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ΜΑΘΗΜΑΤΙΚΑ. — **The Compliance Fourth-Rank Tensor for the Transtropic Material and its Spectral Decomposition**, by *Academician P. S. Theocaris**.

ABSTRACT

The spectral decomposition of the fourth-rank transversely isotropic (transtropic) tensor of an elastic solid constitutes the simplest kind of decomposition which, expressed explicitly in terms of the respective engineering constants of the materials, presents certain advantages over other forms of decomposition useful in engineering applications. Indeed, the eigentensors derived from this decomposition present the quality to decompose the energy of the body in orthogonal components of the second order symmetric tensor space. In this way energy-orthogonal stress states were explicitly determined associating the elastic energy components with these stress tensors, which identified with combinations of the dilatational and distortional strain energies of the field. Thus, it was shown that the four eigentensors of the spectral decomposition define four characteristic states of stress corresponding to two states σ_1 and σ_2 which are shears with σ_2 being a simple shear and σ_1 being a state of stress derived from the superposition of pure shear and simple shear. The other two eigenstates yield a sum $(\sigma_3+\sigma_4)$ which is the orthogonal supplement to the shear subspace of σ_1 and σ_2. These eigenstates constitute an equilateral loading in the plane of isotropy of the material, superimposed with either a prescribed tension (σ_3), or compression (σ_4) along the axis of symmetry of the material.

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I. INTRODUCTION

The definition of energy orthogonal stress states was introduced by Rychlewski [1]. This term denotes stress tensors for the generally anisotropic solid, mutually orthogonal and at the same time colinear with their respective strain tensors. It was shown [2] that, if a given stress tensor was decomposed in energy orthogonal tensors, then these tensors also decompose the elastic energy function. The decomposition of the elastic stiffness or compliance tensor in elementary fourth-rank tensors serves as a means for the energy orthogonal decomposition of the stress tensor, the appropriate decomposition of the fourth-rank tensor being the *spectral one.*

Similar decompositions, non spectral, of the fourth rank tensor were also given by Walpole [3], Srinivasan and Nigan [4] and others, in order to simplify calculations with fourth rank tensors, and obtain invariant expressions for the components of the stiffness or compliance tensors.

Indeed, Walpole has presented for the first time a reduction of the algebra of the fourth-rank tensor to irreducible subalgebras, which were much simpler than the initial one, and therefore they facilitate any operation between fourth-rank tensors. In this general form of decomposition Walpole includes also the spectral decomposition of the fourth-rank tensor, which possesses the already discussed unique properties in refs. [5] and [6]. Subsequently, Walpole decomposes the fourth-rank tensors for all kinds of crystals. However, all the subspaces contained in these analyses do not correspond to the spectral decomposition, except for the trivial cases of the isotropic fourth-rank tensor and the tensor corresponding to the cubic system. Tbus, in refs. [5] and [6], as well as in this communication, the spectral decomposition was applied for the first time for the fourth-rank tensor S_{ijkl} of compliance of the elastic transtropic material and this decomposition was presented explicitly in terms of the engineering constants.

Indeed, in the respective decomposition by Walpole [3] the fourth rank tensor $\mathbf{S}$ is expressed by:

$$\mathbf{S} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + f\mathbf{F} + g\mathbf{G} \tag{1}$$

whereas the spectral decomposition [5, 6] yields:

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 + \lambda_3\mathbf{E}_3 + \lambda_4\mathbf{E}_4 \tag{2}$$

In relation (2) the λ_is (i=1, 2, 3, 4) are the eigentensors of S, whereas in relation (1) the quantities a, b, c are not eigentensors of **S**. The only common property between the two decompositions is the identity of the *idempotent tensors* **F** and **C** with the $\mathbf{E}_3$ and $\mathbf{E}_4$, whereas the $\mathbf{E}_1$ and $\mathbf{E}_2$ are totally different than their respective **A** and **B** eigentensors.

In this paper it was succeeded to decompose spectrally the compliance tensor for a transtropic material, representing fiber reinforced composites and to evaluate its characteristic values. Based on the properties of this decomposition, energy orthogonal stress states were established. It was further shown that the eigen-values of the respective stiffness tensor, when spectrally decomposed, establish new bounds for the values of Poisson's rations of the transtropic material obeying restrictions of thermodynamics.

2. INTERRELATION BETWEEN TENSOR SPACES

Let **L** be the space of symmetric rank-two tensors over $\mathbf{R}^3$. Tensor space **L** together with the ordinary definition of scalar product constitutes a 6-D Euclidean space. We denote by **M** the symmetric tensor square of **L**, that is the space of symmetric fourth-rank tensors. In symbolic notation, the above defined tensor spaces are given as:

$$\mathbf{L} = \mathrm{sym}\ \mathbf{R}^3 \otimes \mathbf{R}^3$$

$$\mathbf{M} = \mathrm{sym}\ \mathbf{L} \otimes \mathbf{L}$$

These tensor spaces constitute the appropriate field for the mathematical description of the *hyperelastic solid*. This solid corresponds to the class of material behaviour, whose elements are characterized by a stress and strain tensor, $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ respectively, $(\boldsymbol{\sigma},\boldsymbol{\epsilon}\,/\boldsymbol{\sigma},\boldsymbol{\epsilon}\in\mathbf{L})$ and by a potential function T (elastic potential), for which it is valid that $=\boldsymbol{\sigma}\partial_\varepsilon T$. For small strains $\boldsymbol{\epsilon}$ and isothermal or adiabatic conditions, this is equivalent to Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{C}\cdot\boldsymbol{\epsilon} = C_{ijkl}\varepsilon_{kl} \qquad (3)$$

By means of the scalar product defined on **L**, the elastic potential is expressed by:

$$2T = \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon} = \sigma_{ij}\varepsilon_{ij} \qquad (4)$$

The fourth-rank tensor **C** (stiffness tensor) acts as a symmetric linear operator on **L**, i.e.:

$$\mathbf{C}:\mathbf{L} \rightarrow \mathrm{L},\ \mathbf{a}.\ \mathbf{C}.\ \mathbf{a} \rightarrow \forall \mathbf{a} \in \mathbf{L} \tag{5α}$$

transforming the space **L** into itself. Equivalently, with respect to the scalar product defined in (3), symmetry of **C** is expressed by:

$$\mathbf{a}\cdot\mathbf{C}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{C}\cdot\mathbf{a}\ \mathrm{A}\mathbf{a},\ \mathbf{b}\ \mathbf{L}. \tag{5a}$$

