ΠΙΝΑΞ ΠΕΡΙΕΧΟΜΕΝΩΝ

NUMERICAL SOLUTION OF CAUCHY TYPE SINGULAR INTEGRAL EQUATIONS

ΠΡΑΓΜΑΤΕΙΑΙ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

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BY

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ΑΘΗΝΑΙ

ΓΡΑΦΕΙΟΝ ΔΗΜΟΣΙΕΥΜΑΤΩΝ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

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ΠΕΡΙΛΗΨΙΣ

Ἰδιόμορφός τις ὁλοκληρωτικὴ ἐξίσωσις τύπου Cauchy πρώτου ἢ δευτέρου εἴδους (ἢ σύστημα τοιούτων ἐξισώσεων) δύναται νὰ ἐπιλυθῇ ἀριθμητικῶς δι᾽ ἀναγωγῆς εἰς σύστημα γραμμικῶν ἐξισώσεων. Διὰ τὴν ἀναγωγὴν ταύτην ἡ ἰδιόμορφος ὁλοκληρωτικὴ ἐξίσωσις ἐφαρμόζεται εἰς ὡρισμένον ἀριθμὸν καταλλήλως ἐκλεγομένων σημείων τοῦ διαστήματος ὁλοκληρώσεως καὶ κατόπιν χρησιμοποιεῖται μέθοδός τις ἀριθμητικῆς ὁλοκληρώσεως διὰ τὴν προσέγγισιν τῶν ὁλοκληρωμάτων τῆς ἐξισώσεως ταύτης. Ἡ μέθοδος αὕτη ἀποτελεῖ γενίκευσιν τῆς εὐρέως χρησιμοποιουμένης μεθόδου ἀριθμητικῆς ἐπιλύσεως ὁλοκληρωτικῶν ἐξισώσεων τύπου Fredholm.

ABSTRACT

A Cauchy type singular integral equation of the first or the second kind (or a system of such equations) can be numerically solved by reduction to a system of linear equations. For this reduction, the singular integral equation is applied at a number of appropriately selected points of the integration interval and then a numerical integration rule is used for the approximation of the integrals in this equation. This method consists a generalization of the corresponding method widely used for the numerical solution of Fredholm integral equations.

NUMERICAL SOLUTION OF CAUCHY TYPE SINGULAR INTEGRAL EQUATIONS

1. INTRODUCTION

An effective method of numerical solution of Fredholm integral equations consists in the reduction of such an equation to a system of linear equations after the integrals occurring in this equation are approximated by sums (through the use of an appropriate numerical integration rule) and the equation is applied at the abscissae used in the numerical integration rule [**1**].

In the case when the kernel of a Fredholm integral equation of the first or the second kind consists of a regular term as well as a Cauchy type singular term (the part of the integral corresponding to the Cauchy type term defined in the principal value sense), then we speak about a Cauchy type singular integral equation or, simpler, a singular integral equation. In this case the above-described method of numerical solution of Fredholm integral equations was believed not to be applicable, since numerical integration techniques could not be used for Cauchy type principal value integrals. The standard technique for the numerical solution of singular integral equations has been their reduction to equivalent Fredholm integral equations of the second kind, according to the developments of Muskhelishvili [**2**], followed by the numerical solution of the latter. The results of Muskhelishvili were generalized by Pogorzelski [**3**] to more complicated cases of singular integral equations, that is to more general classes of the regular part of the kernel. Pogorzelski showed that, in general, the reduction of a singular integral equation of the first or the second kind to a Fredholm integral equation of the second kind is not always possible. It may be noted on this point that the singular integral equations to be considered here belong, in general, to this class.

On the other hand, several investigators succeeded in reducing some Cauchy type singular integral equations of special forms to systems of linear equations by developing numerical integration rules for the evaluation of Cauchy type principal value integrals through expansion of the integrands into a series of orthogonal polynomials. In this way, Kalandiya [4] treated the case of singular integral equations of the first kind of the form:

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\,\frac{\varphi(t)}{t-x}\,dt + \int_{-1}^{1} w(t)\,k(t,x)\,\varphi(t)\,dt = f(x)\,, \quad -1 < x < 1\,, \tag{1.1}$$

where $w(t)$ is a weight function of the special form:

$$w(t) = (1-t)^{\pm 1/2}\,(1+t)^{\pm 1/2}\,, \tag{1.2}$$

$\varphi(t)$ is the unknown function assumed regular along the integration interval $[-1, 1]$, $k(t, x)$ is a regular bounded kernel along the integration interval (with respect to both its variables) and $f(x)$ is a regular function along the same interval.

Furthermore, Erdogan and Gupta [5] have developed a method of numerical evaluation of Cauchy type principal value integrals of the form:

$$I = \int_{-1}^{1} w(t)\,\frac{\varphi(t)}{t-x}\,dt\,, \tag{1.3}$$

with weight function $w(t)$ given by Eq. (1.2), and applied it to the reduction of Eq. (1.1) (or a system of such equations) to a system of linear equations of the form:

$$\sum_{k=1}^{n} A_k \left[\frac{1}{\pi\,(t_k - x_r)} + k\,(t_k,\, x_r)\right] \varphi(t_k) = f(x_r), \quad r = 1, 2, \ldots, m, \tag{1.4}$$

where t_k are the abscissae and A_k the weights used in numerical integration rules of the Gauss-Chebyshev type for regular integrals and x_r are properly selected points of the integration interval (obtained as roots of Chebyshev polynomials), the number m of which may be equal to $(n-1)$, n or $(n+1)$, n being the number of the abscissae used. In the first case one more linear equation, resulting from some physical condition, supplements the system of linear equations (1.4). Nevertheless, Erdogan and Gupta [5] did not realize that their method of numerical evaluation of Cauchy type principal value integrals of the form (1.3) was indeed the Gauss-Chebyshev method, accurate for integrands polynomials of up to $2n$ degree, believing that it was accurate for integrands $\varphi(t)$ polynomials of up to $(n-1)$ degree only. This misunderstanding has not been realized even in two more recent papers on the same subject (Erdogan, Gupta and Cook **[6]**, Erdogan **[7]**), as well as in a paper of Krenk **[8]**, who generalized the results of Erdogan and Gupta **[5]** to the case of singular integral equations of the form:

$$Aw(x)\,\varphi(x) + \frac{B}{\pi}\int_{-1}^{1} w(t)\,\frac{\varphi(t)}{t-x}\,dt + \int_{-1}^{1} w(t)\,k(t,x)\,\varphi(t)\,dt = f(x) \qquad (1.5)$$

with weight function $w(t)$ of the form:

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}, \qquad (1.6)$$

where:

$$\alpha = \frac{1}{2\pi i}\ln\frac{A-iB}{A+iB} + N,\ \ \beta = -\frac{1}{2\pi i}\ln\frac{A-iB}{A+iB} + M,\ \ \alpha+\beta = M+N = -\varkappa, \qquad (1.7)$$

where N and M are integer numbers and the index $\varkappa$ is restricted to the values (-1), 0 and 1.

We can also mention that Krenk reduced the singular integral equation (1.5) to the system of linear equations (1.4) after having used the properties of Jacobi polynomials and implicitly derived the Gauss - Jacobi numerical integration rule for Cauchy type principal value integrals of the form (1.3).

Recently, Theocaris and Ioakimidis **[9]** showed, by using the same method used in Refs. **[5]** to **[7]**, that the methods of numerical evaluation of Cauchy type principal value integrals derived in these references and used for the reduction of the Cauchy type singular integral equation (1.1) to the system of linear equations (1.4) were accurate for integrands $\varphi(t)$ polynomials of up to $2n$ degree. Thus, their characterization as Gauss - Chebyshev methods was completely justified. The same proof used in **[9]** may be easily extended to the case of the Gauss - Jacobi numerical integration rule used by Krenk **[8]** for the numerical solution of the Cauchy type singular integral equation (1.5).

On the other hand, in recent years several papers regarding the numerical evaluation of Cauchy type principal value integrals have appeared. Among them, we can mention the papers of Hunter **[10]**, who generalized the Gauss - Legendre numerical integration rule for the numerical evaluation of Cauchy type principal value integrals, and of Chawla and Ramakrishman **[11]**, who generalized the Gauss - Jacobi and the Gauss - Chebyshev numerical integration rules for the numerical evaluation of the same type of integrals.

Furthermore, Theocaris and Ioakimidis [**12, 13**] developed a quite general method for the evaluation of Cauchy type principal value integrals, permitting the application of any numerical integration rule for regular integrals to the case of Cauchy type principal value integrals.

The results of Hunter [**10**] and of Chawla and Ramakrishman [**11**] have been used by Theocaris and Ioakimidis ([**14**] and [**15**] respectively) for the numerical solution of Cauchy type singular integral equations of the form:

$$A(x)\, w(x)\, \varphi(x) + B(x) \int_{-1}^{1} w(t)\, \frac{\varphi(t)}{t-x}\, dt + \int_{-1}^{1} w(t)\, k(t,x)\, \varphi(t)\, dt = f(x)\,, \qquad (1.8)$$

where $A(x)$ and $B(x)$ are bounded continuous functions in the integration interval $[-1, 1]$, $k(t,x)$ is a regular bounded kernel in the same interval with respect to both its variables t and x, except for $x = \pm 1$ when it may become unbounded like $1/(t-x)$, and $f(x)$ a known function possibly presenting weak singularities near the end-points $x = \pm 1$ or the integration interval. According to Refs. [**14, 15**], a Cauchy type singular integral equation of the form (1.8), with $w(t)$ equal to unity or given by Eq. (1.6) respectively, can be reduced to the following system of linear equations:

$$\sum_{k=1}^{n} A_k \left[\frac{B(x_r)}{t_k - x_r} + k(t_k, x_r) \right] \varphi(t_k) = f(x_r)\,, \qquad r = 1, 2, \ldots, m\,, \qquad (1.9)$$

if the points x_r of application of Eq. (1.8) are properly selected.

