	42757	61224 (+1)
			— 0.10000	00000	00000 (+1)	0.48256	73822	68029 (+1)
				N = 8				
0.97347	23826	44719	0.10000	00000	00000 (+1)	0.17495	16871	49489
0.76963	19742	29721	0.89529	87635	23753	0.35749	80546	89530
0.40469	04077	47603	0.60314	78192	63779	0.38220	98884	11404
— 0.44835	78478	52929 (—1)	0.18480	22729	56475	0.43123	23062	14104
— 0.48469	94034	71737	— 0.27202	54420	74006	0.52364	13786	62506
— 0.82270	42488	21603	— 0.67156	39692	10425	0.71848	33327	64202
— 0.98842	59138	56542	— 0.93013	56349	35727	0.13186	31835	20687 (+1)
			— 0.10000	00000	00000 (+1)	0.46746	46773	08821 (+1)
				N = 10				
0.98412	34854	58994	0.10000	00000	00000 (+1)	0.13522	50955	35114
0.86012	24277	23297	0.93699	84570	22432	0.27393	16091	47562
0.62775	68353	66187	0.75593	83202	81945	0.28488	68391	35484
0.31632	79384	97258	0.47965	12428	34735	0.30505	41275	08229
— 0.34893	36008	76006 (—1)	0.14297	68783	19250	0.33818	90428	14363
— 0.38161	90009	74850	— 0.21163	06538	62569	0.39224	06797	41129
— 0.68013	03258	21962	— 0.53945	71060	88020	0.48600	63199	78138
— 0.89280	29950	41433	— 0.79917	08982	82592	0.67597	79831	04330
— 0.99308	24388	88187	— 0.95807	73245	62530	0.12500	01977	96408 (+1)
			— 0.10000	00000	00000 (+1)	0.44397	81581	25789 (+1)

Table VI (continued)

X (K) A = —0.3	T (I)	W (I) B = —0.9
	N = 4	
0.80926 44035 77995	0.10000 00000 00000 (+1)	0.23051 4350γ 1373ĩ
— 0.14791 69673 15784	0.38395 56694 03722	0.86826 89082 29357
— 0.93659 28532 13504	— 0.63396 56694 03720	0.17643 51941 16793 (+1)
	— 0.10000 00000 00000 (+1)	0.63708 50576 38578 (+1)
	N = 5	
0.89273 48703 34081	0.10000 00000 00000 (+1)	0.15247 39304 81348
0.28597 63099 68428	0.64010 85848 49336	0.54285 90387 07807
— 0.49231 64865 43949	— 0.11277 38886 05443	0.86255 86606 89362
— 0.96484 57782 41704	— 0.79204 05785 96834	0.16730 00649 44476 (+1)
	— 0.10000 00000 00000 (+1)	0.60030 93497 27352 (+1)
	N = 6	
0.93152 57766 44102	0.10000 00000 00000 (+1)	0.11084 09778 32185
0.52331 62860 02455	0.76630 08256 97671	0.38502 27945 52754
— 0.91076 59669 84375 (—1)	0.22746 70776 48851	0.56160 18517 74718
— 0.66743 66801 41838	— 0.39985 80012 66626	0.83925 67835 27716
— 0.97770 33225 45341	— 0.86663 71748 07168	0.16026 00881 67658 (+1)
	— 0.10000 00000 00000 (+1)	0.57346 62487 23284 (+1)