With respect to the symbolism used we note that small boldface letters denote tensors of **L**, capital boldface letters are tensors in **H**-space and normally printed letters denote scalars.
Fulfilment of basic requirements concerning the properties of the strain energy function, T, implies tensor **C** to be positive definite, i.e.,

$$\mathbf{a}\cdot\mathbf{C}\cdot\mathbf{a} > 0\ \forall \mathbf{a} \in \mathbf{L}.$$

As a result, a fourth-rank rensor **S**, (compliance), exists such that the commutative product of **C** and **S** yields the unit element of **M**-space. Then, equivalently from relation (3) and taking also into account that:

$$\mathbf{I}\cdot\mathbf{a} = \mathbf{a} \quad \forall \mathbf{a} \in \mathbf{L}$$

$$\mathbf{C}\cdot\mathbf{S} = \mathrm{C}_{ijmn}\mathrm{s}_{mnk\,l} = \mathbf{S}\cdot\mathbf{C} = \mathbf{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

one has:

$$\boldsymbol{\epsilon} = \mathbf{S}\cdot\boldsymbol{\sigma}. \tag{6}$$

By means of equations (3) and (6), relation (4) is expressed in an alternative form:

$$2\mathrm{T} = \boldsymbol{\sigma}\cdot\mathbf{S}\cdot\mathbf{o} = \boldsymbol{\epsilon}\cdot\mathbf{C}\cdot\boldsymbol{\epsilon}, \tag{7}$$

which, due to the positive definite property of **C** and **S**, assures the strain energy function to be strictly positive.

3. FOURTH-RANK TENSOR ALGEBRA

Consider some properties of the **M**-space tensor in accordance to their interrelation with square matrices, space of rank six. The set of characteristic values of the tensor **S**, say $\lambda_1, \lambda_2, ...\lambda_6$, are real, but not necessarily distinct. Then: $\lambda_1,...,\lambda_m / m \leq 6$ and $\lambda_K \neq \lambda_N$ for $K \neq N$ $(K, N=1,....,m)$ and the tensor **S** disposes a set of roots of its minimum polynomial, which-in factorized form, can be written as:

$$(\mathbf{S}-\lambda_1\mathbf{I})...(\mathbf{S}-\lambda_m\mathbf{I})=0. \tag{8}$$

In what follows, no summation convention is adopted over indices K and N. Then, as it can be readily proved, tensor S is expressed by:

$$\mathbf{S}=\lambda_1\mathbf{E}_1+...+\lambda_m\mathbf{E}_m, \tag{9}$$

while tensors $\mathbf{E}_N$ are the polynomial multipliers of the Lagrange polynomial associated with (8). Thus, tensors $\mathbf{E}_N$ are given by:

$$\mathbf{E}_N=\frac{(\mathbf{S}-\lambda_1\mathbf{I})...\ (\mathbf{S}-\lambda_{N-1}\mathbf{I})\ (\mathbf{S}-\lambda_{N+1}\mathbf{I})...\ (\mathbf{S}-\lambda_m\mathbf{I})}{(\lambda_N-\lambda_1)...\ (\lambda_N-\lambda_{N-1})\ (\lambda_N-\lambda_{N+1})...\ (\lambda_N-\lambda_m)}, \tag{10}$$

and satisfy the following relations:

$$\mathbf{I}=\mathbf{E}_1+...+\mathbf{E}_m$$

$$\mathbf{E}_K\cdot\mathbf{E}_N=0,\ \mathbf{E}^2_K=\mathbf{E}_K\ . \tag{11}$$

By means of the obtained *spectral decomposition of tensor* **S**, as in relation (9), its inverse, say $\mathbf{S}^{-1}$, is simply expressed as:

$$\mathbf{S}^{-1}=\frac{1}{\lambda_1}\mathbf{E}_1+...+\frac{1}{\lambda_m}\mathbf{E}_m\ . \tag{12}$$

The *spectral decomposition* (9) and (12), of the fourth-rank tensors over $\mathbf{R}^3$ is of great help in the sequel for the determination of energy orthogonal stress states, but it is also very important *per se*, since λ_N can be expressed by means of S_{ijkl}, which are invariants with respect to changes of the coordinate system. Then, in formulating solutions of anisotropic elasticity problems, use of the stiffness or compliance tensors in the forms (9) and (12) will significantly simplify calculations, since only tensors $\mathbf{E}_N$ have to transform their components with changes of the coordinate system, and those tensors contain a large number of zeros.

Besides the spectral decomposition of $\mathbf{S}$, there is a possibility of many others which also yield invariant expressions for S_{ijkl}. We note for example the works of Hill [7, 8]. Srinivasan and Nigam [4] and Walpole [3]. The decompositions obtained in these papers are appropriate for the formulation of elastic-inclusion problems or related ones, but they are not offering the powerful properties of the spectral decomposition and its applications.

4. ORTHOGONAL DECOMPOSITION OF SPACE L

Recalling relations (3) and (5a) and taking also into account the decomposition of the unit tensor of $\mathbf{M}$-space, obtained through the spectral decomposition of a tensor $\mathbf{S} \in \mathbf{M}$, one has the following equivalences:

$$I \cdot \mathbf{s} = \mathbf{s}, \quad \mathbf{s} \in \mathbf{L}$$

or:

$$(\mathbf{E}_1 + \ldots + \mathbf{E}_m) \cdot \mathbf{s} = \mathbf{E}_1 \cdot \mathbf{s} + \ldots + \mathbf{E}_m \cdot \mathbf{s}, \quad m \leqq 6$$

Denoting by $S_k \in L$:

$$\mathbf{s}_K = \mathbf{E}_K \cdot \mathbf{S} \quad K = 1, \ldots m$$

it is valid that:

$$\mathbf{S} = \mathbf{s}_1 + \ldots + \mathbf{s}_m \tag{13}$$

Moreover, it can be readily proved that:

$$\mathbf{s}_K \cdot \mathbf{s}_N = 0 \quad \text{for} \quad K \neq N$$

or what is the same:

$$\mathbf{s}_K \perp \mathbf{s}_N, \quad K \neq N \quad . \tag{14}$$

For the tensors $\mathbf{s}_K$, relation (5a) still hilds:

$\mathbf{I}\cdot\mathbf{s}_m=\mathbf{E}_1\cdot\mathbf{s}_m+\ldots+\mathbf{E}_m\cdot\mathbf{s}_m=\mathbf{s}_m$

and the following important relations are also valid:

$\mathbf{E}_K\cdot\mathbf{s}_N=0$ for $K\neq N$

$$\mathbf{E}_K\cdot\mathbf{s}_K=\mathbf{S}_K \tag{15}$$

Then, by means of relations (9) and (15), it can be seen that, for the linear symmetric operator $\mathbf{S}\in\mathbf{M}$ on $\mathbf{L}$, tensors $\mathbf{S}_K\in\mathbf{L}$ are eigen-elements and the associated eigen-values are λ_N, that is:

$$\mathbf{S}\cdot\mathbf{s}_m=(\lambda_1\mathbf{E}_1+\ldots+\lambda_m\mathbf{E}_m)\cdot\mathbf{s}_m=\lambda_m\mathbf{s}_m \tag{16}$$

The analysis already presented by means of relations (13-16) could be summarized in a classic theorem of linear Algebra as follows:

Theorem:

$\mathbf{L}$: sym $\mathbf{R}^3\otimes\mathbf{R}^3$, supplied with a definite positive inner product (scalar).
$\mathbf{S}$: $\mathbf{L}\rightarrow\mathbf{L}$ symmetric linear operator. For, if λ_K is an eigen-value of $\mathbf{S}$ and $\mathbf{L}_{\lambda k}(\mathbf{S})\subset\mathbf{L}$ is the set of all $\mathbf{s}_K\in\mathbf{L}$, such that $\mathbf{S}\cdot\mathbf{s}_K=\lambda_K\mathbf{s}_K$, the following propositions are valid:

1. $\mathbf{L}_{\lambda k}(\mathbf{S})\subset\mathbf{L}$, $\mathbf{S}:\mathbf{L}_{\lambda k}\rightarrow\mathbf{L}$ is onto and 1-1, that is $\mathbf{S}:\mathbf{L}_{\lambda k}\rightarrow\mathbf{L}_{\lambda k}$.

2. $\mathbf{L}=\mathbf{L}_{\lambda 1}(\mathbf{S})+\ldots+\mathbf{L}_{\lambda m}(\mathbf{S})/\mathbf{L}_{\lambda K}(\mathbf{S})$, $\mathbf{L}_{\lambda N}(\mathbf{S})$ being orthogonal for $K\neq N$. $\mathbf{L}_{\lambda K}(\mathbf{S})$ are called eigen subspaces, associated with λ_K.

The proof of the theorem is also given by the same set of relations (13-16). Subspaces $\mathbf{L}_{\lambda K}(\mathbf{S})$ are constructed by means of the action of linear symmetric operators $\mathbf{E}_K$, i.e.:

$$\mathbf{L}_{\lambda K}(\mathbf{S})=(\mathbf{s}_K/\mathbf{s}_K=\mathbf{E}_K\ \mathbf{s},\ \mathbf{S}\in\mathbf{L}). \tag{17}$$

In classic mathematical symbolism this is denoted as follows:

$$\mathbf{L}_{\lambda K}(\mathbf{S})=\mathrm{I_m}\mathbf{E}_K$$

$$(\mathbf{L}-\mathbf{L}_{\lambda K}(\mathbf{S}))=\mathrm{Ker}\mathbf{E}_K, \tag{18}$$

where *Im* denotes the image set of operator $\mathbf{E}_K$, while *Ker* its kernel set.

The above-cited analysis, based on the theory of linear symmetric operators, consists of well established topics of linear algebra, which can be found in any classical textbook, as for example in these of Lang [9], Bishop and Goldberg [10] and others. An equivalent analysis, argumented in a somehow inverse way was also given by Rychlewski [2].

5. ENERGY-ORTHOGONAL STRESS STATES

Consider again the hyperelastic solid solely characterized by its property $\mathbf{C}$ or $\mathbf{S}$. Then, for a given stressing, $\boldsymbol{\sigma}$, of it, Hooke's law makes correspond an associated straining $\boldsymbol{\epsilon}$, or the inverse, that is:

$$\boldsymbol{\epsilon}=\mathbf{S}\cdot\boldsymbol{\sigma}=(\lambda_1\mathbf{E}_1+\ldots+\lambda_m\mathbf{E}_m)\cdot\sigma=\lambda_1\boldsymbol{\sigma}_1+\ldots+\lambda_m\boldsymbol{\sigma}_m. \tag{19}$$

Then, the strain tensor $\boldsymbol{\epsilon}$ can be written as the sum of elementary tensors $\boldsymbol{\epsilon}_K$:

$$\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_1+\ldots+\boldsymbol{\epsilon}_m \tag{20}$$

and the following is valid for these tensors:

$$\boldsymbol{\epsilon}_K=\lambda_K\boldsymbol{\sigma}_K,\ \ k=1,\ldots m \tag{21}$$

where no summation convention was adopted over K, as it has been already stated. The background for the definition of energy orthogonal stress states is now well prepared.

Definition:

Two stress states $\boldsymbol{\sigma}_K$, $\boldsymbol{\sigma}_N$, $\in\mathbf{L}$ of the hyperelastic solid $\mathbf{C}$ are called energy orthogonal if:

1. $\boldsymbol{\sigma}_K\cdot\boldsymbol{\sigma}_N=0$

2. $\boldsymbol{\sigma}_K\cdot\boldsymbol{\epsilon}_N=0$

It will be useful, in the sequel, if we remind a classic theorem of the linear symmetric operator theory, appropriately translated in tensor symbolism.

Theorem:

Let $\mathbf{S}:\mathbf{L}\rightarrow\mathbf{L}$ a symmetric linear operator, and $\boldsymbol{\sigma}_N$ a non-zero eigen-element of it. If $\boldsymbol{\sigma}_K\in\mathbf{L}$ and $\boldsymbol{\sigma}_K\cdot\boldsymbol{\sigma}_N=0$, then:

$$\mathbf{S}\cdot\boldsymbol{\sigma}_K\perp\boldsymbol{\sigma}_N\ .$$

The proof of the theorem is trivial.

From the definition of energy-orthogonal stress states and the just above cited theorem the following considerations are in order:

1. The orthogonal decomposition (13) of **L**-space is on the same time energy orthogonal. That is, eigen-elements $\boldsymbol{\sigma}_K$ of the solid **S** are energy orthogonal.

2. If $\boldsymbol{\sigma}_K$ is an eigen-element of **S**, then tensors $\boldsymbol{\sigma}_K$ and $\boldsymbol{\epsilon}_K$ are colinear.