At this point, it may be noted that the special case of Eq. (1.8) where $A(x) \equiv 0$ was treated by Erdogan, Gupta ann Cook [**6**] and Erdogan [**7**] under the assumption that the constanrs α and β entering Eq. (1.6) for the weight function $w(t)$ are different from zero and, of course, greater than (-1). These authors have reduced Eq. (1.8) (with $A(x) \equiv 0$) to the system of linear equations (1.9), possibly supplemented by one more linear equation, after having used the Gauss-Jacobi numerical integration rule for regular integrals also for Cauchy type principal value integrals without any justification. Recently, it was shown by Theocaris and Ioakimidis [**15**] that the selection of the points x_r of application of Eq. (1.8) as proposed by Erdogan, Gupta and Cook [**6, 7**] was not the correct one. This fact resulted in a very slow rate of convergence of the results obtained from the numerical solution of Eq. (1.8) to their correct values with increasing values of n.

Besides the Gauss-Chebyshev, Gauss-Legendre and Gauss-Jacobi numerical integration rules, several other Gauss-type numerical integration rules have been used by Theocaris and Ioakimidis [**16**, **12**] for the numerical solution of Cauchy type singular integral equations of the form:

$$A(x)\, w(x)\, \varphi(x) + B(x) \int_a^b w(t) \frac{\varphi(t)}{t-x}\, dt + \int_a^b w(t)\, k(t, x)\, \varphi(t)\, dt = f(x), \qquad a < x < b, \tag{1.10}$$

where $w(x)$ is a properly determined weight function and $[a, b]$ the integration interval associated with the numerical integration rule in use.

Although the Gauss-type numerical integration rules seem to be the best practically useful numerical integration rules for the evaluation of regular or Cauchy type principal value integrals, nevertheless in most cases when a Cauchy type singular integral equation of the form (1.10) is to be solved, special interest is placed upon the values of the unknown function $\varphi(t)$ at the end-points a and/or b of the integration interval. In this way, if a Gauss-type numerical integration rule is used for the numerical solution of Eq. (1.10), an application of the interpolation (or rather extrapolation) methods will be afterwards necessary in order that $\varphi(a)$ and/or $\varphi(b)$ be determined. But, since Gauss-type numerical integration rules are accurate for functions $\varphi(t)$ polynomials of up to $(2n-1)$ degree, while the interpolation (or extrapolation) methods are accurate for functions $\varphi(t)$ polynomials of only up to $(n-1)$ degree, it is evident that interpolation (or extrapolation) introduces considerable error in the values $\varphi(a)$ and/or $\varphi(b)$, which are of particular interest. In this case, the use of Radau-type (or semi-closed-type) numerical integration rules, containing among their abscissae one of the end-points a and b, or of Lobatto-type (or closed-type) numerical integration rules, containing among their abscissae both end-points a and b, seem to be the best possibilities although a slight reduction in the accuracy of numerical integration is possible, since these types of numerical integration rules are accurate for functions $\varphi(t)$ polynomials of up to $(2n-2)$ and $(2n-3)$ degrees respectively [**17**].

For the numerical solution of Cauchy type singular integral eguations, Lobatto-type numerical integration rules were introduced for the first time by Theocaris and Ioakimidis, who treated the cases of the Lobatto-Chebyshev rule [**9**] and the Lobatto-Legendre rule [**14**]. The results contained in Ref. [**9**] have been subsequently taken into account by Krenk [**18**], who derived by a complicated method the Lobatto-Jacobi method for the numerical solution of Eq. (1.5), under restrictions (1.6) and (1.7). A much simpler derivation of the same method was made by Ioakimidis and Theocaris [**19**, **20**]. Moreover, the same authors applied the Lobatto-Jacobi rule for the numerical solution of Eq. (1.5) not subject to restrictions (1.6) and (1.7). Furthermore, the method of constructing Lobatto-type rules from the corresponding Gauss-type rules and the corresponding interpolation methods are given in Refs. [**12**] and [**16**].

On the other hand, some Radau-type numerical integration rules have been applied to the numerical solution of singular integral equations by Theocaris and Ioakimidis, who used for this purpose the Radau-Legendre rule [**14**] and the Radau-Jacobi rule [**19**] and gave [**12**, **16**] the method of constructing Radau-type rules from the corresponding Gauss-type rules, as well as the corresponding interpolation methods.

In this paper, a general treatment of the methods of reduction of singular integral equations of the form (1.10) to systems of linear equations of the form (1.9) will be made, in a way independent of the numerical integration rule used, which will be assumed completely known in advance. This treatment will be based on the application of numerical integration rules to the evaluation of Cauchy type principal value integrals. Several generalizations to some cases of Cauchy type singular integral equations of special type, like singular integral equations along contours or singular integral equations associated with complex singularities at the end-points of the integration interval, will be also made.

2. NUMERICAL INTEGRATION RULES FOR CAUCHY TYPE INTEGRALS

A first method for the computation of a Cauchy type principal value integral I is by using its definition as a principal value integral [**2**, **3**]:

$$I = \int_a^b \frac{g(t)}{t-x}\, dt = \lim_{\varepsilon \to 0} \left[\int_a^{x-\varepsilon} \frac{g(t)}{t-x}\, dt + \int_{x+\varepsilon}^{b} \frac{g(t)}{t-x}\, dt \right], \tag{2.1}$$

where ε is a positive number, tending to zero. A sufficiently broad class of functions $g(t)$, for which the integral (2.1) exists for x inside the interval (a, b) but not coinciding with its end-points a or b, is the class of functions satisfying the Hölder condition [**2**, **3**] in every simple subinterval of the interval $[a, b]$ not containing the points a or b. These functions $g(t)$ are permitted to present weak singularities at the points a and b. In the sequel, it will be assumed that the functions $g(t)$ belong to this class.

Now, if we define the complex function $\Phi(z)$ by:

$$\Phi(z) = \int_a^b \frac{g(t)}{t-z}\, dt \tag{2.2}$$

and apply the second Plemelj's formula [**2**, **3**] to the boundary values $\Phi^+(x)$ and $\Phi^-(x)$ of this function at a point x of the integration interval (a, b), we find that:

$$I = \Phi(x) = \frac{1}{2}\left[\Phi^+(x) + \Phi^-(x)\right] \tag{2.3}$$

or, in another writing:

$$I = \int_a^b \frac{g(t)}{t-x}\, dt = \frac{1}{2}\left[\lim_{\substack{z\to x\\ \operatorname{Im} z>0}} \int_a^b \frac{g(t)}{t-z}\, dt + \lim_{\substack{z\to x\\ \operatorname{Im} z<0}} \int_a^b \frac{g(t)}{t-z}\, dt\right]. \tag{2.4}$$

This equation denotes that a Cauchy type principal value integral may be computed as the sum of two Cauchy type integrals which are not principal value integrals. Now we can apply an appropriate numerical integration method, taking also into account the logarithmic or weak power singularities of the function $g(t)$ near the points a and b, if such singularities exist, for the approximation of the integrals in the right-hand side of Eq. (2.4). If this rule is of the form:

$$\int_a^b w(t)\, \varphi(t)\, dt = \sum_{k=1}^{n} A_k\, \varphi(t_k) + E_n, \tag{2.5}$$

we see that:

$$\int_a^b w(t) \frac{\varphi(t)}{t-x}\, dt = \sum_{k=1}^{n} A_k \frac{\varphi(t_k)}{t_k - x} - \varphi(x) \frac{q_n(x)}{p_n(x)} + E_n, \tag{2.6}$$

where $w(t)$ is the weight function due to the singularities of $g(t)$, that is:

$$g(t) = w(t)\, \varphi(t), \tag{2.7}$$

where $\varphi(t)$ is a regular function along the integration interval. Also in Eq.

(2. 6) the point x is assumed not to coincide with anyone of the abscissae t_k or the end-points a and b of the integration interval. Furthermore, in Eq. (2. 6) the fuction $p_n(z)$ is the polynomial of degree n:

$$p_n(z) = \prod_{k=1}^{n} (z - t_k) \tag{2. 8}$$

associated with the integration rule in use and the function $q_n(z)$ is defined by:

$$q_n(z) = -\int_a^b w(t) \frac{p_n(t)}{t-z} \, dt. \tag{2. 9}$$

When $z \in (a, b)$, the integral (2. 9) exists only in the principal value sense and hence it should be interpreted as a Cauchy type principal value integral and computed by using Eq. (2. 1) or Eq. (2. 4). It should be also mentioned that the term of the error term E_n, as in Eq. (2. 5), due to the pole of the integrands in the integrals of the right-hand side of Eq. (2. 4) was computed as pointed out in Ref. **[21]** and extracted from E_n. This part is the second term of the right-hand side of Eq. (2. 6), where the second Plemelj's formula was also taken into account for the boundary values of the function $q_n(z)$. This means that in Eq. (2. 6) $q_n(x)$ is a Cauchy type principal value integral.

In this way, the error term E_n in Eq. (2. 6) may be computed as if the integral of its left-hand side were a regular integral and all methods of error estimation in numerical integration rules remain applicable. One last remark concerns the class of functions $\varphi(t)$ for which the error term in Eq. (2. 6) vanishes. On this subject, it can be easily seen that, if the numerical integration rule applied for the calculation of the integral of the left-hand side of Eq. (2. 6) is exact, in the case of regular integrals, for integrands $\varphi(t)$ polynomials of up to p degree (where generally $n-1 \leqslant p \leqslant 2n-1$), then this rule will be exact, in the case of Cauchy type principal value integrals of the form (2. 6), when the function $\varphi(t)$ is a polynomial of up to $(p-1)$ degree. This fact follows immediately from the methods of error estimation **[21]**. We can thus say that, when integration rules are applied to Cauchy type principal value integrals, then more accuracy is generally obtained, after the same computational effort, than that obtained in the case of regular integrals if in both cases the same function $\varphi(t)$ is considered.

