ΜΑΘΗΜΑΤΙΚΑ.— **Heegaard splittings of differentiable manifolds,** *by George M. Rassias**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φίλωνος Βασιλείου.

1. H. Weyl [**1**] had defined abstract differentiable manifolds in 1912. The importance of this geometric object was indicated by the work of H. Whitney [**1**]. L. S. Pontryagin had developed methods for using differentiable functions and mappings to prove topological problems.

Much information about the topological structure of a differentiable manifold has been obtained from the use of a special class of differentiable functions, so called Morse functions, in honor of M. Morse, who invented the topological theory of differentiable functions on a closed manifold, known as Morse theory. This theory has proved to be a very powerful idea in differential topology especially in the structure of differentiable manifolds.

Of course, the basic idea about Morse theory had been obtained by Poincaré [**1**]. There have been notable applications of Morse theory in solving various problems in and outside Mathematics, by many mathematicians — especially R. Bott [**1**], J. Milnor [**1**, **2**], S. Smale [**1** - **3**], and R. Thom [**1**, **2**].

S. Smale [**3**] verified the importance of Morse theory in attacking difficult topological problems by proving a fundamental theorem on the structure of differentiable manifolds, the so-called h-cobordism theorem, which has several important applications, including the proof of the generalized Poincaré conjecture (i.e. any homotopy n-sphere, $n > 4$, is homeomorphic to the n-sphere), and also of the generalized Schoenflies conjecture.

The h-cobordism theorem is one of the most important and interesting theorems of the field of topology, especially of Differential topology. Through this work, S. Smale generalized the well known construction of any closed orientable surface in the form of a sphere with handles by inventing the construction of attaching handles to higher

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* ΓΕΩΡΓΙΟΥ ΡΑΣΣΙΑ, **Heegaard διαμερισμοὶ διαφορισίμων πολλαπλοτήτων.**

dimensional manifolds. *The present work is a part of lectures given by the author at the International Congress of Mathematicians in Helsinki (Finland), the period of August 15-23, 1978.*

2. Let M be a closed (i.e. compact without boundary) C^∞ differentiable manifold and $f: M \to R$ be a C^∞ differentiable function on M.

Definitions. A point $p \in M$ is *a critical point* of f iff the induced map $f_*: TM_p \to TR_{f(p)}$ is zero, where TM_p is the tangent space of M at p. In other words, choosing a local coordinate system $(x_1, \ldots x_n)$ in a neighborhood W of p, we have that

$$\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0.$$

A real number $a \in R$ is said to be *a critical value* of f if $f^{-1}(a)$ contains a critical point of f. If $f^{-1}(a)$ contains no critical points, then a is *a regular value* of f. (Of course, every number not belonging in the image of f is a regular value of f). A critical point p of f is said to be *non-degenerate* iff the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$ is non-singular.

In other words, this matrix defines a symmetric bilinear form on the tangent space TM_p. This bilinear form is the *Hessian* of f at p. The *index* of a critical point p of f is defined to be the maximal dimension of a subspace of TM_p on which the Hessian of f is negative definite.

A C^∞ differentiable function $f: M \to R$ is said to be *a Morse function* if f has only non-degenerate critical points. Morse theory studies the connection between the topological structure of a differentiable manifold M and the Morse functions defined on M. By *a handlebody of genus* $p (\geqslant 0)$, we mean a 3-dimensional disk D^3 with p solid handles attached on D^3. It is known (Seifert-Threlfall [**1**]) that any closed, orientable 3-dimensional manifold can be obtained from two handlebodies of the same genus by identifying the boundaries.

A closed orientable 3-dimensional manifold is of *genus* p $(\geqslant 0)$ if it can be obtained by identifying the boundaries of two handlebodies of genus p, but cannot be obtained by identifying the boundaries of two handlebodies of genus less than p. A *Heegaard Splitting* of genus p ($\geqslant 0$) of a closed, orientable 3-dimensional manifold, is a decomposition of M as $M = H \cup H'$, such that $H \cap H' = \partial H = \partial H'$, H, H' are handlebodies of the same genus p ($\geqslant 0$) and $H \cap H'$ is a closed orientable surface of genus p. Now, we are going to give *a Morse theoretic proof* of the existence of a Heegaard splitting for any closed 3-dimensional manifold.

Theorem 1. Every closed C^∞ differentiable orientable 3-dimensional manifold M admits a Heegaard splitting.

Proof. By S. Smale [**2**], there exists a nice Morse function $f: M \to R$, having exactly one critical point of index 0, and one critical point of index 3.