3. For an isotropic fourth-rank tensor **S**, of a linear hyperelastic solid, although while for a tensor $\boldsymbol{\sigma}$ and the associated $\boldsymbol{\epsilon}$ colinearity is not in general the case, there exists two energy-orthogonal stress state, $\boldsymbol{\sigma}_p$, $\boldsymbol{\sigma}_D$ defined as:

$$\boldsymbol{\sigma}_p = \frac{1}{3} \operatorname{tr}\boldsymbol{\sigma} \; \mathbf{1} \otimes 1, \quad \mathbf{1} = \delta_{ij} \tag{22}$$

and:

$$\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_D = \boldsymbol{\sigma}, \quad \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_D = 0 \tag{23}$$

or which, of course, it is valid that σ_p is parallel to $\boldsymbol{\epsilon}_p$ and $\boldsymbol{\sigma}_D$ is parallel to $\boldsymbol{\epsilon}_D$. A similar decomposition for the transversely isotropic **S**-tensor with $\forall \boldsymbol{\sigma} \in \mathbf{L}$ is the subject of the present paper.

4. The energy-orthogonal decomposition provides a powerful means for the decomposition of the elastic potential T of generally anisotropic solids, as for the case of isotropic ones. Then, any quadratic limit condition of elastic behaviour can be provided with a bound on some of the parts of the decomposed elastic energy, or, in other words, each quadratic yielding condition has a definite energy interpretation [1].

For, if we consider relation (4) defining the potential function T, one has:

$$\begin{aligned} 2T(\boldsymbol{\sigma}) &= \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} = \boldsymbol{\sigma} \cdot \mathbf{S} \cdot \boldsymbol{\sigma} = (\boldsymbol{\sigma}_1 + \dots + \boldsymbol{\sigma}_m) \cdot (\lambda_1 \mathbf{E}_1 + \dots \lambda_m \mathbf{E}_m) \cdot (\boldsymbol{\sigma}_1 + \dots + \boldsymbol{\sigma}_m) = \\ &= \lambda_1 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 + \dots + \lambda_m \boldsymbol{\sigma}_m \cdot \boldsymbol{\sigma}_m . \end{aligned} \tag{24}$$

Recalling relation (21), the expression for the potential function can be recast as follows:

$$2T(\boldsymbol{\sigma}) = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\epsilon}_1 + \dots + \boldsymbol{\sigma}_m \boldsymbol{\epsilon}_m$$

or what is the same:

$$T(\boldsymbol{\sigma}_1+\ldots+\boldsymbol{\sigma}_m)=T\boldsymbol{\sigma}_{(1)}+\ldots+T(\boldsymbol{\sigma}_m)=\frac{1}{2}\lambda_1 tr\boldsymbol{\sigma}_1^2+\ldots+\lambda_m tr\boldsymbol{\sigma}_m^2). \qquad (25)$$

Thus, any characteristic state $\boldsymbol{\sigma}_N$ has its own elastic potential $T(\boldsymbol{\sigma}_N)$, which does not depend on the action of other $\boldsymbol{\sigma}_K$s. This decomposition of the elastic energy is as simple as this of the isotropic solid and is valid for any class of anisotropy. For example, triclinic crystals have $m=6$ By means of relation (25) or (24) the condition for positive definite T attains its most rational form. That is simply:

$$\lambda_1 > 0,\ldots,\lambda_m > 0. \qquad (26)$$

6. SPECTRAL DECOMPOSITION OF THE TRANSVERSELY ISOTROPIC FOURTH RANK TENSOR

Consider now as an example the decomposition of the compliance tensor of **S** a transversely isotropic linear elastic solid. Of course any other similar tensor related with different material properties will decompose in the same manner.

We suppose the Cartesian coordinate system which the stress and strain tensor components are referred to, being oriented along the principal material directions, with 33-axis normal to the isotropic (transverse) plane. Using engineering constants with subscript (T) to denote elastic properties on the isotropic plane and subscript (L) the corresponding ones on the normal (longitudinal) plane components of the **S**-tensor, associated with the adopted Cartesian system, are given by:

$$\begin{aligned}
&S_{1111}=S_{2222}=1/E_T\\
&S_{3333}=1/E_L\\
&S_{1122}=S_{2211}=\nu_T/E_T\\
&S_{1133}=S_{3311}=S_{2233}=S_{3322}=-\nu_L/E_L \qquad (27)\\
&S_{2323}=S_{2332}=S_{3223}=S_{3232}=1/4G_L\\
&S_{1313}=S_{1331}=S_{3113}=S_{3131}=1/4G_L\\
&S_{1212}=S_{1221}=S_{2112}=S_{2121}=1/4G_T.
\end{aligned}$$

All remaining S_{IJKl} are zero. Furthermore, between engineering constants of the transverse plane holds the well-known isotropic relation:

$1/2G_T=(1+\nu_T)/E_T$.

The characteristic values of the associated square matrix of rank six to tensor **S**, defined by (27), were found to be:

$\lambda_1=(1+\nu_T)/E_T=1/2G_T$

$\lambda_2=1/2G_L$

$\lambda_3=(1-\nu_T)/2E_T+1/2E_L+\{[(1-\nu_T)/2E_T-1/2E_L]^2+2\nu_L^2/E_L^2\}^{1/2}$

$$\lambda_4=(1-\nu_T)/2E_T+1/2E_L-\{[(1-\nu_T)/2E_T-1/2E_L]^2+2\nu_L^2/E_L^2\}^{1/2} \tag{28}$$

That is, two of its characteristic values, namely λ_1 and λ_2, are of multiplicity two. Then, the minimum polynomial of tensor S is a *quartic*, and has as roots $\lambda_1, \lambda_2, \lambda_3$ and λ_4. The associated four idempotent tensors of the spectral decomposition of **S** are given by:

$$\mathbf{E}_1=\overset{1}{E}_{ijkl}=\frac{1}{2}(b_{ik}b_{jl}+b_{jk}b_{il}-b_{ij}b_{kl})$$

$$\mathbf{E}_2=\overset{2}{E}_{ijkl}=\frac{1}{2}(b_{ik}a_{jl}+b_{il}a_{jk}+b_{jl}a_{ik}+b_{jk}a_{il})$$

$$\mathbf{E}_3=\overset{3}{E}_{ijkl}=\mathbf{f}\otimes\mathbf{f}=f_{ij}f_{kl} \tag{29}$$

$$\mathbf{E}_4=\overset{4}{E}_{ijkl}=\mathbf{g}\otimes\mathbf{g}=g_{ij}g_{kl},\ \mathbf{a},\mathbf{b},\mathbf{f},\mathbf{g}\in\mathbf{L}.$$