3. APPLICATION TO THE NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS

Now it is quite possible to try to numerically solve a Cauchy type singular integral equation of the form :

$$A(x)\, w(x)\, \varphi(x) + B(x) \int_a^b w(t) \frac{\varphi(t)}{t-x}\, dt + \int_a^b w(t)\, k(t,x)\, \varphi(t)\, dt = f(x), \quad a < x < b, \tag{3.1}$$

In Eq. (3.1) the integration interval $[a, b]$ may be finite or infinite. In both cases the first integral of the left-hand side is defined in the principal value sense. Also the funtions $A(x)$ and $B(x)$ are assumed continuous, bounded and different from zero along the integration interval $[a, b]$, the weight function $w(t)$ is assumed to be positive along the integration interval and satisfying the Hölder condition in every simple subinterval of this interval not containing its end-points a and b. At these points the function $w(t)$ may present any type of logarithmic or weak power singularities. It must be noted that the right-hand side function $f(x)$, although assumed to satisfy the Hölder condition in every simple subinterval of the integration interval (a, b) not containing the end-points a and b, nevertheless is permitted to present logarithmic or weak power singularities near these points a and b. Also the kernel $k(t, x)$ is supposed to be a continuous function with respect to both its variables in the whole interval $[a, b]$ without singularities at the end-points of this interval, except when $x = a$ or $x = b$. Then it may become unbounded like $1/(t-x)$.

In this way, the weight function $w(t)$ must be selected in such a way so that its behaviour near the points a and b be compatible with the behaviour of the functions $A(x)$, $B(x)$ and $f(x)$ near these points. This function $w(t)$ was introduced in order that the really unknown function $\varphi(t)$ be considered without any singularity along the whole integration interval $[a, b]$. It is further assumed that it satisfies the Hölder condition along this interval. A method of finding the weight function $w(t)$, which basically depends on its behaviour near the points a and b is presented in Refs. [**6**, **7**] for a special case of the singular integral equation (3.1). For more general cases, a generalization of this method is completely possible if the developments of Refs. [**2**, **3**, **22**] are taken into account.

Under these assumptions and the final assumption that the integrals occurring in Eq. (3. 1) exist for every value of the variable x inside the open interval (a, b), we can apply a method of numerical integration of the form (2. 5), with the same weight function $w(t)$ and integration interval $[a, b]$ as in Eq. (3. 1), for the approximation of the second integral of this equation and of the form (2. 6) for the approximation of its first integral. Thus, we find after ignoring the error terms :

$$A(x)\, w(x)\, \varphi(x) + B(x) \left[\sum_{k=1}^{n} A_k \frac{\varphi(t_k)}{t_k - x} - \varphi(x) \frac{q_n(x)}{p_n(x)} \right] + \\ + \sum_{k=1}^{n} A_k\, k(t_k, x)\, \varphi(t_k) = f(x). \tag{3. 2}$$

Equation (3. 2) is an approximation of Eq. (3. 1) generally satisfactory enough. Furthermore, we can apply this equation at a number of points x_r inside the open interval (a, b) and, in this way, reduce it to a system of linear equations. If Eq. (3. 1) were a Fredholm integral equation, the points x_r could be selected the same with the points t_k. But, in our case, such a selection is evidently impossible because in this case the numerical integration rule (2. 6) is not valid. It is easy to see that, in our case, the most appropriate selection of the points of application of Eq. (3. 2) is to select them as roots of the following, generally transcendental, equation :

$$A(x)\, w(x) - B(x) \frac{q_n(x)}{p_n(x)} = 0. \tag{3. 3}$$

If x_r $(r = 1, 2, \ldots, m)$ are the roots of this equation, we obtain the following system of linear equations :

$$\sum_{k=1}^{n} A_k \left[\frac{B(x_r)}{t_k - x_r} + k(t_k, x_r) \right] \varphi(t_k) = f(x_r), \quad r = 1, 2, \ldots, m. \tag{3. 4}$$

If $m = n$, by solving this system of linear equations we can find the values of the unknown function $\varphi(t)$ at the points t_k $(k = 1, 2, \ldots, n)$ and, afterwards, by using the methods of interpolation, and perhaps of extrapolation too, an approximate expression of the function $\varphi(t)$ along the whole integration interval $[a, b]$.

If $m > n$, then we can take into account only n of the linear equations of the system (3. 4), ignoring the remaining equations, although this technique introduces considerable errors in the values of the function $\varphi(t)$ near the points x_r, for which the corresponding linear equations have been ignored. One better solution of this problem is to change the numerical integration rule used and to use another rule with a reduced number m of points x_r. Such a rule could be, for example, a rule containing among its abscissae t_k the end-points a and b of the integration iuterval.

Finally, if $m < n$, then the system of linear equations (3. 4) must be supplemented by one more linear equation, so that its solution be possible. These supplementary equations are in most cases possible to be found and belong, in general, to two kinds: On the one hand, it is possible that some of the unknown values $\varphi(t_k)$ of the unknown function $\varphi(t)$ at some abscissae t_k be known from physical considerations whence one more equation results, or, better, the number of unknowns in the system of linear equations (3. 4) is reduced by one. On the other hand, in a lot of cases the unknown function $\varphi(t)$ should satisfy, besides Eq. (3. 1), one more integral condition of the form:

$$\int_a^b w(t)\, k_0(t)\, \varphi(t)\, dt = C, \tag{3. 5}$$

where the kernel $k_0(t)$ is assumed to be a continuous function in the interval $[a, b]$, without singularities near the points a or b, and C is a constant. After applying a numerical integration rule of the form (2. 5) for the integral of Eq. (3. 5), we find after ignoring the error term:

$$\sum_{k=1}^{n} A_k\, k_0(t_k)\, \varphi(t_k) = C. \tag{3. 6}$$

This equation can be taken into consideration for the solution of the system of linear equations (3. 4).

We must also remark that, in general, conditions of the form (3. 5) resulting from physical considerations should be taken into account even if, in this way, some of the points x_r should be ignored in the system of linear equations (3. 4).

Although the number m of the points x_r of application of Eq. (3. 1) has been completely investigated in some cases of specific Gauss-, Radau- and Lobatto-type numerical integration rules **[9, 14, 15, 19, 20]**, in general, such an investigation is quite difficult, since it requires to find the number of roots x_r of the generally transcendental equation (3. 3) inside the integration interval (a, b). Here we will restrict ourselves to a simple remark assuring in most cases the existence of a sufficient number of such points x_r. To achieve it, we consider Eq. (2. 6), expressing the general form of a numerical integration rule for the case of a Cauchy type principal value integral, applied for $\varphi(t) \equiv 1$. In this case, the error term E_n vanishes and, because of the definition (2. 9) of the functions $q_n(z)$, we obtain :

$$q_0(x) = \sum_{k=1}^{n} \frac{A_k}{x - t_k} + \frac{q_n(x)}{p_n(x)}. \tag{3. 7}$$

Then Eq. (3. 3), the roots x_r of which are sought, takes the form:

$$A(x)\, w(x) - B(x)\, q_0(x) = B(x) \sum_{k=1}^{n} \frac{A_k}{t_k - x}. \tag{3. 8}$$

Since in most cases all the weights A_k are positive numbers, the functions $A(x)$ and $B(x)$ have been assumed bounded, different from zero and continuous along the whole integration interval and the function $q_0(x)$, defined by Eq. (2. 9) for $n = 0$, satisfies the Hölder condition inside the integration interval (a, b), except its end-points a and b, (since an analogous behaviour has been assumed for the weight function $w(t)$), it is evident that Eq. (3. 8) has an odd number of roots x_r (and probably only one such root) inside each one subinterval (t_k, t_{k+1}) of the integration interval (a, b) defined by two successive abscissae t_k and t_{k+1}. Hence, there exist at least $(n-1)$ roots in total. In some special cases, an analogous reasoning assures the existence of one or two more roots x_r in the subintervals (a, t_1) and (t_n, b) (where $t_1 < t_2 < \ldots < t_n$). Nevertheless, this depends on the behaviour of the weight function $w(t)$ near the end-points a and b of the integration interval.

Finally, we can remark that the developments of this section can be easily generalized to the case when, instead of a single singular integral equation of the form (3. 1), we have to solve a system of such singular integral equations.

4. SOME SPECIAL CASES

Several special cases of the theory developed in sections 2 and 3 and regarding the numerical evaluation of Cauchy type principal value integrals and its application to the numerical solution of Cauchy type singular integral equations are of particular practical interest.

As in the case of regular integrals and the corresponding Fredholm integral equations, we are interested in highly accurate numerical integration rules. Such rules are the rules based on orthogonal polynomials and especially the rules of the Gauss-, Radau- and Lobatto-type, which are exact in the evaluation of regular integrals of the form (2.5) for integrands $\varphi(t)$ polynomials of degrees $(2n-1)$, $(2n-2)$ and $(2n-3)$ respectively **[17]** and in the case of Cauchy type principal value integrals of the form (2.6) for integrands $\varphi(t)$ polynomials of degrees $(2n)$, $(2n-1)$ and $(2n-2)$ respectively. The development of these general and very useful numerical integration rules for the case of Cauchy type principal value integrals and their application to the numerical solution of Cauchy type singular integral equations can be found in Refs. **[16]** and **[12]**.

