ΜΑΘΗΜΑΤΙΚΑ.— **On the holonomy theorem,** *by Efstathios Vassiliou**.
᾿Ανεκοινώθη ὑπὸ τοῦ ᾿Ακαδημαϊκοῦ κ. Ὄθωνος Πυλαρινοῦ.

INTRODUCTION

One of the nicest results of the theory of finite-dimensional connections is, certainly, the so-called Holonomy Theorem due to E. Cartan and rigorously proved first by Ambrose - Singer [**1**]. It describes, roughly speaking, the Lie algebra of the holonomy groups of a given finite-dimensional connection, in terms of the curvature form of the connection.

The purpose of the present note is to give an outline of the way in which we can obtain an analogous result within the infinite-dimensional framework.

Terminology and notations are those found in standard books such as [**2**], [**3**], [**4**], [**7**], where we refer for background theory. For the sake of simplicity, differentiability is of class C^{∞} (smoothness). Throughout we use the vector bundle technique, so that the classical result gets a refreshing form.

The author wishes to express his acknowledgement to Prof. A. Mallios, of the University of Athens, for his helping discussions and his valuable encouragement.

1. PRELIMINARIES

Let $l = (P, G, B, \pi)$ be a principal fibre bundle (p. f. b, for short) with connected and paracompact base B. A smooth connection on l is a G-splitting of the exact sequence of vector bundles (u.b's)

$$(\star) \qquad O \longrightarrow P \times \mathbf{g} \xrightarrow{\ v\ } TP \xrightarrow{\ T\pi !\ } \pi^*(TB) \rightarrow O$$

i. e there exists either a G - v . b-morphism $V : TP \rightarrow P \times \mathbf{g}$ such that $V \circ v = id_{P\times \mathbf{g}}$, or a G - v · b-morphism $c : \pi^*(TB) \rightarrow TP$ such that $T\pi ! \circ c = id_{\pi^*(TB)}$. Here $\mathbf{g}$ denotes the Lie algebra of G, $T\pi !$ is the v · b-morphism defined by the universal property of pull-backs and v is given (fiberwise) by $v(p, X) = T_e \varphi(p, .) . (X)$, if φ denotes the (right)

* ΕΥΣΤΑΘΙΟΥ ΒΑΣΙΛΕΙΟΥ, **Περὶ τοῦ θεωρήματος τῆς ὁλονομίας.**

action of G on P and $\mathbf{g} \cong T_e G$. The maps V and v are G-morphisms in the sense they preserve the action of G on the bundles of (⋆).

The existence of a connection on l implies that $TP = VP \oplus HP$, where VP and HP denote respectively the vertical subbundle Im(v) and the horizontal subbundle Im(c) of TP. Hence, each $u \in TP$ has the unique expression $u = u^v + u^h$, where the exponents v and h denote the vertical and horizontal parts respectively. It is standard that the existence of a connection is equivalent to the definition of a $\mathbf{g}$-valued connection form ω on P, satisfying the well-known properties (cf. [**3**]). Here we set $\omega_p(u) := pr_2 \circ V \cdot (u)$, for each $p \in P$ and $u \in T_p P$. Following the classical pattern, we also define the curvature form Ω related with ω by the structural equation

$$\Omega = d\omega + \frac{1}{2} \cdot [\omega, \omega].$$

The holonomy group $\Phi(p)$ at p is the set of $s \in G$ such that p and $p \cdot s$ can be joined by a horizontal curve. Similarly, the restricted holonomy group $\Phi^o(p)$ at p is the set of $s \in G$ such that p and $p \cdot s$ can be joined by a horizontal curve with O-homotopic projection. Finally, the holonomy bundle $Q = P(p)$ at p is the set of points $q \in P$ which can be joined with p by a horizontal curve.

Since the connectedness of the base B makes the choice of the reference point inessential, we omit it throughout the rest of the note.

Let **R** be the set of all $\Omega_q \cdot (X_q, Y_q)$, where $q \in Q$ and X_q, Y_q are arbitrary elements of $H_q P$. Note that, in virtue of the structural equation, we obtain the expression $\Omega_q \cdot (X_q, Y_q) = -\omega([X, Y]^v \cdot (q))$, where X and Y are the horizontal vector fields extending X_q and Y_q respectively. The Lie algebra **h** generated by **R** is called the holonomy algebra (of the given connection).