The second-rank symmetric tensors **a** and **b**, figuring in relations for $\mathbf{E}_1$ and $\mathbf{E}_2$, and are defined by:

$$\mathbf{a}=\mathbf{k}\otimes\mathbf{k}$$

$$\mathbf{a}+\mathbf{b}=\mathbf{I}, \tag{30}$$

with **k** being the unit vector of $\mathbf{R}^3$, associated with the 33-direction of the Cartesian coordinate system. Tensors **f** and **g** are also axisymmetric and they depend on the components of tensor **S**. They are given by:

$$\mathbf{f}=\frac{1}{\sqrt{2}}\cos\omega\,\mathbf{b}+\sin\omega\,\mathbf{a}$$

$$\mathbf{g}=\frac{1}{\sqrt{2}}\sin\omega\,\mathbf{b}-\cos\omega\,\mathbf{a} \tag{31}$$

with:

$$\cos 2\omega = \frac{[(1-\nu_T)/2E_T - 1/2E_L]}{[((1-\nu_T)/2E_T - 1/2E_L)^2 + 2\nu^2_T/E^2_L]^{1/2}}$$

where ω is a characteristic angle which together with the four eigenvalues λ_1, λ_2, λ_3, λ_4 define explicity and in the simplest possible way the mechanical behaviour of the transtropic material. Angle ω is called the eigenangle of the material.

It is worthwhile mentioning again that for the characteristic values given by relations (28) and the corresponding idempotent tensors of relation (29) it is valid and can be readily checked that:

$$\mathbf{I} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4$$

$$\mathbf{S} = \lambda_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 + \lambda_3 \mathbf{E}_3 + \lambda_4 \mathbf{E}_4.$$

It is interesting to note that relations (28) and (33) correspond to the decomposition of the isotropic elastic solid in the case for which it is valid that: $E_L = E_T$, $G_L = G_T$, $\nu_L = \nu_T$. Then, relations (33) are written as follows:

$$\mathbf{I} = \mathbf{E}_p + \mathbf{E}_D, \quad \mathbf{S} = \lambda_p \mathbf{E}_p + \lambda_D \mathbf{E}_D \qquad (34)$$

with:

$$\lambda_p = 1/3K, \quad \lambda_D = 1/2G, \quad \mathbf{E}_p = \frac{1}{3}\, \mathbf{I} \otimes \mathbf{I}, \quad K = E/3(1\text{-}2\nu. \qquad (35)$$

7. CHARACTERISTIC STATES OF L FOR TRANSTROPIC S

Let define the orthogonal subspaces of $\mathbf{L}$ in terms of which the space of second-rank symmetric tensors, $\mathbf{L}$, is expressed as their direct sum, and which also constitute characteristic states of tensor $\mathbf{S}$, that is they satisfy the following relations:

$$\mathbf{S} \cdot \boldsymbol{\sigma}_m = \lambda_m \boldsymbol{\sigma}_m \qquad (36)$$

with m running from 1 to 4, and λ_m given by relations (28). These stress states are simply defined by equations of the form:

$$\boldsymbol{\sigma}_m = \mathbf{E}_m \cdot \boldsymbol{\sigma} \qquad (37)$$

with $\mathbf{E}_m$ given by relations (29). Denoting by $\boldsymbol{\sigma}$ the contracted stress tensor, which in the form of a 6-D vector is written as follows:

$$\boldsymbol{\sigma} = [\sigma_1, \ \sigma_2, \ \sigma_3, \ \sigma_4, \ \sigma_5, \ \sigma_6]^T , \qquad (38)$$

carrying out the calculations implied by relations (37), one finally has:

$$\boldsymbol{\sigma}_1= \left[\frac{1}{2}(\sigma_1\text{-}\sigma_2), \frac{1}{2}(\sigma_1\text{-}\sigma_2),\ 0,\ 0,\ 0,\ \sigma_6\right]^T$$

$$\boldsymbol{\sigma}_2=[0,\ 0,\ 0,\ \sigma_4,\ \sigma_5,\ 0]^T \tag{39}$$

$$\boldsymbol{\sigma}_3= \left[\frac{1}{\sqrt{2}}(\sigma_1+\sigma_2)\cos\omega+\sigma_3\sin\omega\right]\left[\frac{1}{\sqrt{}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega,\ 0,\ 0,\ 0\right]^T$$

$$\boldsymbol{\sigma}_4= \left[\frac{1}{\sqrt{2}}(\sigma_1+\sigma_2)\sin\omega\text{-}\sigma_3\cos\omega\right]\left[\frac{1}{\sqrt{2}}\sin\omega, \frac{1}{\sqrt{2}}\sin\omega,\text{-}\cos\omega,\ 0,\ 0,\ 0\right]^T$$

It is of course valid that $\boldsymbol{\sigma}=\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3+\boldsymbol{\sigma}_4$.

As it may be derived from relations (3), the characteristic states of stress, which correspond to the spectral decomposition of the compliance tensor **S** of a transtropic material, decompose the random stress tensor in a constantly prescribed manner. That is, states $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are *shears*, with $\boldsymbol{\sigma}_2$ simple shear and $\boldsymbol{\sigma}_1$ a superposition of pure and simple shear. The sum of $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ is the orthogonal supplement to the shear subspace of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ These two states, i.e., $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$, constitute equilateral stressing in the plane of isotropy and prescribed tension or compression along the material axis of symmetry.

Then, for a loading σ, for which it is valid that:

$\boldsymbol{\sigma} \in L_{\lambda K}$ (**S**)

the corresponding *coaxial* strain tensor and elastic energy are given by the following simplified relations:

$$\boldsymbol{\epsilon}=\lambda_K\boldsymbol{\sigma}$$

$$2T=\lambda_K\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}. \tag{40}$$

For a state of random stressing, which does not belong to any of the subspace $\mathbf{L}_{\lambda K}$ (**S**), the strain tensor and the elastic energy are given also in simplified form, i.e., relations (19) and (25), after performing decomposition (39).

As it is known from isotropic elasticity, the strain energy density at any given stress, $\boldsymbol{\sigma}$, can be separated into two parts, the *voluminal and the distortional*, accounting for recoverable elastic energy stored by dilation and distortion of the solid respectively.

Such a separation for the anisotropic solid, with identified parts as previously, is not in general conceivable. By means of decompositions of the stress tensor in the form of (39), it is possible, however, to distinguish some loadings or classes of anisotropic materials, for which such a decomposition of the elastic energy in dilatational and distortional parts constitutes a well defined process.