				N = 7				
0.95257	57481	71134	0.10000	00000	00000 (+1)	0.85494	40213	91629 (—1)
0.66190	84453	24626	0.83666	18446	47939	0.29314	12707	31263
0.18871	15194	44970	0.44072	31835	33686	0.41037	12787	17818
— 0.33582	34361	84202	— 0.76363	46825	99616 (—1)	0.55767	09212	10800
— 0.76648	13621	36311	— 0.57139	04816	14832	0.81569	10188	68720
— 0.98461	10639	41204	— 0.90740	89560	84610	0.15463	33555	44506 (+1)
			— 0.10000	00000	00000 (+1)	0.55252	83329	48397 (+1)
				N = 8				
0.96524	05478	49365	0.10000	00000	00000 (+1)	0.68680	75325	49026 (—1)
0.74866	74219	78016	0.87963	00898	69654	0.23368	44562	24794
0.37992	40434	65199	0.57917	86180	03128	0.31964	44883	71509
— 0.65735	15598	68812 (—1)	0.16118	99671	52633	0.41423	68818	03314
— 0.49750	25893	29922	— 0.28914	16079	66807	0.54867	08581	09485
— 0.82743	53819	14106	— 0.68007	02663	60142	0.79437	88238	00017
— 0.98874	40682	50252	— 0.93203	68006	98467	0.14999	06731	87810 (+1)
			— 0.10000	00000	00000 (+1)	0.53547	82783	15467 (+1)
				N = 10				
0.97905	17429	59661	0.10000	00000	00000 (+1)	0.48103	77855	83037 (—1)
0.84619	86576	84520	0.92702	98059	35276	0.16231	95667	67626
0.60910	00645	44132	0.73911	01704	01606	0.21674	89670	95905
0.29706	79206	15968	0.46021	03367	95155	0.26842	03561	71878
— 0.51413	28569	46835 (—1)	0.12474	18757	89397	0.32798	04904	14562
— 0.39337	09503	40104	— 0.22592	48512	66975	0.40717	89501	46850
— 0.68664	05099	98614	— 0.54854	95995	38272	0.52892	82537	41683
— 0.89507	57249	56185	— 0.80335	57000	58611	0.75872	23798	88149
— 0.99323	16880	06572	— 0.95897	64237	71862	0.14267	38060	45417 (+1)
			— 0.10000	00000	00000 (+1)	0.50888	44973	35767 (+1)

Table VI (continued)

X (K) A = −0.1	T (I)	W (I) B = −0.9
	N = 4	
0.77069 50866 32802	0.10000 00000 00000 (+1)	0.14207 13917 63439
— 0.18785 12879 42328	0.33355 85071 70123	0.80702 47313 22645
— 0.94020 16019 72955	— 0.65355 85071 70123	0.19423 88574 62052 (+1)
	— 0.10000 00000 00000 (+1)	0.72749 22686 92392 (+1)
	N = 5	
0.86863 63695 25043	0.10000 00000 00000 (+1)	0.84696 40662 82041 (—1)
0.24698 49431 88274	0.60295 71144 76534	0.45769 31254 33204
— 0.51277 98098 90477	— 0.14481 95927 91201	0.88545 88228 42993
— 0.96640 93933 38236	— 0.80099 46645 42476	0.18743 17462 22533 (+1)
	— 0.10000 00000 00000 (+1)	0.68642 41567 50079 (+1)
	N = 6	
0.91521 60818 34127	0.10000 00000 00000 (+1)	0.56694 73749 80632 (—1)
0.49120 87398 05436	0.73911 69789 41872	0.29971 71144 37000
— 0.11777 73418 75109	0.19593 03066 30473	0.53865 90092 35008
— 0.67885 37200 93961	— 0.41920 11706 30897	0.89815 87926 75190
— 0.97851 60960 08183	— 0.87140 16704 97003	0.18102 93110 15035 (+1)
	— 0.10000 00000 00000 (+1)	0.65628 84620 63490 (+1)