Some of the rules of the Gauss-, Radau- and Lobatto-type associated with the classical systems of orthogonal polynomials can be used in a lot of practical applications and, in this way, special attention has been paid on them. These rules were derived in Refs. **[16]** and **[12]** and based on the general theory developed in these references. Also their properties during their application to the numerical solution of singular integral equations have been completely investigated, in some cases in separate references. These special highly accurate numerical integration rules and the corresponding methods of numerical solution of singular integral equations are the following:

i) The Gauss-, Radau- and Lobatto-Legendre rules associated with the Legendre polynomials ane their modified forms. These rules assume as integration interval the interval $[-1, 1]$ and as weight function the function $w(t) = 1$ (see Refs. **[16]**, **[12]**, **[14]**).

ii) The Gauss- and Lobatto-Chebyshev rules associated with the Chebyshev polynomials. These rules assume as integration interval the iterval $[-1, 1]$ and as weight function the function $w(t) = (1-t)^{\pm 1/2}(1+t)^{\pm 1/2}$ (see Refs. **[16]**, **[12]**, **[9]**).

iii) The Gauss-, Radau- and Lobatto-Jacobi numerical integration rules associated with the Jacobi polynomials, whose special cases are both the Legendre and the Chebyshev polynomials. These rules assume as integration interval the interval $[-1, 1]$ and as weight function the function $w(t) =$ $= (1 - (1-t)^{\alpha}(1+t)^{\beta}$ $(\alpha, \beta > -1)$ (see Refs. **[16]**, **[12]**, **[15]**, **[18]**, **[19]**, **[20]**).

iv) The Gauss-Laguerre numerical integration rule associated with the Laguerre polynomials. This rule assumes as integration interval the interval $[0, \infty)$ and as weight function the function $w(t) = \exp(-t)$ (see Refs. [**16**], [**12**]).

v) The Gauss-Hermite numerical integration rule and its modified form associated with the Hermite polynomials. This rule assumes as integration interval the interval $(-\infty, \infty)$ and as weight function the function $w(t) = \exp(-t^2)$ (see Refs. [**16**], [**12**], [**23**]).

When using these rules, associated with the classical orthogonal polynomials, to the numerical solution of Cauchy type singular integral equations of the form (3.1), we can completely investigate (under the assumption that the functions $A(x)$ and $B(x)$ reduce to constants) the number and location of the points x_r of application of Eq. (3.1) during its numerical solution, based on a theorem of Sturm's type reported by Porter [**24**]. A complete analysis, based on this theorem, in the case of the Gauss-, Radau- and Lobatto-Legendre numerical integration rules can be found in Ref. [**14**] and in the case of the Gauss-, Radau- and Lobatto-Jacobi numerical integration rules in Refs. [**15**], [**19**] and [**20**].

Furthermore, recently Theocaris and Tsamasphyros [**25**] generalized the rules associated with the Jacobi polynomials to the case when these rules include among their abscissae arbitrary points of the real axis outside the integration interval $[-1, 1]$. Thus, the Radau- and Lobatto-Jacobi rules including among the abscissae used one or both end-points of the integration interval are derived as special cases. Although the results of such a generalization of the previously mentioned rules associated with the Jacobi polynomials is of some theoretical interest, nevertheless, it is of limited usefulness in the case of Cauchy type singular integral equations occurring in practical applications, both because the values of the unknown function in a singular integral equation is seldom of any interest outside the integration interval and because, even in the opposite case, no assurance that a sufficient number of points x_r of application of the singular integral equation inside the integration interval $(-1, 1)$ can be proved.

One can finally mention that in the case of Gauss-, Radau- and Lobatto-type rules, once the unknown function has been determined at the abscissae t_k used in such a rule, then an approximate expression of this function along the whole integration interval can be easily obtained as a series of the orthogonal polynomials associated with the rule in use. Such formulae have been given by Krenk [**26**] in the cases of the Gauss-Chebyshev and the Gauss-Jacobi rule and by Theocaris and Ioakimidis [**16**], [**12**], in the general case of an arbitrary Gauss-, Radau- or Lobatto-type rule.

5. APPLICATION TO CRACK PROBLEMS

The methods of numerical solution of Cauchy type singular integral equations presented in this paper are applicable to a broad class of Applied Mechanics and Engineering problems where Cauchy type singular integral equations may arise.

Among the problems normally reduced to such integral equations are crack problems examined in the framework of Plane Elasticity [**12, 27**]. Several general methods for reducing such a problem to a Cauchy type singular integral equation can be found in Ref. [**12**]. In most cases, the most interesting result from the numerical solution of the Cauchy type singular integral equation associated with a crack problem is the values of the unknown function at the end-points of the crack, which are proportional to the stress intensity factors at these points. In this way, the Radau- and especially the Lobatto-type methods, including among the abscissae used one or both end-points of the integration interval respectively, should be preferred over the Gauss-type rules.

Among the crack problems reduced to Cauchy type singular integral equations and numerically solved in several special cases of practical interest, we can mention the following problems :

i) The problem of a simple smooth curvilinear crack in an infinite isotropic or anisotropic elastic medium (see Refs. [**12**], [**18**]).

ii) The problem of a periodic array of simple smooth curvilinear cracks in an infinite isotropic elastic medium (see Refs. [**12**], [**29**]).

iii) The problem of a doubly-periodic array of simple smooth curvilinear cracks in an infinite isotropic elastic medium (see Refs. [**12**], [**30**]).

iv) The problem of a star-shaped array of simple smooth curvilinear cracks (or a star-shaped crack) in an infinite isotropic elastic medium (see Refs. [**12**], [**31**]).

v) The problem of a symmetrical or asymmetric cruciform crack in an infinite isotropic elastic medium (see Refs. [**12**], [**9**], [**32**]).

vi) The problem of an edge crack in an isotropic elastic half-plane (see Ref. [**12**]).

vii) The problem of a branched crack in an infinite isotropic elastic medium (see Refs. [**12**], [**33**], [**34**]).

viii) The problem of a periodic array of semi-infinite cracks in an infinite isotropic elastic medium (see Refs. [**12**], [**23**]).

One can also notice on crack problems occurring in Plane Elasticity that :

i) All general problems of curvilinear cracks of arbitrary shape cannot be solved in closed-form or by the methods of integral transforms. The most appropriate method for their solution is their reduction to Cauchy type singular integral equations.

ii) The Lobatto- (or Radau-) type methods, when used for the numerical solution of singular integral equations resulting in crack problems, have the advantages that no extrapolation is required for the determination of the stress intensity factors at the crack tips and at the same time the results obtained are much more accurate than the results obtained when using Gauss-type methods, with the same number of abscissae, or methods based on the expansion of the unknown function to a series of orthogonal polynomials. The Lobatto- (or Radau-) type methods for the numerical solution of Cauchy type singular integral equations have been introduced for the first time by the present authors.

Furthermore, one can notice that several other problems of Mathematical Physics, based more or less on the concept of Green's functions (like the problem of the flow past a curvilinear arc), can be reduced to Cauchy type singular integral equations and solved in exactly the same way as crack problems.

6. THE CASE OF CONTOURS

In several applications we have to numerically solve Cauchy type singular integral equations of the form :

$$A(t_0)\, g(t_0) + B(t_0) \int_L \frac{g(t)}{t-t_0}\, dt + \int_L K(t, t_0)\, g(t)\, dt = f(t_0) \tag{6.1}$$

along a contour L. In Eq. (6. 1) the kernel $K(t, t_0)$ and the right-hand side function $f(t_0)$ are assumed to be regular functions of the points t, t_0 of the contour L.

In order to numerically solve a Cauchy type singular integral equation of the form (6. 1), we are based on the modified form of the trapezoidal rule for periodic functions [**35**] :

$$\int_0^{2\pi} g(s)\, ds = \frac{\pi}{n} \sum_{k=0}^{2n-1} g(s_k) + E_n\,, \qquad s_k = \frac{k\pi}{n}\,, \tag{6.2}$$

and the corresponding formula valid for the case of Cauchy type principal value integrals of periodic functions and derived by Chawla and Ramakrishnan [**36**] :

$$\int_0^{2\pi} g(s) \cot \frac{s-\sigma}{2}\, ds = \frac{\pi}{n} \sum_{k=0}^{2n-1} g(s_k) \cot \frac{s_k - \sigma}{2} + 2\pi g(\sigma) \cot n\sigma\,, \qquad s_k = \frac{k\pi}{n}. \tag{6.3}$$

Since the function $g(t)$ in Eq. (6. 1) can be considered as a periodic function of a real variable s or σ (e. g. the arc-length) varying along the contour L, one can solve Eq. (6. 1) on the basis of Eqs. (6. 2) and (6. 3) in a completely analogous way to that used for the case of Eq. (3. 1) and developed in section 3. Thus, when selecting the abscissae s_k along L as shown in Eqs. (6. 2) and (6. 3), it is sufficient to select the points σ_r of application of Eq. (6. 1) in such a way that :

$$A\big(t_0(\sigma_r)\big) + \pi B\big(t_0(\sigma_r)\big) \cot n\sigma_r = 0\,. \tag{6.4}$$

On this point, one can notice that Eq. (6. 4) is analogous to Eq. (3. 3), valid for the case of Cauchy type singular integral equations along a part of the real axis.

Although the above-described method of numerical solution of Eq. (6. 1) is applicable only when the ratio $B(t_0)/A(t_0)$ is a real function, nevertheless special techniques can be used even when this ratio is not a real function. For example, when $A(t_0)$ is a real constant and $B(t_0)$ an imaginary constant (or vice-versa), one can apply Eq. (6. 1) at first by using the abscissae s_k and the points of application σ_r given by:

$$s_k = \frac{k\pi}{n}, \quad \sigma_r = \frac{(2r-1)\pi}{2n}, \quad k, r = 0, 1, \ldots, 2n-1, \tag{6.5}$$

and then by using the points σ_r as abscissae and the points s_k as points of application of Eq. (6. 1). In this way, one obtains a system of $2n$ linear equations immediately reduced to a system of n linear equations as in the case when $A(t_0)$ and $B(t_0)$ were real functions.

The method of numerical solution of Cauchy type singular integral equations along contours was applied to the numerical solution of the first fundamental problem of Plane Elasticity for a finite region bounded by a smooth contour of arbitrary shape, or an infinite region with a hole determined by a contour of arbitrary shape, with quite satisfactory results [**37**].