Let consider again the transtropic solid and its characteristic stress states, given by relations (39). The associated with $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ strain tensors, $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are related with pure distortion of the form, without any volume change. This is obvious, since the only normal strain components are those of tensor $\boldsymbol{\epsilon}_1$, for which it is valid that:

$$\varepsilon_1^1 + \varepsilon_2^1 = 0, \varepsilon_3^1 = 0$$

and of course:

$$\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 0.$$

Thus, the following part of the elastic energy of a transversely isotropic solid:

$$2T_d = \lambda_1 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 + \lambda_2 \boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_2 \tag{41}$$

is *purely distortional elastic energy.*

The remaining parts of the decomposition (39), i.e., $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$, they are not associated solely with distortional or dilatational elastic energy. Their respective tensors $\boldsymbol{\epsilon}_3$ and $\boldsymbol{\epsilon}_4$ produce both a volume change and a shape distortion. However, for some loading configurations, or special material characteristics, the work produced by the stresses $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ could be identified with a dilatational strain energy, or a distortional one.

Consider as an example the transversely isotropic materials which satisfy the following equation:

$$(1-\nu_T)/E_T = (1-\nu_L)/E_L \tag{42}$$

where ν_L, ν_T, E_L and E_T can take any value, where the moduli E_L, E_T must be positive and ν_L, ν_T assume values for which all λ_K are positive, in order to maintain the positive definite character of the elastic potential function, T.

Then, by means of relations (28), (32) and (39) it can be proved that the work of stresses $\boldsymbol{\sigma}_4$ is expressed by:

$$\lambda_4 \boldsymbol{\sigma}_4 \cdot \boldsymbol{\sigma}_4 \tag{43}$$

This quantity expresses a dilatational strain energy whereas, on the other hand, the work of $\boldsymbol{\sigma}_3$ is a distrortional one.

8. CONCLUSIONS

The energy-orthogonal decomposition of the space of secondrank symmetric tensors, and especially of the stress tensor $\boldsymbol{\sigma}$, was obtained by means of the spectral decomposition of the symmetric fourth-rank tensor, $\mathbf{S}$, which defines, in an unambiguous manner, the positive definite elastic energy function:

$$2T = \boldsymbol{\sigma} \cdot \mathbf{S} \cdot \boldsymbol{\sigma}.$$

The decomposition of tensor $\boldsymbol{\sigma}$ thus obtained for the transversely isotropic solid, gave four orthogonal (-energy) stress states, which decompose in a clear and radical manner the elastic energy function.

Two of them, i.e., $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, are solely associated with a distortional elastic energy, whereas the remaining two denote in general both voluminal and distortional elastic energies.

An interesting geometric interpretation arises for the energy-orthogonal stress states, if we consider the «projections» of $\boldsymbol{\sigma}_K$ in the principal 3-D stress-space. Then, the characteristic state $\boldsymbol{\sigma}_2$ vanishes, whereas stress states $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ are represented by three mutually orthogonal vectors oriented along directions with the following associated unit vectors:

$$\mathbf{e}_1: \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$

$$\mathbf{e}_3: \left(\frac{1}{\sqrt{2}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega\right) \tag{43}$$

$$\mathbf{e}_4: \left(\frac{1}{\sqrt{2}}\sin\omega, \sin\omega, -\cos\omega\right).$$

The eigenangle ω was defined by relation (32), where it was expressed by means of the components S_{ijkl} of the initial Cartesian coordinate system.

It is of interest noting that vectors $\mathbf{e}_3$ and $\mathbf{e}_4$ lie on the plane $\sigma_1 = \sigma_2$, where by $(0, \sigma_1 \sigma_2 \sigma_3)$ we denote the stress principal Cartesian coordinate system. Vector $\mathbf{e}_3$ subtends with axis σ_3 an angle $\left(\frac{\pi}{2} - \omega\right)$, whereas vector $\mathbf{e}_4$ subtends with the same axis angle (π-ω). Vector $\mathbf{e}_1$ is perpendicular to axis σ_3 and to the plane $\sigma_1=\sigma_2$ and lies on the deviatoric π-plane.

Let the initial coordinate system ($0, \sigma_1\sigma_2\sigma_3$) transform to the one dictated by the directions of $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_4$, with axis σ_3 having the direction of $\mathbf{e}_3$ and axis σ_1 this of $\mathbf{e}_1$. If the new coordinate system is denoted by ($0,\sigma_1\sigma_2\sigma_3$), then it is obvious that the expression for the elastic energy function becomes:

$$2T = \lambda_1 \bar{\sigma}_1^2 + \lambda_4 \bar{\sigma}_2^2 + \lambda_3 \bar{\sigma}_3^2. \tag{44}$$

By normalizing the total energy 2T giving the value $2T=1$, equation (44) represents an *ellipsoid*, centred at the origin 0 of the coordinate system and having as axes of symmetry the directions, $\mathbf{e}_1$, $\mathbf{e}_3$ and $\mathbf{e}_4$. The lenghts of the semi-axes of the ellipsoid along the axes of the coordinate system are respectively $1/\sqrt{\lambda_1}$, $1/\sqrt{\lambda_4}$ and $1/\sqrt{\lambda_3}$.

Moreover, if the fourth-rank tensor **S** describes the isotropic solid, then relation (44) represents an *ellipsoid of revolution* with its major semi-axis along $\mathbf{e}_4$, having the direction of the hydrostatic axis, i.e., $\sigma_1=\sigma_2=\sigma_3$, and the equal two semi-axes lying on the deviatoric π-plane. In this case, $\lambda_1=\lambda_3=\lambda_D$ and $\lambda_4=\lambda_p$ with λ_p and λ_D given by relations (35). The representation of the elastic energy for the isotropic solid by the ellipsoid of revolution is due to Beltrami according to Stassi d' Alia [11].

Thus, the energy orthogonal stress states, decomposing a given stressing $\boldsymbol{\sigma}$, were shown to also decompose appropriately the elastic energy function, as described by relation (25), and when represented geometrically in a principal stress space, they lie along the directions or the semi-axes of the ellipsoid reprenented by relation (44), which is the geometric representation of the elastic energy function, when it is normalized to $2T=1$.

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ΠΕΡΙΛΗΨΙΣ

Ὁ τανυστὴς τετάρτης τάξεως ἐνδόσεως ἐγκαρσίως ἰσοτρόπων μέσων καὶ ἡ φασματικὴ ἀποσύνδεσίς του.