It is already known (cf. [**6**]) that the holonomy groups Φ and Φ^o are infinite-dimensional Lie subgroups of G, i.e. they are endowed with a manifold structure such that the natural injections $i : \Phi \longrightarrow G$ and $j : \Phi^o \longrightarrow G$ are C^∞-morphisms. Also, it is known that $l^o = (Q, \Phi^o, B, \pi)$ is a p.f.b. The infinite-dimensional version of the holonomy theorem, however, relies heavily upon the following conditions :

(C. 1) : $i : \Phi^\circ \dashrightarrow G$ is an immersion

(C. 2) : $\mathbf{h}$ is a direct summand of $\mathbf{g}$ (i. e. $\mathbf{h}$ is a closed subspace of $\mathbf{g}$ admitting a closed supplement in $\mathbf{g}$).

Naturally, the above conditions are automatically satisfied in the finite-dimensional framework. This is not the case here, where they play an important role.

An immediate consequence of (C. 1) is that Q is an immersed submanifold of P and we can define a connection on l°, by an appropriate restriction of c or V. (C. 2) is needed later, in the definition of the bundle V^* (cf. below).

2. THE HOLONOMY THEOREM

With the above conventions we state:

The holonomy theorem: *The holonomy algebra* $\mathbf{h}$ *generates* Φ°.

The proof of the theorem is based on the following lemmas:

Lemma 1. *Let* A^* *be the fundamental vector field of* $A \in \mathbf{g}$. *If* $A^*_q \in V_q Q$, *for some* $q \in Q$, *then* $A^*_q \in V_q Q$, *for all* $q \in Q$.

Proof: The definition of A^* (cf. [3]) implies that $A^*_q = v(q, A)$. Let us denote by $\mathbf{g}^\circ$ the Lie algebra of Φ°. Since $v(q, .) : \mathbf{g}^\circ \rightarrow V_q Q$ is a topological vector space isomorphism, we deduce that $A \in \mathbf{g}^\circ$; hence, $A^*_q \in V_q Q$, for every $q \in Q$. ■

Lemma 2. *If* $A \in \mathbf{h}$ *and* $q \in Q$, *then* $A^*_q \in V_q Q$.

Proof: We prove the lemma in the special case of

$$A = \omega([X, Y]^v \cdot (q)).$$

In fact, since $v \circ V = id_{VP}$ on VP, we obtain

$$A^*_q = v(q, A) = [X, Y]^v \cdot (q) \in V_q Q.$$

Combining Lemma 1 and the previous case, we complete the proof. ■

In the sequel we define the sets $V^*_q := \{A^*_q : A \in \mathbf{h}\}$ and $V^* := \bigcup_{q \in Q} V^*_q$. In virtue of (C. 2), V^* is a vector subbundle of TQ. In particular $V^*_q = v(\{q\} \times \mathbf{h})$. The above lemmas now yield the following:

Lemma 3. V^* *coincides with the vertical subbundle* VQ *of* TQ.

Proof: Following essentially [**1**], we show that $V^* \oplus HP$ is an integrable subbundle of TQ and each horizontal curve joining p and q lies in the maximal integrable submanifold N through p; hence, $Q \subset N$ and $T_q Q \subset T_q N = V_q^* \oplus H_q P$. On the other hand, it is immediate that $V_q^* \oplus H_q Q \subset T_q Q$ thus $TQ = V^* \oplus HP$, which completes the proof. ∎

The proof of the Theorem is now an immediate consequence of Lemma 3. Indeed, we have that

$$v(\{q\} \times \mathbf{g}^\circ) = V_q Q = V_q^* = v(\{q\} \times \mathbf{h})$$

which yields $\mathbf{h} = \mathbf{g}^\circ$, since v is a topological vector space isomorphism on the fibers. ∎

As a corollary, we check that **h** is the Lie algebra of Φ, as well. This is the case, since $\Phi^\circ \subset \Phi$ and **h** is stable under Ad(s), for each $s \in \Phi$ (cf. [**5**]).

Π Ε Ρ Ι Λ Η Ψ Ι Σ

Εἰς τὴν ἐργασίαν ταύτην ἐπιτυγχάνεται ἡ ὑπὸ ὡρισμένας προϋποθέσεις ἐπέκτασις εἰς συνοχὰς μὴ πεπερασμένης διαστάσεως τοῦ θεωρήματος τῆς ὁλονομίας τῶν Ambrose - Singer τοῦ ἀφορῶντος εἰς τὰς συνοχὰς πεπερασμένης διαστάσεως, συμφώνως πρὸς τὸ ὁποῖον ἡ ἄλγεβρα Lie τῶν ὁμάδων ὁλονομίας συνοχῆς πεπερασμένης διαστάσεως ἐκφράζεται διὰ τῆς μορφῆς καμπυλότητος τῆς συνοχῆς.