7. THE CASE OF COMPLEX SINGULARITIES

The numerical solution of Cauchy type singular integral equations of the form (3. 1) in the case when either then functions $A(x)$ and $B(x)$ or the weight function $w(x)$ are complex is of particular interest in a lot of cases. Since the investigation of Eq. (3. 1) in this general case is difficult, we will limit ourselves to the case of Eq. (1. 5), under restrictions (1. 6) and (1. 7). In this case, the Gauss-, Radau- and Lobatto-Jacobi rules can be used, at least when the ratio A/B is a real constant.

In the opposite case, it was proved by Theocaris and Ioakimidis [**38**], both theoretically and through numerical applications, that it is permissible (under some usually fulfilled conditions) to use as points x_r of application of Eq. (1. 5) points lying in the complex plane outside the integration interval. That is, although in general no points of application of Eq. (1. 5) can be determined in the integration interval $(-1, 1)$, since the weight function $w(x)$ as well as the corresponding Jacobi polynomials result to have complex coefficients, nevertheless, we can work exactly as in the case of a real weight function $w(x)$ and Jacobi polynomials with real coefficients without any change in formulae used.

It may be noted that this technique, which seems a little strange, has been used for the first time by the present authors and proved quite effective in the numerical solution of Eq. (1.5) with complex singularities. Such equations could be solved up to now only after an expansion of the unknown function to a series of the corresponding Jacobi polynomials, as proposed by Erdogan, Gupta and Cook [**6**, **7**]. The method proposed by these authors, compared to the method proposed here, requires much more computational effort and is much less accurate for the same number of linear equations used. Of course, in the case when the values of the unknown function at the end-points of the integration interval are of interest (e. g. when stress intensity factors are to be computed), the Lobatto-Jacobi rule should be preferred over the Gauss-Jacobi rule. In this case, the end-points of the integration interval are included among the abscissae used although all other abscissae and points of application of Eq. (1.5) are, in general, complex numbers and lie outside the integration interval.

We could add on this point that Theocaris and Tsamasphyros [**25**] proposed in the case of complex singularities that either the points of application of Eq. (1.5) be arbitrarily selected in the integration interval $(-1, 1)$ and interpolation methods be used for the expression of the unknown function at these points, or the imaginary part of the complex singularities be ignored. Of course, it is evident that the method proposed here should be preferred over both these methods.

Finally, the case of more general classes of Cauchy type singular integral equations with complex singularities is under investigation and it is hoped that the extension of the method proposed here to the numerical solution of such equations (based on their application at points of the complex plane generally lying outside the integration interval), will be possible.

8. CONCLUSIONS

In this section, we will in brief summarize the results obtained by the present authors in this paper, as well as in the papers previously referenced, on the subjects of numerical evaluation of Cauchy type principal value integrals, of numerical solution of Cauchy type singular integral equations and of the determination of stress intensity factors in Plane Elasticity crack problems :

i) It was shown that the previously available methods of numerical solution of Cauchy type singular integral equations, developed in Refs. [4] to [8] and reducing such an equation to a system of linear equations with unknowns the values of the unknown function at the abscissae used were exact for integrands polynomials of degree up to $(2n-1)$ and not $(n-1)$ as it was believed. In this way, they were completely equivalent to the corresponding methods based on Gaussian integrations rules.

ii) The same method used in Refs. [4] to [7] for the derivation of the Gauss-Chebyshev method of numerical solution of Cauchy type singular integral equations was used for the derivation of the Lobatto-Chebyshev method of numerical solution of singular integral equations, accurate for integrands polynomials of degree up to $(2n-3)$.

iii) The Lobatto-type methods of numerical solution of Cauchy type singular integral equations are very convenient for the numerical solution of such equations arising in Plane Elasticity crack problems, since, in this way, the determination of the stress intensity factors at the crack tips can be made easily and accurately.

iv) It was shown that all usually used numerical integration rules for regular integrals, with an arbitrary number and location of their abscissae, can be extended to the case of Cauchy type principal value integrals, with no change in their abscissae and weights if one more term, corresponding to the pole of the integrand inside the integration interval, be taken into account.

v) By an appropriate choice of the points of application of a Cauchy type singular integral equation, the additional term mentioned previously vanishes.

In this way, such an equation can be numerically solved, as if it were a regular Fredholm integral equation, by reduction to a system of linear equations, possibly supplemented by one or more additional conditions.

vi) The Gauss-, Radau- and Lobatto-type numerical integration rules are, in general, the best rules to be used for the numerical solution of Cauchy type singular integral equations, in the same way as in the case of Fredholm integral equations. By using special forms of interpolation formulae, the unknown function in a Cauchy type singular integral equation, once determined at the abscissae used, can be expressed in a polynomial form along the whole integration interval.

vii) Several elementary theorems can be used for the determination of a minimum number of points of application of a Cauchy type singular integral equation. In the case of rules of the Gauss-, Radau- and Lobatto-type and associated with the classical systems of orthogonal polynomials, an exact investigation of the number and location of the points of application of a Cauchy type singular integral equation can be made, under the additional assumption that the ratio of the coefficients of the free term and the Cauchy type principal value term of this equation is a constant. In this frequently encountered case, it can be shown that the points of application of the integral equation alternate with the abscissae used.

viii) In the case of Cauchy type singular integral equations along contours, the modified form of the trapezoidal rule for regular and Cauchy type principal value integrals of periodic functions can be used in such a way that the numerical solution of such an equation can be made in a manner similar to that used for singular integral equations with an integration interval lying on the real axis.

ix) In the case of Cauchy type singular integral equations with complex singularities, it is possible, for a sufficiently broad class of equations of this type, that the points of their application used lie, in general, outside the integration interval. The same is true for the abscissae used. In this case, the methods used for Cauchy type singular integral equations with real singularities can be extended, without modifications, to the case of Cauchy type singular integral equations with complex singularities.

x) The methods of numerical solution of Cauchy type singular integral equations presented in this paper, as well as in a series of papers by the present authors, make it clear that the numerical solution of such an equation is by no means impossible and in most cases it requires about the same amount of computational effort as the numerical solution of a regular Fredholm integral equation. Thus, it is believed that in future a lot of problems of Mathematical Physics will be solved by reduction to Cauchy type singular integral equations, which combine accuracy and effectiveness, and not by the presently used less accurate (like the finite-elements) or less effective (like the integral transforms) methods.

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ΝΕΑΙ ΜΕΘΟΔΟΙ ΑΡΙΘΜΗΤΙΚΗΣ ΕΠΙΛΥΣΕΩΣ ΙΔΙΟΜΟΡΦΩΝ ΟΛΟΚΛΗΡΩΤΙΚΩΝ ΕΞΙΣΩΣΕΩΝ

Μία ἐκ τῶν μεθόδων διὰ τὴν ἀριθμητικὴν ἐπίλυσιν ὁλοκληρωτικῶν ἐξισώσεων τύπου Fredholm συνίσταται εἰς τὴν ἀναγωγὴν μιᾶς τοιαύτης ἐξισώσεως εἰς σύστημα γραμμικῶν ἐξισώσεων, ἔνθα τὰ ἐνυπάρχοντα εἰς τὴν ἐξίσωσιν ὁλοκληρώματα προσεγγίζονται δι᾽ ἀθροισμάτων, κατόπιν ἐφαρμογῆς καταλλήλων μεθόδων ἀριθμητικῆς ὁλοκληρώσεως, καὶ ἡ ἐξίσωσις ἐφαρμόζεται εἰς τὰς τετμημένας τὰς χρησιμοποιηθείσας εἰς τὴν μέθοδον ἀριθμητικῆς ὁλοκληρώσεως.

Εἰς τὴν περίπτωσιν ὅπου ὁ πυρὴν τῆς ὁλοκληρωτικῆς ἐξισώσεως τύπου Fredholm, πρώτου ἢ δευτέρου εἴδους, ἀποτελεῖται ἀπὸ ὁμαλὸν ὅρον καὶ ἰδιόμορφον ὅρον τύπου Cauchy, ὁμιλοῦμεν περὶ ἰδιομόρφου ὁλοκληρωτικῆς ἐξισώσεως τύπου Cauchy. Εἰς τὴν περίπτωσιν ταύτην ἡ προηγουμένως ἀναφερθεῖσα μέθοδος ἀριθμητικῆς ἐπιλύσεως ἐπιστεύετο ὅτι δὲν ἠδύνατο νὰ ἐφαρμοσθῇ. Ἡ ὑπάρχουσα τεχνικὴ διὰ τὴν ἀριθμητικὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων (τύπου Cauchy) συνίστατο εἰς τὴν ἀναγωγήν των εἰς ἰσοδυνάμους ὁλοκληρωτικὰς ἐξισώσεις τύπου Fredholm δευτέρου εἴδους καὶ εἰς τὴν ἐν συνεχείᾳ ἀριθμητικὴν ἐπίλυσιν τῶν τελευταίων. Ἡ μέθοδος αὕτη εἰσαχθεῖσα ὑπὸ τοῦ Muskhelishvili, ἐγενικεύθη ὑπὸ τοῦ Pogorzelski διὰ πολυπλοκωτέρας περιπτώσεις ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων. Οὗτος ἀπέδειξεν ὅτι ἡ ἀναγωγὴ ἰδιομόρφου ὁλοκληρωτικῆς ἐξισώσεως πρώτου ἢ δευτέρου εἴδους εἰς ὁλοκληρωτικὴν ἐξίσωσιν τύπου Fredholm δευτέρου εἴδους δὲν εἶναι πάντοτε δυνατή. Σημειοῦται ἐνταῦθα ὅτι αἱ ἐν χρήσει εἰς τὴν Μηχανικὴν ἰδιόμορφοι ὁλοκληρωτικαὶ ἐξισώσεις, διὰ τὰς ὁποίας ἐνδιαφερόμεθα ἡμεῖς ἐνταῦθα, ἀνήκουν εἰς τὴν κατηγορίαν αὐτήν.