Τὸ γνωστὸν κριτήριον ἀστοχίας τῶν ὑλικῶν τὸ διατυπωθὲν ὑπὸ τῶν Huber, Mises καὶ Hencky (HMH) εἰς τὸν χῶρον τῶν κυρίων τάσεων εἶναι κατάλληλον δι' ἐλαστικῶς ἰσότροπα στερεὰ μὴ παρουσιάζοντα φαινόμενα διαφορῶν ἀντοχῆς εἰς ἐφελκυσμὸν καὶ θλῖψιν. Τὸ κριτήριον αὐτὸ εἰς τὸν χῶρον τῶν κυρίων τάσεων ἀπεικονίζεται ὡς κύλινδρος κυκλικῆς ἐγκαρσίας διατομῆς μὲ ἄξονα συμμετρίας τὸν ὑδροστατικὸν ἄξονα εἰς τὸν χῶρον τῶν κυρίων τάσεων.

Τὸ κριτήριον αὐτὸ ἐπεξετάθη ὑπὸ τοῦ Hill, τὸ 1948, δι' ὀρθότροπα στερεά, μὴ παρουσιάζοντα πάλιν φαινόμενα διαφόρου ἀντοχῆς εἰς ἐφελκυσμὸν καὶ θλῖψιν. Εἰς τὴν περίπτωσιν τῶν ὀρθοτρόπων στερεῶν ὁ τόπος αὐτὸς διαρροῆς καὶ ἀστοχίας παρίσταται ὑπὸ ἐλλειπτικοῦ κυλίνδρου μὲ ἄξονα συμμετρίας πάλιν τὸν ὑδροστατικὸν ἄξονα.

Δεδομένου ὅτι ἅπαντα τὰ στερεὰ τὰ χρησιμοποιούμενα εἰς τὰς ἐφαρμογὰς παρουσιάζουν τὸ φαινόμενον τῆς διαφορᾶς ἀντοχῆς εἰς ἐφελκυσμὸν καὶ θλῖψιν, τὰ κριτήρια αὐτὰ μόνον δι' ἰδανικὰς περιπτώσεις εἶναι κατάλληλα.

Κριτήρια τὰ ὁποῖα λαμβάνουν ὑπ' ὄψιν των τὸ φαινόμενον διαφόρου ἀντοχῆς (ΦΔΑ) εἶναι τὰ κριτήρια τὰ παριστώμενα ἀπὸ μονοχώνους ἐπιφανείας, ὡς τὸ κριτήριον Coulomb (κωνικὴ ἐπιφάνεια, 1773), τὸ κριτήριον Hoffman (1967) καὶ τὸ κριτήριον Tsai-Wu (1971). Ἐκτὸς τούτων, τὰ ὁποῖα εἴτε θεωροῦνται εἰδικαὶ περιπτώσεις τῶν κατωτέρω ἐμφανιζομένων, εἴτε εἶναι ἀπηρχαιωμένα καὶ ἀκατάλληλα, τελευταίως εἰσήχθησαν ὑπὸ τοῦ συγγραφέως τὰ ἐκ περιστροφῆς συμμετρικὰ παραβολοειδῆ καὶ τὰ ἐλλειπτικὰ παραβολοειδῆ δι' ἰσότροπα ἢ ὀρθότροπα στερεὰ ἀντιστοίχως. Καὶ αἱ δύο μονόχωνοι αὐταὶ ἐπιφάνειαι διατηροῦν ὡς ἄξονα συμμετρίας ἄξονα παράλληλον πρὸς τὸν ὑδροστατικόν, ὁ ὁποῖος μεταπίπτει καὶ ταυτίζεται πρὸς τὸν ὑδροστατικὸν ἄξονα διὰ τὰ παραβολοειδῆ ἐκ περιστροφῆς, κατάλληλα δι' ἰσότροπα μέσα.

Εἶναι γνωστὸν ὅτι διὰ τὰ ἰσότροπα στερεὰ ἡ πυκνότης τῆς ἐλαστικῆς ἐνεργείας παραμορφώσεων (ΠΕΠ) δύναται νὰ χωρισθῆ εἰς δύο ὅρους, ἤτοι εἰς τὴν ἐνέργειαν μεταβολῆς τοῦ ὄγκου ἢ διογκωτικὴν ἐνέργειαν καὶ εἰς τὴν στροφικὴν ἐνέργειαν. Πράγματι, διὰ δύο τυχαίας φορτίσεις $\boldsymbol{\sigma}_a$ καὶ $\boldsymbol{\sigma}_b$, εὑρισκομένας ἐντὸς τῆς ἐλαστικῆς περιοχῆς, ἡ ὁλικὴ (ΠΕΠ) $T(\boldsymbol{\sigma}_v+\boldsymbol{\sigma}_D)$ δίδεται ἀπὸ τὴν σχέσιν:

$T(\boldsymbol{\sigma}_V + \boldsymbol{\sigma}_D) = T(\boldsymbol{\sigma}_V) + T(\boldsymbol{\sigma}_D)$

᾿Αλλὰ ἡ (ΠΕΠ) ἑκάστης φορτίσεως, βάσει τοῦ νόμου τοῦ Hooke, δίδεται ὡς $T(\boldsymbol{\sigma}_V) = \boldsymbol{\sigma}_V \cdot \mathbf{S} \cdot \boldsymbol{\sigma}_V$ καὶ $T(\boldsymbol{\sigma}_D) = \boldsymbol{\sigma}_D \cdot \mathbf{s} \cdot \boldsymbol{\sigma}_D$
ὅπου $\mathbf{s}$ ἐκφράζει τὸν τανυστὴν 4ης τάξεως ἐνδόσεως τοῦ ὑλικοῦ.
Τὸ ἄθροισμα τῶν (ΠΕΠ) δίδει:
$T(\boldsymbol{\sigma}_V + \boldsymbol{\sigma}_D) = \boldsymbol{\sigma}_V \cdot \mathbf{S} \cdot \boldsymbol{\sigma}_V + \boldsymbol{\sigma}_D \cdot \mathbf{S} \cdot \boldsymbol{\sigma}_D + 2\boldsymbol{\sigma}_V \cdot \mathbf{S} \cdot \boldsymbol{\sigma}_D$
᾿Αποδεικνύεται ὅτι μόνον διὰ τὰ ἰσότροπα στερεὰ $\boldsymbol{\sigma}_V \cdot \mathbf{s} \cdot \boldsymbol{\sigma}_D = 0$ καὶ ἑπομένως εἶναι δυνατὸς ὁ διαχωρισμὸς τῶν δύο μορφῶν τῶν ΠΕΠ, ὡς ἀνωτέρω ἀνεφέρθη.