Thus we may assume that the critical values of f are 0, 1, 2, 3. Now, we consider the inverse image N by f, of a regular value a, $1 < a < 2$, and we have that N is a 2-dimensional submanifold of M. Consider $H = \{x \in M / f(x) \leqslant a\}$ which is diffeomorphic to

$$\underbrace{(S^1 \times D^2) \# (S^1 \times D^2) \# \ldots \# (S^1 \times D^2)}_{p \text{ times}}$$

After that, consider $H' = \{x \in M / f(x) \geqslant a\}$ which is diffeomorphic to

$$\underbrace{(S^1 \times D^2) \# (S^1 \times D^2) \# \ldots \# (S^1 \times D^2)}_{p \text{ times}}$$

Then, $H \cap H' = \{x \in M / f(x) = a\}$ is diffeomorphic to

$$\underbrace{(S^1 \times S^1) \# (S^1 \times S^1) \# \ldots \# (S^1 \times S^1)}_{p \text{ times}}$$

Note that H and H' have the same number of copies of $S^1 \times D^2$ since the Morse function f defined on M has the same number of critical points of index 1 and 2. This is true, because $0 = x(M) = \sum_{i=1}^{3} (-1)^i c_i$, and by assumption $c_0 = c_3 = 1$. Thus, $c_1 = c_2$.

Hence, M admits a Heegaard splitting. Q. E. D.

In the following theorem, we show the existence of a Heegaard splitting for compact 3-dimensional manifolds with non-empty boundary.

Theorem 2. Every compact 3-dimensional manifold M with non-empty (orientable) boundary $\partial M \neq \emptyset$ admits a kind of Heegaard splitting, i. e.

$$M = H \cup H', \quad H \cap H' = \partial H \cap \partial H'$$

where H, H′ are handlebodies of the *same* genus.

Proof. *Case 1:* ∂M is a connected orientable surface of genus n.

Then, there exists a Morse function $f : \partial M \to [0,1]$ having critical points $p_1, p_2, \ldots, p_{2n}$, and a Morse function

$$g : \partial M \times [0,1) \to [0,1]$$

such that the critical points $p_1, \ldots, p_{2n}$ are *not* critical points of g. Then, extend g to a Morse function

$$G : M \to [0,1]$$

such at $G = g$ on $\partial M \times [0,1)$.

Consider $G^{-1}\left(\frac{1}{2}\right)$ which is a compact 2-dimensional manifold with boundary, so that the critical values corresponding to the critical points of index 1, are less than the critical values corresponding to the critical points of index 2.

Note, $$\partial\left(G^{-1}\left(\frac{1}{2}\right)\right) = G^{-1}\left(\frac{1}{2}\right) \cap \partial M .$$

Then, $G^{-1}\left(\frac{1}{2}\right)$ decomposes M into two submanifolds H, H′ having an equal number of critical points of index 1, and of index 2 respectively. In other words, H, H′ have the *same* genus since H, H′ intersect on the same surface namely $\partial H \cap \partial H' = G^{-1}\left(\frac{1}{2}\right)$.

Note, $$\partial H = G^{-1}\left(\frac{1}{2}\right) \cup \left(\partial M \cap G^{-1}\left(0, \frac{1}{2}\right]\right) \text{ and}$$

$$\partial H' = G^{-1}\left(\frac{1}{2}\right) \cup \left(\partial M \cap G^{-1}\left(\frac{1}{2}, 1\right]\right).$$

Thus,

$$H \cap H' = \partial H \cap \partial H' = G^{-1}\left(\frac{1}{2}\right).$$

Hence, M admits the requested Heegaard splitting.

Case 2: ∂M is disconnected (i. e. consists of a number of closed connected orientable 2-dimensional manifolds). In this case, we repeat the same argument as in the previous case for each of the connected boundary components, and so we get again the requested Heegaard splitting. Q. E. D.

Theorem 3. Let M be an open (i. e. noncompact without boundary) connected manifold of dimension n. Then, there exists a Morse function f on M such that

a. The function f has no critical points of index n.
b. The function f is proper, and,
c. The function f has a single critical point of index 0.

Proof. We can write $M = \bigcup_{i=1}^{\infty} M_i$, $\partial M_i \neq \varnothing$, $M_i \subset \operatorname{int} M_{i+1}$. Then, $M_{i+1} - \operatorname{int} M_i$ is a compact manifold with boundary. Now, since $M_i \subset \operatorname{int} M_{i+1}$ it follows that

$$\partial (M_{i+1} - \operatorname{int} M_i) = \partial M_{i+1} + \partial M_i .$$

Consider the cobordism

$$W_i = (M_{i+1} - \operatorname{int} M_i ;\ \partial M_{i+1},\ \partial M_i) .$$

Then, there exists a Morse function f_i on W_i (i. e. $f_i : M_{i+1} - \operatorname{int} M_i \rightarrow [i, i+1]$ such that $f_i^{-1}(i) = \partial M_i$, $f_i^{-1}(i+1) = \partial M_{i+1}$ and all the critical points of f_i lie in $(M_{i+1} - \operatorname{int} M_i) - \partial M_i - \partial M_{i+1}$ and are nondegenerate).

Then, the function defined on $M (f : M \rightarrow R)$ which equals f_i on W_i for each i is a proper Morse function.

We cancel critical points of index 0 against an equal number of critical points of index 1. So, all critical points of index 0, except a single one, are cancelled.