Ἐξ ἄλλου, διάφοροι ἐρευνηταὶ κατώρθωσαν νὰ ἀναγάγουν ὡρισμένας ἰδιομόρφους ὁλοκληρωτικὰς ἐξισώσεις τύπου Cauchy εἰς συστήματα γραμμικῶν ἐξισώσεων, διὰ χρήσεως μεθόδων ἀριθμητικῆς ὁλοκληρώσεως διὰ τὸν ὑπολογισμὸν κυρίων τιμῶν ὁλοκληρωμάτων τύπου Cauchy, τῇ βοηθείᾳ ἀναπτύξεως τῆς ὁλοκληρωτέας ποσότητος εἰς ὀρθογώνια πολυώνυμα. Οὕτω ὁ Kalandiya εἰς τὴν Σοβιετικὴν Ἕνωσιν ἀνέπτυξε σχετικὴν μέθοδον διὰ τὴν ἀριθμητικὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων πρώτου εἴδους τοῦ τύπου :

$$\frac{1}{\pi}\int_{-1}^{1} w(t)\,\frac{\varphi(t)}{t-x}\,dt+\int_{-1}^{1} w(t)\,k(t,x)\,\varphi(t)\,dt=f(x),\quad -1<x<1, \qquad (\text{E.}1)$$

ὅπου ἡ συνάρτησις $w(t)$ εἶναι ἡ συνάρτησις βάρους, ἡ ὁποία ἐθεωρήθη ἔχουσα τὴν μορφήν :

$$w(t) = (1-t)^{\pm 1/2}(1+t)^{\pm 1/2}, \tag{E.2}$$

$\varphi(t)$ εἶναι ἡ ἄγνωστος συνάρτησις, θεωρουμένη ὁμαλὴ εἰς τὸ διάστημα ὁλοκληρώσεως $[-1, 1]$, $k(t, x)$ ὁμαλὸς πεφραγμένος πυρὴν ἐντὸς τοῦ διαστήματος ὁλοκληρώσεως καὶ $f(x)$ ὁμαλὴ συνάρτησις ἐντὸς τοῦ αὐτοῦ διαστήματος.

Ἐπίσης οἱ Erdogan καὶ Gupta εἰς τὰς Η.Π.Α. ἀνέπτυξαν μέθοδον ἀριθμητικοῦ ὑπολογισμοῦ κυρίων τιμῶν ὁλοκληρωμάτων τύπου Cauchy τῆς μορφῆς :

$$I = \int_{-1}^{1} w(t)\frac{\varphi(t)}{t-x}\,dt, \tag{E.3}$$

μὲ τὴν αὐτὴν συνάρτησιν βάρους $w(t)$. Ἐφήρμοσαν δὲ τὴν μέθοδον ταύτην διὰ τὴν ἀναγωγὴν τῆς ἐξισώσεως (Ε. 1) εἰς σύστημα γραμμικῶν ἐξισώσεων τῆς μορφῆς :

$$\sum_{k=1}^{n} A_k\left[\frac{1}{\pi(t_k - x_r)} + k(t_k, x_r)\right]\varphi(t_k) = f(x_r), \quad r = 1, 2, \ldots, m, \tag{E.4}$$

ὅπου t_k εἶναι αἱ τετμημέναι καὶ A_k τὰ βάρη τὰ χρησιμοποιούμενα εἰς τὴν μέθοδον ἀριθμητικῆς ὁλοκληρώσεως Gauss-Chebyshev διὰ κοινὰ ὁλοκληρώματα καὶ x_r εἶναι καταλλήλως ἐπιλεγόμενα σημεῖα εἰς τὸ διάστημα ὁλοκληρώσεως (λαμβανόμενα ὡς ρίζαι πολυωνύμων Chebyshev), ὁ ἀριθμὸς m τῶν ὁποίων δύναται νὰ εἶναι ἴσος πρὸς $(n-1)$, n ἢ $(n+1)$, τοῦ ἀριθμοῦ n δηλοῦντος τὸ πλῆθος τῶν τετμημένων, αἱ ὁποῖαι ἐλήφθησαν εἰς τὸ διάστημα ὁλοκληρώσεως. Εἰς τὴν περίπτωσιν ὅπου $m = n-1$, ἀπαιτεῖται μία ἐπὶ πλέον γραμμικὴ σχέσις, προκύπτουσα ἔκ τινος φυσικῆς συνθήκης τοῦ προβλήματος, ἡ ὁποία συμπληρώνει τὸ σύστημα τῶν γραμμικῶν ἐξισώσεων.

Ἐν τούτοις, οἱ Erdogan καὶ Gupta δὲν ἀντελήφθησαν ὅτι ἡ μέθοδός των ἀριθμητικῆς ὁλοκληρώσεως ἦτο πράγματι ἡ μέθοδος Gauss - Chebyshev, ἡ ὁποία εἶναι ἀκριβὴς δι᾽ ὁλοκληρώματα μὲ πολυώνυμα βαθμοῦ μέχρι $2n$, καὶ ἐπίστευον ὅτι ἦτο ἀκριβὴς δι᾽ ὁλοκληρώματα μὲ πολυώνυμα βαθμοῦ μέχρι $(n-1)$. Ἡ παρεξήγησις αὐτὴ συνεχίσθη εἰς σειρὰν δημοσιεύσεων ὑπὸ τοῦ καθηγητοῦ Erdogan καὶ τῶν συνεργατῶν του, συμπεριελήφθη δὲ καὶ εἰς δημοσίευσιν τοῦ Δανοῦ ἐρευνητοῦ Krenk, γενικεύσαντος τὰ ἀποτελέσματα τῶν Erdogan καὶ Gupta εἰς πολυπλοκωτέρας ὁλοκληρωτικὰς ἐξισώσεις τῆς μορφῆς :

$$Aw(x)\,\varphi(x) + \frac{B}{\pi}\int_{-1}^{1} w(t)\frac{\varphi(t)}{t-x}\,dt + \int_{-1}^{1} w(t)\,k(t,x)\,\varphi(t)\,dt = f(x) \tag{E.5}$$

μὲ συνάρτησιν βάρους $w(t)$ τῆς μορφῆς:

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}, \qquad (E.6)$$

ὅπου:

$$a = \frac{1}{2\pi i}\ln\frac{A-iB}{A+iB}+N,\ \beta = -\frac{1}{2\pi i}\ln\frac{A-iB}{A+iB}+M,\ \alpha+\beta=M+N=-\varkappa, \qquad (E.7)$$

ὅπου N καὶ M εἶναι ἀκέραιοι ἀριθμοί, ὁ δὲ δείκτης $\varkappa$ λαμβάνει τὰς τιμὰς (-1), 0 καὶ 1.

Ἀναφέρομεν ὅτι ἡ τεχνικὴ τοῦ Krenk ἐβασίσθη εἰς ἰδιότητας τῶν πολυωνύμων Jacobi καὶ ὅτι οὗτος συνήγαγε, χωρὶς νὰ τὸ ἀντιληφθῇ, τὴν λεγομένην Gauss-Jacobi μέθοδον ἀριθμητικῆς ὁλοκληρώσεως διὰ κυρίας τιμὰς ἰδιομόρφων ὁλοκληρωμάτων τύπου Cauchy,

Τελευταίως, ὁ ὁμιλῶν μετὰ τοῦ κ. Ἰωακειμίδη ἀπέδειξαν, χρησιμοποιήσαντες τὴν αὐτὴν μέθοδον μὲ τὸν Erdogan καὶ τοὺς συνεργάτας του, ὅτι αἱ μέθοδοι ἀριθμητικοῦ ὑπολογισμοῦ τῆς κυρίας τιμῆς ἰδιομόρφων ὁλοκληρωμάτων τύπου Cauchy, αἱ χρησιμοποιηθεῖσαι ὑπὸ τοῦ Erdogan καὶ τῶν συνεργατῶν του, ἦσαν ἀκριβεῖς δι' ὁλοκληρωτέας συναρτήσεις $\varphi(t)$ πολυώνυμα βαθμοῦ μέχρι $2n$ καὶ οὐχὶ μέχρι $(n-1)$, ὡς παρεδέχοντο ὁ Erdogan καὶ οἱ συνεργάται του, ἀποδείξαντες οὕτως ὅτι πράγματι ἡ τεχνικὴ ἡ ἐφαρμοσθεῖσα ὑπὸ τούτων συνέπιπτε μὲ τὴν μέθοδον Gauss-Chebyshev καὶ γενικώτερον μὲ τὴν μέθοδον Gauss-Jacobi.

Ἐξ ἄλλου ἄλλοι ἐρευνηταί, ὡς ὁ Hunter εἰς τὴν Ἀγγλίαν καὶ οἱ Chawla καὶ Ramakrishnan εἰς τὴν Ἰνδίαν, ἐγενίκευσαν τὰς μεθόδους ἀριθμητικῆς ὁλοκληρώσεως Gauss-Legendre ἀφ' ἑνός, καὶ Gauss-Jacobi καὶ Gauss-Chebyshev ἀφ' ἑτέρου, διὰ τὸν ἀριθμητικὸν προσδιορισμὸν κυρίων τιμῶν ὁλοκληρωμάτων τύπου Cauchy. Περαιτέρω, ὁ ὁμιλῶν μετὰ τοῦ κ. Ἰωακειμίδη ἀνέπτυξαν γενικὴν μέθοδον ἀριθμητικοῦ ὑπολογισμοῦ ὁλοκληρωμάτων τύπου Cauchy, ἐπιτρέπουσαν τὴν ἐφαρμογὴν οἱασδήποτε μεθόδου ἀριθμητικῆς ὁλοκληρώσεως ἐφαρμοζομένης εἰς κοινὰ ὁλοκληρώματα καὶ εἰς ἰδιόμορφα ὁλοκληρώματα.