				N = 7				
0.94085	28476	92624	0.10000	00000	00000 (+1)	0.40839	42955	36896 (—1)
0.63621	00244	92070	0.81623	64146	06155	0.21339	68942	04204
0.16231	37232	98538	0.41316	00457	63436	0.37036	97604	01324
— 0.35363	49517	77211	— 0.99226	21810	84351 (—1)	0.56795	73644	43489
— 0.77340	26504	80733	— 0.58357	88859	45283	0.89213	26935	11483
— 0.98508	60216	32591	— 0.91022	77199	52236	0.17547	47310	85850 (+1)
			— 0.10000	00000	00000 (+1)	0.63269	63931	65783 (+1)
				N = 8				
0.95642	53062	94898	0.10000	00000	00000 (+1)	0.30946	64848	16778 (—1)
0.72800	68251	44303	0.86383	07372	23107	0.16058	72771	21863
0.35601	25289	98959	0.55585	00214	61376	0.27331	72926	76481
— 0.85715	47390	88598 (—1)	0.13852	13128	77907	0.40232	29128	91205
— 0.50967	45273	47895	— 0.30545	32455	07834	0.57791	86685	79069
— 0.83191	92578	46593	— 0.68814	25252	38306	0.88009	02749	56232
— 0.98904	52051	52118	— 0.93383	70704	47020	0.17068	84572	90680 (+1)
			— 0.10000	00000	00000 (+1)	0.61343	39737	01720 (+1)
				N = 10				
0.97356	06703	46165	0.10000	00000	00000 (+1)	0.19687	71990	59230 (—1)
0.83229	35816	60552	0.91685	36513	24942	0.10139	88465	89402
0.59082	32774	23121	0.72250	07243	07451	0.16912	37764	72565
0.27837	30227	49082	0.44126	53904	19949	0.23955	11966	77665
— 0.67360	46848	39326 (—1)	0.10709	62529	54415	0.32070	17653	35378
— 0.40467	50304	88923	— 0.23969	62970	08359	0.42464	12666	35711
— 0.69288	80788	09679	— 0.55728	42876	99171	0.57699	33979	11850
— 0.89725	37641	78857	— 0.80736	84945	46018	0.85240	94572	01312
— 0.99337	46342	49457	— 0.95983	77279	88504	0.16288	31335	30802 (+1)
			— 0.10000	00000	00000 (+1)	0.58330	68622	59270 (+1)

TABLE VII

Collocation points abscissas and weights for the Gauss - Laguerre numerical integration rule $w(x) = x^{\alpha} \exp(-x)$ $\alpha = -0.9\ (02) -0.7$ $n = 4\ (1)\ 8, 10.$

X (K)	T (I) A = −0.9	W (I)
	N = 4	
0.35914 33754 55156	0.25907 77815 97432 (−1)	0.88892 37184 75298 (+1)
0.20230 57578 73572 (+1)	0.10163 33231 92219 (+1)	0.58904 86627 56540
0.53343 71462 95893 (+1)	0.34308 74722 70107 (+1)	0.34879 57099 89796 (−1)
0.11736 57649 81479 (+2)	0.79268 84267 21699 (+1)	0.34228 01602 28392 (−3)
	N = 5	
0.28731 48371 42518	0.20777 15131 92881 (−1)	0.87382 89241 24244 (+1)
0.15972 27267 69638 (+1)	0.80899 75361 34602	0.70278 23530 89744
0.40788 91971 08905 (+1)	0.26749 00020 62407 (+1)	0.70111 72063 28495 (−1)
0.81490 83089 24566 (+1)	0.58690 26089 96340 (+1)	0.23127 60116 11556 (−2)
0.15361 86632 39238 (+2)	0.11126 29920 19586 (+2)	0.11623 58758 61307 (−4)
	N = 6	
0.23950 21025 97005	0.17342 74091 62196 (−1)	0.86104 58012 50877 (+1)
0.13222 72349 21621 (+1)	0.67256 50982 20954	0.78743 14118 75148
0.33271 18158 74294 (+1)	0.22009 00258 35410 (+1)	0.10871 22810 33294
0.64462 63393 09349 (+1)	0.47284 81828 27537 (+1)	0.67747 22140 24804 (−2)
0.11135 46059 16006 (+2)	0.85451 61044 14663 (+1)	0.13090 24113 82447 (−3)
0.19035 08722 06004 (+2)	0.14435 54903 00867 (+2)	0.36869 98889 94601 (−6)