Now, by a duality argument (i. e. turning the manifold upside down) we cancel all critical points of index n against an equal number of critical points of index (n — 1) until we eliminate all critical points of index n.

We are repeating this process for each cobordism W_i so that f_i has the above properties and f_{i+1} has the additional property that $f_{k+1} = f_k$ on W_k for each $1 \leqslant k \leqslant i$. Taking $f: M \rightarrow R$ so that $f = f_i$ on W_i for each $i = 1, \ldots, \infty$, f has the requested properties. Q. E. D.

Corollary 1. The fundamental group of any open 2-dimensional manifold is a free group on a finite or countable set of generators. Another combinatorial proof of this corollary is given in Ahlfors-Sario [**1**].

Corollary 2. Any open 2-dimensional manifold which is simply-connected, is homeomorphic to $\mathbf{R}^2$.

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ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν παροῦσαν ἐργασίαν ἀποδεικνύονται τὰ ἀκόλουθα τρία θεωρήματα. Τὰ δύο πρῶτα θεωρήματα ἀναφέρονται εἰς τὴν ἐφαρμογὴν τῆς θεωρίας τοῦ Morse διὰ τὴν ἀπόδειξιν τῆς ὑπάρξεως ἑνὸς διαμερισμοῦ κατὰ τὴν σημασίαν τοῦ Heegaard, ὀνομαζομένου Heegaard splitting, διὰ πᾶσαν C^∞ διαφορίσιμον, προσανατολίσιμον, συμπαγῆ τριδιάστατον πολλαπλότητα ἄνευ συνόρου, ὡς ἐπίσης διὰ τὴν ἀπόδειξιν ὑπάρξεως ἑνὸς καταλλήλου Heegaard splitting διὰ πᾶσαν C^∞ διαφορίσιμον, προσανατολίσιμον, συμπαγῆ πολλαπλότητα μετὰ συνόρου.

Τὰ ἀνωτέρω δύο συμπεράσματα ἀποτελοῦν θεμελιωδέστατα καὶ σπουδαιότατα θεωρήματα πρὸς πληρεστέραν κατανόησιν τοῦ πεδίου τῆς δομῆς τῶν τριδιαστάτων πολλαπλοτήτων.

Τὸ τρίτον θεώρημα ἀναφέρεται εἰς τὴν ἀπόδειξιν ὑπάρξεως μίας χρησίμου συναρτήσεως τοῦ Morse ἐπὶ πάσης C^∞ διαφορισίμου, συναφοῦς, καὶ μὴ συμπαγοῦς n-διαστάτου πολλαπλότητος ἄνευ συνόρου διὰ πάντα φυσικὸν ἀριθμὸν n. Ἡ σπουδαιότης τοῦ ἀνωτέρω θεωρήματος προκύπτει ἐκ δύο πολὺ ἐνδιαφερόντων πορισμάτων αὐτοῦ εἰς τὸ πεδίον τῶν διδιαστάτων πολλαπλοτήτων.

Τὰ συμπεράσματα τῆς παρούσης ἐργασίας ἀποτελοῦν ἓν μέρος ἐκ τῶν θεωρημάτων ἀνακοινωθέντων εἰς τὸ International Congress of Mathemati-

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REFERENCES

L. V. Ahlfors and L. Sario, Riemann Surfaces, Princeton, N. J., Princeton University Press, 1960.

R. Bott, Lectures on Morse Theory, Bonn, 1960.

J. Milnor, Morse Theory, Ann. Math. Stud., No. 51, Princeton, 1963.

———, Whitehead Torsion, Bull. A. M. S. 72 (1966) 358 - 426.

H. Poincaré, Journal de Math. Pure et Appliquee, 3 (1885).

G. M. Rassias, Heegaard Splittings and the Poincaré Conjecture, Practica of the Academy of Athens, 1978.

———, Counterexamples to a Conjecture of René Thom, International Congress of Mathematicians, Helsinki, 1978.

S. M. Reifenberg, Rassias Theory and Its Importance to a Problem of Hilbert, International Congress of Mathematicians, Helsinki, 1978.

H. Seifert and W. Threlfall, Lehrbuch der Topologie, N. Y. Chelsea 1947.

S. Smale, The Generalized Poincaré Conjecture in Higher Dimensions, Bull A. M. S. 66 (1960) 373 - 375.

———, The Generalized Poincaré Conjecture in Dimensions Greater Than Four, Ann. of Math. (2) 74 (1961) 391 - 406.

———, On the Structure of Manifolds, Amer. J. of Math., Vol. 84, 1962, 387 - 399.

R. Thom, Structural Stability and Morphogenesis, Benjamin, N. Y., 1975.

———, Sur Une Partition en Cellule Associee a Une Fonction Sur Une Variete, C. R. Acad. Sci., Paris, 228 (1949).

H. Weyl, Die Idee der Riemannschen Flaache, Teubner, 3rd Edit. Stuttgart, 1955.

H. Whitney, Differentiable Manifolds, Ann. of Math (2) 37 (1936).