Οὕτως ἐπετεύχθη ἀριθμητικὴ ἐπίλυσις ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τῆς μορφῆς:

$$A(x)\,w(x)\,\varphi(x) + B(x)\int_{-1}^{1} w(t)\,\frac{\varphi(t)}{t-x}\,dt + \int_{-1}^{1} w(t)\,k(t,x)\,\varphi(t)\,dt = f(x), \qquad (E.8)$$

ὅπου αἱ $A(x)$ καὶ $B(x)$ εἶναι πεφραγμέναι συνεχεῖς συναρτήσεις εἰς τὸ διάστημα ὁλοκληρώσεως $[-1, 1]$, $k(t, x)$ εἶναι ὁμαλὸς πεφραγμένος πυρὴν εἰς τὸ αὐτὸ διάστημα ὡς πρὸς ἀμφοτέρας τὰς μεταβλητὰς t καὶ x, ἐκτὸς τῶν τιμῶν $x = \pm 1$, καὶ $f(x)$ γνωστὴ συνάρτησις, πιθανῶς παρουσιάζουσα ἀσθενεῖς ἰδιομορφίας παρὰ τὰ ἄκρα τοῦ διαστήματος ὁλοκληρώσεως.

Συμφώνως πρὸς τὴν μέθοδον ταύτην ἡ ἀνωτέρω ἐξίσωσις δύναται νὰ ἀναχθῇ εἰς τὸ κάτωθι σύστημα γραμμικῶν ἐξισώσεων :

$$\sum_{k=1}^{n} A_k \left[\frac{B(x_r)}{t_k - x_r} + k(t_k, x_r) \right] \varphi(t_k) = f(x_r), \qquad r = 1, 2, \ldots, m, \qquad \text{(E. 9)}$$

ἀρκεῖ τὰ σημεῖα x_r ἐφαρμογῆς τῆς ὁλοκληρωτικῆς ἐξισώσεως νὰ ἔχουν καταλλήλως ἐπιλεγῆ.

Ἐν συνεχείᾳ ἐδείχθη ὅτι ἡ ἐκλογὴ τῶν σημείων x_r ἐφαρμογῆς τῆς ὁλοκληρωτικῆς ἐξισώσεως, ὡς προετάθη ὑπὸ τῶν Erdogan, Gupta καὶ Cook διὰ τὴν εἰδικὴν περίπτωσιν ὅπου $A(x) \equiv 0$, δὲν ἦτο ἡ ὀρθή, ὁδηγοῦσα εἰς βραδυτάτην σύγκλισιν τῶν ἀποτελεσμάτων.

Ἐκτὸς τῶν μεθόδων ἀριθμητικῆς ὁλοκληρώσεως Gauss - Legendre, Gauss - Chebyshev καὶ Gauss - Jacobi, ἐχρησιμοποιήθησαν καὶ ἕτεραι μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως διὰ τὴν ἐπίλυσιν τῆς κατωτέρω γενικωτάτης ἰδιομόρφου ὁλοκληρωτικῆς ἐξισώσεως :

$$A(x)\, w(x)\, \varphi(x) + B(x) \int_a^b w(t) \frac{\varphi(t)}{t - x}\, dt + \int_a^b w(t)\, k(t, x)\, \varphi(t)\, dt = f(x), \qquad a < x < b, \qquad \text{(E. 10)}$$

ὅπου ἡ συνάρτησις βάρους $w(x)$ προσδιορίζεται καταλλήλως εἰς τὸ διάστημα ὁλοκληρώσεως $[a, b]$.

Παρὰ τὸ γεγονὸς ὅτι αἱ μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως τύπου Gauss φαίνονται νὰ εἶναι αἱ καταλληλότεραι μέθοδοι ἀριθμητικοῦ ὑπολογισμοῦ κυρίων τιμῶν ἰδιομόρφων ὁλοκληρωμάτων τύπου Cauchy, ἐν τούτοις εἰς πολλὰς περιπτώσεις κατὰ τὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ἐνδιαφέρει εἰδικῶς ἡ τιμὴ τῆς ἀγνώστου συναρτήσεως $\varphi(t)$ εἰς τὰ ἀκραῖα σημεῖα a καὶ b τοῦ διαστήματος ὁλοκληρώσεως τῆς ἰδιομόρφου ὁλοκληρωτικῆς ἐξισώσεως. Εἰς τὴν περίπτωσιν αὐτήν, ἐὰν χρησιμοποιηθοῦν αἱ μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως τύπου Gauss, ἀπαιτεῖται νὰ ἐφαρμοσθῇ μέθοδός τις παρεμβολῆς, ἢ μᾶλλον προεκβολῆς, διὰ τὸν καθορισμὸν τῶν τιμῶν $\varphi(a)$ καὶ $\varphi(b)$ τῆς ἀγνώστου συναρτήσεως $\varphi(t)$ εἰς τὰ ἄκρα a καὶ b τοῦ διαστήματος ὁλοκληρώσεως. Ἀλλά, δεδομένου ὅτι αἱ μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως τύπου Gauss εἶναι ἀκριβεῖς διὰ συναρτήσεις $\varphi(t)$ πολυώνυμα βαθμοῦ μέχρι καὶ $(2n - 1)$, ἐνῷ αἱ μέθοδοι τῆς παρεμβολῆς ἢ προεκβολῆς εἶναι ἀκριβεῖς διὰ συναρτήσεις $\varphi(t)$ πολυώνυμα βαθμοῦ μέχρι $(n-1)$ μόνον, εἶναι προφανὲς ὅτι αἱ μέθοδοι παρεμβολῆς ἢ προεκβολῆς εἰσάγουν σημαντικὰ σφάλματα εἰς τὰς τιμὰς $\varphi(a)$ καὶ $\varphi(b)$ τῆς συναρτήσεως $\varphi(t)$, αἱ ὁποῖαι παρουσιάζουν εἰδικὸν ἐνδιαφέρον.

Εἰς τὴν περίπτωσιν αὐτὴν ἡ χρησιμοποίησις τῶν μεθόδων ἀριθμητικῆς ὁλοκληρώσεως τύπου Radau, ἢ ἡμικλείστου τύπου, περιλαμβανουσῶν μεταξὺ τῶν τετμημένων των ἓν ἐκ τῶν ἄκρων τοῦ διαστήματος ὁλοκληρώσεως a ἢ b, ἢ τῶν μεθόδων τύπου Lobatto, ἢ κλειστοῦ τύπου, περιλαμβανουσῶν μεταξὺ τῶν τετμημένων των ἀμφότερα τὰ ὅρια ὁλοκληρώσεως, παρουσιάζεται ὡς ἡ καλυτέρα δυνατὴ λύσις, καθ᾽ ὅσον ἡ ἀκρίβεια τῆς ἀριθμητικῆς ὁλοκληρώσεως διὰ τῶν μεθόδων αὐτῶν ἐλαττοῦται ὀλίγον μόνον, ἀφοῦ αἱ μέθοδοι αὗται εἶναι ἀκριβεῖς διὰ συναρτήσεις $\varphi(t)$ πολυώνυμα βαθμοῦ μέχρι $(2n-2)$ καὶ $(2n-3)$ ἀντιστοίχως.

Αἱ μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως τύπου Radau καὶ Lobatto εἰσήχθησαν διὰ τὸν ὑπολογισμὸν κυρίων τιμῶν ὁλοκληρωμάτων τύπου Cauchy τὸ πρῶτον ὑπὸ τοῦ ὁμιλοῦντος καὶ τοῦ κ. Ἰωακειμίδη εἰς σειρὰν ἤδη δημοσιευθέντων ἄρθρων εἰς τὸ ἐξωτερικόν.

Εἰς τὴν ἀνακοίνωσιν αὐτὴν παρουσιάζεται γενικὴ μέθοδος ἀναγωγῆς ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τῆς γενικῆς μορφῆς (Ε. 10) εἰς συστήματα γραμμικῶν ἐξισώσεων τῆς μορφῆς (Ε. 9) κατὰ τρόπον ἀνεξάρτητον τῆς χρησιμοποιουμένης μεθόδου ἀριθμητικῆς ὁλοκληρώσεως, ἥτις θεωρεῖται γνωστὴ ἐκ τῶν προτέρων. Ἐκ τῆς μεθόδου αὐτῆς προέκυψαν γενικεύσεις ἐφαρμοζόμεναι εἰς τὴν ἀριθμητικὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τύπου Cauchy εἰδικῶν μορφῶν, ὅπως π. χ. ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων κατὰ μῆκος κλειστῶν καμπύλων ἢ ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων συνδεομένων μὲ μιγαδικὰς ἰδιομορφίας εἰς τὰ ἄκρα τοῦ διαστήματος ὁλοκληρώσεως.

Διὰ τῆς διεξοδικῆς ταύτης μελέτης, ἡ ὁποία ἐπεξετάθη διὰ πρώτην φορὰν καὶ εἰς τὴν λίαν ἐνδιαφέρουσαν περίπτωσιν, ὅπου αἱ ἰδιόμορφοι ὁλοκληρωτικαὶ ἐξισώσεις περιέχουν μιγαδικὰς ἰδιομορφίας εἰς τὰ ἄκρα τοῦ διαστήματος ὁλοκληρώσεως, προέκυψαν τὰ ἑξῆς κυρίως συμπεράσματα :

1) Ἀπεδείχθη ὅτι αἱ προϋπάρχουσαι μέθοδοι ἀριθμητικοῦ ὑπολογισμοῦ κυρίων τιμῶν ἰδιομόρφων ὁλοκληρωμάτων τύπου Cauchy καὶ ἐφαρμοζόμεναι εἰς τὴν ἀριθμητικὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τύπου Cauchy εἶναι ἀκριβεῖς δι᾽ ὁλοκληρουμένας συναρτήσεις πολυώνυμα βαθμοῦ $(2n-1)$ καὶ οὐχὶ $(n-1)$, ὡς ἐπιστεύετο μέχρι σήμερον. Κατὰ συνέπειαν εἶναι ἀπολύτως ἰσοδύναμοι πρὸς τὰς ἀντιστοίχους μεθόδους τὰς βασιζομένας εἰς μεθόδους ὁλοκληρώσεως τύπου Gauss.