	N = 7	
0.20536 29983 41707	0.14882 68905 81309 (—1)	0.85000 95062 09346 (+1)
0.11291 78494 07629 (+1)	0.57577 29268 38072	0.85193 38087 91951
0.28179 62496 49916 (+1)	0.18729 45923 63012 (+1)	0.14715 95881 84442
0.53789 74716 86761 (+1)	0.39795 25178 84313 (+1)	0.13767 64426 76794 (—1)
0.90263 05525 66825 (+1)	0.70465 11188 36071 (+1)	0.54501 86889 57396 (—3)
0.14243 12386 00487 (+2)	0.11385 65885 29302 (+2)	0.65655 45245 23729 (—5)
0.22743 46083 18369 (+2)	0.17823 70324 03396 (+2)	0.11096 99258 89865 (—7)
	N = 8	
0.17975 69252 16368	0.13033 85728 66258 (—1)	0.84032 66392 07882 (+1)
0.98580 25615 85796	0.50346 02097 34729	0.90214 59004 59557
0.24476 49029 09880 (+1)	0.16315 50144 42472 (+1)	0.18373 18641 50728
0.46320 39749 94804 (+1)	0.34435 79433 77284 (+1)	0.22883 69627 75134 (—1)
0.76606 22503 97354 (+1)	0.60300 34104 61902 (+1)	0.14416 43517 34506 (—2)
0.11762 16154 03866 (+2)	0.95558 69544 25233 (+1)	0.37902 18433 16068 (—4)
0.17441 73706 17078 (+2)	0.14350 75027 04787 (+2)	0.29968 00686 85111 (—6)
0.26478 89930 10109 (+2)	0.21271 72243 54311 (+2)	0.32036 87820 77983 (—9)
	N = 10	
0.14389 46439 53594	0.10440 00710 15863 (—1)	0.82397 95947 40720 (+1)
0.78672 96555 85787	0.40252 92421 87010	0.97402 21964 44984
0.19421 36069 71312 (+1)	0.12988 12815 49899 (+1)	0.24887 98818 78892
0.36415 39445 45340 (+1)	0.27211 44134 99959 (+1)	0.45362 47601 56081 (—1)
0.59372 46719 22246 (+1)	0.47105 39757 19458 (+1)	0.51138 60840 29568 (—1)
0.89130 38791 55219 (+1)	0.73331 76246 72739 (+1)	0.32305 23358 53483 (—3)
0.12707 18411 29668 (+2)	0.10695 92485 61766 (+2)	0.10151 23664 80723 (—4)
0.17568 90596 71175 (+2)	0.14981 57149 89211 (+2)	0.13200 43668 99586 (—8)
0.24038 60021 87934 (+2)	0.20545 28456 98867 (+2)	0.50464 45998 07931 (—9)
0.34010 63935 55603 (+2)	0.28300 57687 13065 (+2)	0.24256 36116 52749 (—12)

Table VII (continued)

X (K)	T (I) A = —0.7	W (I)
	N = 4	
0.47160 25596 33298	0.82704 29759 12019 (—1)	0.24778 31658 31905 (+1)
0.22293 56629 15334 (+1)	0.11775 02187 10197 (+1)	0.47939 49955 37430
0.56251 64541 83042 (+1)	0.36799 93484 10139 (+1)	0.33976 56532 53543 (—1)
0.12115 64878 85376 (+2)	0.82598 00031 20544 (+1)	0.36576 85057 54504 (—3)
	N = 5	
0.37892 60456 18006	0.66635 24308 34577 (—1)	0.23585 87628 39497 (+1)
0.17672 62078 77379 (+1)	0.94119 30832 52310	0.56266 40165 28902
0.43192 11704 97384 (+1)	0.28807 82101 28433 (+1)	0.67837 16173 73769 (—1)
0.84565 51830 27893 (+1)	0.61429 62310 49216 (+1)	0.24669 58537 74097 (—2)
0.15742 80002 38729 (+2)	0.11468 42726 18877 (+2)	0.13222 48859 91680 (—4)
	N = 6	
0.31680 48075 53864	0.55795 62805 17835 (—1)	0.22598 99722 25431 (+1)
0.14670 70183 58606 (+1)	0.78468 91252 81390	0.62007 41730 68785
0.35327 04851 51480 (+1)	0.23767 31962 19828 (+1)	0.10424 78169 57890
0.67088 44836 27496 (+1)	0.49628 71990 36251 (+1)	0.71981 89358 68780 (—2)
0.11454 37021 32650 (+2)	0.88357 25339 81604 (+1)	0.14864 60651 46540 (—3)
0.19417 49544 79387 (+2)	0.14784 18595 42901 (+2)	0.43998 27706 18433 (—6)