Πρὸς τοῦτο θεωρεῖται μία Banach - νηματικὴ δέσμη (P, G, B, π) μὲ συνεκτικὴν καὶ παρασυμπαγῆ βάσιν B καὶ συνοχὴν ω, παριστῶνται δὲ διὰ τῶν συμβόλων :

- **g** : ἡ ἄλγεβρα Lie - Banach τῆς ὁμάδος δράσεως G,
- Ω : ἡ μορφὴ καμπυλότητος τῆς συνοχῆς ω,
- HP : ἡ ὁριζοντία ὑποδέσμη τῆς ἐφαπτομένης δέσμης TP,
- Φ(p) : ἡ ὁμὰς ὁλονομίας εἰς τὸ σημεῖον p,
- $\Phi^\circ(p)$: ἡ περιωρισμένη ὁμὰς ὁλονομίας εἰς τὸ σημεῖον p,
- Q(p) : ἡ δέσμη ὁλονομίας, δηλ. τὸ σύνολον τῶν $q \in P$ τῶν συνδεομένων μετὰ τοῦ p δι' ὁριζοντίας καμπύλης,
- **h** : ἡ ἄλγεβρα ὁλονομίας, δηλ. ἡ ἄλγεβρα Lie ἡ παραγομένη ὑπὸ στοιχείων τῆς μορφῆς $\Omega_q(X_q, Y_q)$ δι' ὅλα τὰ ζεύγη (X_q, Y_q) ἐν HP, ὅταν τὸ q διατρέχῃ τὴν Q(p) καὶ ἀποδεικνύεται τὸ ἑξῆς :

Θεώρημα τῆς ὁλονομίας : *Ἡ ἄλγεβρα ὁλονομίας* **h** *παράγει τὰς ὁμάδας ὁλονομίας Φ καὶ Φ° ὅταν α) ἡ ἄλγεβρα ὁλονομίας* **h** *εἶναι κλειστὸς ὑποχῶρος τῆς* **g** *ἔχων κλειστὸν τοπολογικὸν συμπλήρωμα ἐν* **g** *καὶ β) ἡ ὁμὰς ὁλονομίας φ° εἶναι ἐμφυτευμένη ὑποομὰς τῆς* G.

Εἰς τὸ θεώρημα τοῦτο ἀντὶ τῶν Φ(p), Φ°(p) γράφονται ἀντιστοίχως Φ, Φ°, παραλειπομένου, λόγῳ τῆς συνεκτικότητος τῆς βάσεως, τοῦ σημείου ἀναφορᾶς p.

Αἱ προϋποθέσεις α καὶ β ἀποτελοῦν συνθήκην ἱκανὴν ἵνα ἰσχύῃ τὸ ἀνωτέρω θεώρημα, ἡ συνθήκη δ' αὕτη πληροῦται πάντοτε εἰς τὴν περίπτωσιν συνοχῆς πεπερασμένης διαστάσεως καὶ εἰς τὴν περίπτωσιν ταύτην τὸ ὑπὸ τοῦ θεωρήματος τούτου ἐκφραζόμενον ἀποτέλεσμα ταυτίζεται μετὰ τοῦ ὑπὸ τοῦ θεωρήματος τῶν Ambrose - Singer ἐκφραζομένου.

REFERENCES

1. W. Ambrose - I. Singer, A theorem on holonomy, Trans. Amer. Math. Soc. 75 (1953), 428 - 443.
2. N. Bourbaki, Variétés différentielles et analytiques (Fascicule de résultats) §§ 1 - 7, Hermann, Paris 1967.
3. S. Kobayashi - K. Nomizu, Foundations of differential geometry, Vol. 1, Interscience, N. York 1963.
4. S. Lang, Introduction to differentiable manifolds, Interscience, N. York 1967.
5. M. Lazard, Groupes différentiables, Univ. Paris (Sorbonne) 1965.
6. L. Maxim, Connections compatible with Fredholm structures on Banach manifolds, Anal. Stiint. Univ. «A. Cuza» Iasi, t. XVIII (1972), 389 - 400.
7. K. Nomizu, Lie groups and differential geometry, Publications Math. Soc. Japan (2), 1956.