2) Ἡ αὐτὴ διαδικασία ἡ ἀναπτυχθεῖσα πρὸς εὕρεσιν τῆς μεθόδου Gauss-Chebyshev ἐχρησιμοποιήθη καὶ διὰ τὴν ἀνάπτυξιν τῆς μεθόδου ἀριθμητικῆς ἐπιλύ-

σεως ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων Lobatto - Chebyshev, ἡ ὁποία εἶναι ἀκριβὴς διὰ πολυώνυμα βαθμοῦ μέχρι $(2n - 3)$.

3) Αἱ μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως τύπου Lobatto εἶναι κατάλληλοι διὰ τὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων, αἱ ὁποῖαι ἀπαντῶνται εἰς προβλήματα ἐλαστικότητος, καὶ δὴ ρηγματωμένων πλακῶν, καθ' ὅσον αὗται ἐπιτρέπουν τὸν ἀπ' εὐθείας, ἄνευ προεκβολῆς, προσδιορισμὸν τῶν συντελεστῶν ἐντάσεως τάσεων εἰς τὰ ἄκρα τῶν ρωγμῶν.

4) Ἀπεδείχθη ὅτι ὅλαι αἱ μέθοδοι, αἱ ὁποῖαι συνήθως χρησιμοποιοῦνται διὰ τὸν ἀριθμητικὸν ὑπολογισμὸν συνήθων ὁλοκληρωμάτων, μὲ τυχαῖον ἀριθμὸν καὶ θέσιν τῶν τετμημένων των, δύνανται νὰ ἐπεκταθοῦν καὶ εἰς τὴν περίπτωσιν κυρίων τιμῶν ἰδιομόρφων ὁλοκληρωμάτων τύπου Cauchy ἄνευ ἀλλαγῆς τῶν τετμημένων καὶ τῶν βαρῶν, ἀρκεῖ μόνον νὰ ληφθῇ ὑπ' ὄψιν εἷς ἐπὶ πλέον ὅρος, ἀντιστοιχῶν εἰς τὸν πόλον τῆς ὁλοκληρουμένης συναρτήσεως ἐντὸς τοῦ διαστήματος ὁλοκληρώσεως.

5) Διὰ καταλλήλου ἐκλογῆς τῶν σημείων ἐφαρμογῆς τῶν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ὁ ἐπὶ πλέον οὗτος ὅρος δύναται νὰ μηδενισθῇ. Κατ' αὐτὸν τὸν τρόπον αἱ τοιαῦται ἐξισώσεις δύνανται νὰ ἐπιλυθοῦν ἀριθμητικῶς, ὡς ἐὰν ἦσαν κοιναὶ ὁλοκληρωτικαὶ ἐξισώσεις τύπου Fredholm, δι' ἀναγωγῆς εἰς συστήματα γραμμικῶν ἐξισώσεων δυνάμενα κατὰ περίπτωσιν νὰ συμπληρωθοῦν διὰ μιᾶς ἢ περισσοτέρων προσθέτων συνθηκῶν.

6) Αἱ μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως τῶν τύπων Gauss, Radau καὶ Lobatto ἀποτελοῦν κατ' ἀρχὴν τὰς καλυτέρας μεθόδους διὰ τὴν ἀριθμητικὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων, ὡς ἀκριβῶς ἰσχύει καὶ διὰ συνήθεις ὁλοκληρωτικὰς ἐξισώσεις. Δι' ἐφαρμογῆς εἰδικῶν τύπων παρεμβολῆς ἡ ἄγνωστος συνάρτησις εἰς τὴν ἰδιόμορφον ὁλοκληρωτικὴν ἐξίσωσιν τύπου Cauchy, εὐθὺς ὡς προσδιορισθῇ εἰς ὡρισμένας τετμημένας, δύναται νὰ ἐκφρασθῇ ὑπὸ μορφὴν πολυωνύμου ἐφ' ὅλου τοῦ διαστήματος ὁλοκληρώσεως.

7) Διάφορα στοιχειώδη θεωρήματα δύνανται νὰ ἐφαρμοσθοῦν διὰ τὸν καθορισμὸν τοῦ ἐλαχίστου ἀριθμοῦ τῶν σημείων ἐφαρμογῆς μιᾶς ἰδιομόρφου ὁλοκληρωτικῆς ἐξισώσεως τύπου Cauchy. Εἰς τὴν περίπτωσιν τῶν μεθόδων Gauss, Radau καὶ Lobatto ἐν συνδυασμῷ μὲ τὰ κλασσικὰ συστήματα ὀρθογωνίων πολυωνύμων δύναται νὰ ἐπιτευχθῇ ὁ πλήρης καθορισμὸς τοῦ ἀριθμοῦ καὶ τῆς θέσεως τῶν σημείων ἐφαρμογῆς τῆς ἰδιομόρφου ὁλοκληρωτικῆς ἐξισώσεως ὑπὸ τὴν πρόσθετον προϋπόθεσιν ὅτι ὁ λόγος τῶν συντελεστῶν τοῦ ἐλευθέρου ὅρου καὶ τοῦ ὁλοκληρώματος τύπου Cauchy τῆς ἐξισώσεως εἶναι σταθερός. Εἰς τὴν συνήθη αὐτὴν περίπτωσιν εἰς τὰς ἐφαρμογὰς δύναται νὰ δειχθῇ ὅτι τὰ σημεῖα ἐφαρμογῆς τῆς ὁλοκληρωτικῆς ἐξισώσεως ἐναλλάσσονται μὲ τὰς χρησιμοποιουμένας τετμημένας.

8) Εἰς τὴν περίπτωσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τύπου Cauchy κατὰ μῆκος κλειστῶν καμπύλων, ἡ τροποποιημένη μορφὴ τῆς μεθόδου τῶν τραπεζίων διὰ κοινὰ ὡς καὶ ἰδιόμορφα ὁλοκληρώματα περιοδικῶν συναρτήσεων δύναται νὰ χρησιμοποιηθῇ κατὰ τοιοῦτον τρόπον, ὥστε ἡ ἀριθμητικὴ ἐπίλυσις τῶν ἐξισώσεων αὐτῶν νὰ δύναται νὰ ἐπιτευχθῇ κατ᾽ ἀνάλογον τρόπον ἐκείνου, ὁ ὁποῖος χρησιμοποιεῖται δι᾽ἰδιομόρφους ὁλοκληρωτικὰς ἐξισώσεις μὲ διάστημα ὁλοκληρώσεως κείμενον ἐπὶ τοῦ πραγματικοῦ ἄξονος.

9) Εἰς τὴν περίπτωσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τύπου Cauchy μετὰ μιγαδικῶν ἰδιομορφιῶν εἰς τὰ ἄκρα τοῦ διαστήματος ὁλοκληρώσεως εἶναι δυνατόν, διὰ σχετικῶς εὐρεῖαν τάξιν ἐξισώσεων τοῦ τύπου αὐτοῦ, νὰ κεῖνται τὰ σημεῖα ἐφαρμογῆς τῆς ὁλοκληρωτικῆς ἐξισώσεως ἐκτὸς τοῦ διαστήματος ὁλοκληρώσεως. Τὸ αὐτὸ δύναται νὰ ἰσχύῃ καὶ διὰ τὰς τετμημένας. Εἰς τὴν περίπτωσιν αὐτὴν αἱ ἤδη χρησιμοποιηθεῖσαι μέθοδοι διὰ τὴν ἀριθμητικὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τύπου Cauchy δύνανται νὰ ἐπεκταθοῦν ἄνευ τροποποιήσεων καὶ διὰ τὴν περίπτωσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τύπου Cauchy μὲ μιγαδικὰς ἰδιομορφίας.

10) Αἱ μέθοδοι ἀριθμητικῆς ἐπιλύσεως ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων τύπου Cauchy αἱ παρουσιασθεῖσαι εἰς τὴν ἀνακοίνωσιν ταύτην, ὡς καὶ εἰς προηγούμενα ἄρθρα τοῦ ὁμιλοῦντος καὶ τοῦ συνεργάτου του, καθιστοῦν φανερὸν ὅτι ἡ ἀριθμητικὴ ἐπίλυσις τοιούτων ἐξισώσεων κατ᾽ οὐδένα τρόπον εἶναι ἀδύνατος, εἰς πλείστας δὲ περιπτώσεις ἀπαιτεῖ τὸ αὐτὸ πλῆθος ἀριθμητικῶν ὑπολογισμῶν ὡς καὶ εἰς τὴν περίπτωσιν συνήθων ὁλοκληρωτικῶν ἐξισώσεων τύπου Fredholm. Οὕτω πιστεύεται ὅτι εἰς τὸ μέλλον σημαντικὸς ἀριθμὸς προβλημάτων τῆς μαθηματικῆς Φυσικῆς θὰ δύνανται νὰ ἐπιλύωνται δι᾽ ἀναγωγῆς των εἰς ἰδιομόρφους ὁλοκληρωτικὰς ἐξισώσεις τύπου Cauchy, αἱ ὁποῖαι συνδυάζουν ἀκρίβειαν καὶ ἀποτελεσματικότητα, εἰς ἀντικατάστασιν τῶν ὀλιγώτερον ἀποτελεσματικῶν καὶ ἀκριβῶν μεθόδων τῶν πεπερασμένων στοιχείων ἢ τῶν ὁλοκληρωτικῶν μετασχηματισμῶν, μεθόδων εὑρισκομένων εἰς γενικὴν χρῆσιν σήμερον.