	N = 7	
0.27223 11795 11110	0.47989 61977 97201 (—1)	0.21764 97968 31152 (+1)
0.12553 40804 53073 (+1)	0.67314 84024 43645	0.66018 06181 88243
0.29978 91114 20452 (+1)	0.20265 43411 92029 (+1)	0.13971 79646 91005
0.56090 98797 57253 (+1)	0.41848 94455 44807 (+1)	0.14547 33736 57858 (—1)
0.93047 68344 56631 (+1)	0.73009 17843 53495 (+1)	0.61726 50226 12584 (—3)
0.14570 45863 37162 (+2)	0.11689 27033 69036 (+2)	0.78203 42336 91614 (—5)
0.23127 07167 04580 (+2)	0.18177 23592 99697 (+2)	0.13766 08843 80238 (—7)
	N = 8	
0.23867 65537 73985	0.42099 93011 15852 (—1)	0.21047 50042 48597 (+1)
0.10976 17370 51711 (+1)	0.58953 15813 99385	0.68847 64180 01185
0.26077 52862 30690 (+1)	0.17679 82290 63894 (+1)	0.17264 85944 71278
0.48372 43219 92553 (+1)	0.36265 60534 18104 (+1)	0.24021 88658 32121 (—1)
0.79088 60320 77761 (+1)	0.62569 42069 94004 (+1)	0.16266 19154 32211 (—2)
0.12052 61593 64822 (+2)	0.98252 23591 58641 (+1)	0.45055 55620 94730 (—4)
0.17775 58257 36144 (+2)	0.14662 55985 10890 (+2)	0.37102 48260 21982 (—6)
0.26863 51385 91125 (+2)	0.21629 10015 10536 (+2)	0.41059 03351 26309 (—9)
	N = 10	
0.19150 21258 08655	0.33803 05004 33038 (—1)	0.19867 34660 28956 (+1)
0.87787 09305 33025	0.47240 10174 98780	0.72268 84678 04533
0.20734 91423 99168 (+1)	0.14104 12723 55670 (+1)	0.22910 55983 16870
0.38106 34517 11782 (+1)	0.28716 38674 40828 (+1)	0.46931 89590 33271 (—1)
0.61422 15047 31452 (+1)	0.48977 69315 90732 (+1)	0.57140 01838 75098 (—2)
0.91525 82569 48537 (+1)	0.75555 58143 15807 (+1)	0.38167 39799 97440 (—3)
0.12980 62956 14206 (+2)	0.10952 45903 86008 (+2)	0.12520 31713 78753 (—4)
0.17876 42604 72595 (+2)	0.15271 96232 22697 (+2)	0.16857 33511 62009 (—6)
0.24381 93875 54580 (+2)	0.20870 33512 55985 (+2)	0.66373 76470 69510 (—9)
0.34396 84401 57373 (+2)	0.28663 66058 89586 (+2)	0.32772 63257 29130 (—12)

TABLE VIII

Collocation Points, Abscissas and Weights for the Generalized Radau - Laguerre Numerical Integration Rule $w(x) = x^A \exp(-x)$, $A = -0.9$, $N = 4 (1) 10$.

X (K) A = −0.9	T (I)	W (I)
	N = 4	
0.75445 24937 42202 (−1)	0.00000 00000 00000	0.79710 99873 20380 (+1)
0.12219 99212 44421 (+1)	0.46412 43410 63911	0.14240 33732 05318 (+1)
0.40736 31935 70476 (+1)	0.23957 60656 30588 (+1)	0.11668 68632 56783
0.10004 21249 54263 (+2)	0.64400 95002 63020 (+1)	0.16872 30154 96811 (−2)
	N = 5	
0.58897 60429 84372 (−1)	0.00000 00000 00000	0.77766 82803 12566 (+1)
0.94366 55682 99390	0.36103 46404 08722	0.15328 43680 90203 (+1)
0.30466 65234 41904 (+1)	0.18276 98586 61073 (+1)	0.19520 04163 54562
0.67595 80989 04659 (+1)	0.46574 12096 72726 (+1)	0.87200 81662 25524 (−2)
0.13585 93504 69651 (+2)	0.95539 54676 25328 (+1)	0.60716 62422 76588 (−4)
	N = 6	
0.48299 65766 28075 (−1)	0.00000 00000 00000	0.76241 98826 59378 (+1)
0.76967 97066 52436	0.29553 36183 00379	0.16006 55117 31280 (+1)
0.24496 76093 59491 (+1)	0.14820 90285 52841 (+1)	0.26673 17697 58644
0.52670 53462 11025 (+1)	0.36981 98782 30359 (+1)	0.21382 37061 58681 (−1)
0.96605 15831 61901 (+1)	0.72187 43396 85948 (+1)	0.53760 75066 11291 (−3)
0.17229 37267 59920 (+2)	0.12805 43391 70081 (+2)	0.20069 81039 85767 (−5)

	N = 7	
0.40932 67096 92770 (—1)	0.00000 00000 00000	0.74992 11960 58405 (+1)
0.65026 52286 66474	0.25019 46587 92559	0.16451 57162 53304 (+1)
0.20536 48096 44090 (+1)	0.12481 58951 89591 (+1)	0.32904 60051 75641
0.43500 97535 59436 (+1)	0.30808 89592 90121 (+1)	0.38170 24393 92830 (—1)
0.77443 44308 36065 (+1)	0.58903 01023 12746 (+1)	0.18935 27686 19582 (—2)
0.12706 81287 65873 (+2)	0.99788 37148 49237 (+1)	0.28736 45100 37042 (—4)
0.20915 94159 79603 (+2)	0.16151 61862 47905 (+2)	0.62299 51902 83648 (—7)
	N = 8	
0.35514 96423 91306 (—1)	0.00000 00000 00000	0.73935 89256 91385 (+1)
0.56310 61857 42072	0.21693 67964 84587	0.16753 77255 44950 (+1)
0.17700 48613 95400 (+1)	0.10787 43779 69583 (+1)	0.38255 08412 23081
0.37176 91275 02548 (+1)	0.26457 18095 48438 (+1)	0.57434 10732 89371 (—1)
0.65215 90832 13012 (+1)	0.50022 21630 55431 (+1)	0.44130 01937 74850 (—2)
0.10403 79003 92678 (+2)	0.83070 00006 15770 (+1)	0.14185 60578 74661 (—3)
0.15858 92649 15846 (+2)	0.12881 79929 44104 (+2)	0.13779 15115 05423 (—5)
0.24634 63867 79271 (+2)	0.19567 58039 82127 (+2)	0.18426 30708 75562 (—8)
	N = 10	
0.28080 74697 97950 (—1)	0.00000 00000 00000	0.72220 65208 21866 (+1)
0.44425 54113 31812	0.17139 74504 26322	0.17110 61764 10982 (+1)
0.13892 09117 51670 (+1)	0.84919 67937 60880	0.46770 47657 41883
0.28918 11138 11964 (+1)	0.20683 16220 27365 (+1)	0.98617 06601 47413 (—1)
0.50011 60543 00369 (+1)	0.38665 18207 70231 (+1)	0.13052 06007 03714 (—1)
0.77979 64766 61889 (+1)	0.63068 70196 21098 (+1)	0.97040 05612 02458 (—3)
0.11417 16774 07620 (+2)	0.94931 09130 71365 (+1)	0.35891 37222 58946 (—4)
0.16103 56827 65248 (+2)	0.13604 30579 97686 (+2)	0.55008 57190 82588 (—6)
0.22389 47834 50175 (+2)	0.18989 93573 35462 (+2)	0.24926 65414 40095 (—8)
0.32142 19522 70464 (+2)	0.26550 34946 75974 (+2)	0.14457 87246 39444 (—11)