Προγενέστεραι προσπάθειαι διαφόρων ἐπιστημόνων κατέληξαν εἰς τὸ συμπέρασμα ὅτι διὰ τὸ ἀνισότροπον σῶμα ἡ ἀποσύνδεσις τῶν δύο μορφῶν ἐνεργείας δὲν εἶναι δυνατὴ καὶ τοιουτοτρόπως ἡ προσπάθεια γενικεύσεως τοῦ κριτηρίου διαρροῆς τῶν Huber-Mises-Hencky καὶ καθιερώσεως τῆς στροφικῆς ἐνεργείας παραμορφώσεων ὡς κρισίμου μεγέθους διὰ τὴν ἔναρξιν τῆς διαρροῆς τοῦ ἀνισοτρόπου μέσου ἀπέτυχε, δεδομένου ὅτι ἀπεδείχθη ὅτι μόνον εἰς εἰδικὰς περιπτώσεις, ὅπου τὰ μέτρα ἐλαστικότητος τοῦ σώματος εἶναι τὰ αὐτὰ πρὸς τὰς τρεῖς κυρίας διευθύνσεις τῶν τάσεων (κυβικὰ κρυσταλλικὰ συστήματα) εἶναι δυνατὴ ἡ τοιαύτη ἀποσύνδεσις.

Κατόπιν τῆς θεωρητικῆς ἀποδείξεως ὑπὸ τοῦ Rychlewski ὅτι διὰ τὸ ἀνισότροπον σῶμα ἰσχύει ἡ δυνατότης ἀποσυνδέσεως τοῦ ἐλαστικοῦ του δυναμικοῦ σὲ στροφικὴ καὶ διογκωτικὴ (ΠΕΠ) παρουσιάζεται ἐδῶ ἡ τοιαύτη ἀποσύνδεσις, ἐκτὸς ἐκείνων ποὺ παρουσιάζουν καταλλήλως συζευγμένας ἐλαστικὰς ἰδιότητας, καὶ εἰς τὰς περιπτώσεις τῆς φασματικῆς ἀναλύσεως τοῦ τανυστοῦ τετάρτης τάξεως ἐνδόσεως τοῦ μέσου.

Περιοριζόμενοι εἰς τὴν ἐνδιαφέρουσαν περίπτωσιν τῶν ἐγκαρσίως ἰσοτρόπων μέσων, ποὺ χρησιμοποιοῦνται πολλαπλῶς σήμερον εἰς τὰς ἐφαρμογὰς περιγράφομεν τὴν φασματικὴν ἀποσύνδεσιν τοῦ τανυστοῦ ἐνδόσεως $\mathbf{S}$ τοῦ μέσου καὶ δίδομεν τὰς ρίζας τοῦ ἐλαχίστου πολυωνύμου του μὲ τοὺς ἀντιστοιχοῦντας ἰδιοτανυστάς.

Πράγματι, ἐὰν ὁ τανυστὴς τετάρτης τάξεως τῆς ἐνδόσεως $\mathbf{S}$ τοῦ μεσου ἀναλυθῇ εἰς τοὺς ἰδιοτανυστὰς τοῦ $\boldsymbol{\sigma}_k$, τοὺς ἀντιστοιχοῦντας εἰς τὰς ἰδιοτιμὰς λ_k, ἀποδεικνύεται ὅτι διὰ τὰς φορτίσεις $\boldsymbol{\sigma}_k$ οἱ ἀντίστοιχοι τανυσταὶ παραμορφώσεως $\boldsymbol{\epsilon}_k$ εἶναι ἀνάλογοι ἀντιστοίχως, μὲ λόγον ἀναλογίας τὴν ἀντίστοιχον ἰδιοτιμὴν λ_k τὴν συνηρτημένην μὲ ἕκαστον ἰδιοτανυστὴν $\boldsymbol{\sigma}_k$. Περαιτέρω, οἱ ἰδιοτανυσταὶ $\boldsymbol{\sigma}_k$ ἐπαληθεύουν τὴν σχέσιν:

$$\boldsymbol{\sigma}_k \cdot \mathbf{S} \cdot \boldsymbol{\sigma}_N = 0 \qquad k \neq N \tag{1}$$

ὁπότε συνάγεται ἡ κατωτέρω ἰδιότης των.

Οἱ ἰδιοτανυσταὶ $\boldsymbol{\sigma}_k$ ἐπαληθεύουν τὴν σχέσιν $\boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_N = 0$ διὰ τιμὰς τῶν $k \neq N$, καὶ εἰς τὴν περίπτωσιν αὐτὴν τὸ ἐλαστικὸν δυναμικὸν $T(\sigma_1+\sigma_2+...\sigma_N)$ ἰσοῦται ἀντιστοίχως πρὸς τὸ ἄθροισμα:

$T(\sigma_1)+T(\sigma_2)+...+T(\sigma_N)$

Φορτίσεις διὰ τανυστῶν ἐχόντων τὰς ἀνωτέρω ἰδιότητας ὠνομάσθησαν ἐνεργειακῶς ὀρθογώνιοι φορτίσεις καὶ τοῦτο εἶναι δυνατὸν δι' εἰσαγωγῆς τοῦ γενικευμένου νόμου Hooke $\boldsymbol{\epsilon}_k=\lambda_k\cdot\boldsymbol{\sigma}_k$ εἰς τὴν συνθήκην (1) νὰ ἀποδείξωμεν ὅτι:

$\boldsymbol{\sigma}_k\cdot\mathbf{S}\cdot\boldsymbol{\sigma}_N=\boldsymbol{\sigma}_k\cdot\boldsymbol{\epsilon}_N=0$ διὰ $k\neq N$

Ἡ σχέσις αὐτὴ ἀποδεικνύει ὅτι ὁ τανυστὴς τάσεως $\boldsymbol{\sigma}_k$ εἶναι ὀρθογώνιος πρὸς τὸν τανυστὴν $\boldsymbol{\epsilon}_N$ καὶ ἑπομένως τὸ ἐσωτερικόν του γινόμενον ἰσοῦται πρὸς τὸ μηδέν, ἤτοι ἡ τάσις $\boldsymbol{\sigma}_k$ δὲν παράγει ἔργον μὲ τὴν παραμόρφωσιν $\boldsymbol{\epsilon}_N$ (διὰ $k\neq N$).

Διὰ καρτεσιανὸν πλαίσιον ἀναφορᾶς πρὸς τὸ ὁποῖον ἀναφέρομεν τὰς συνιστώσας τῶν ἀνωτέρω τανυστῶν, καὶ τοῦ ὁποίου αἱ διευθύνσεις ταυτίζονται μὲ τὰς κυρίας διευθύνσεις τοῦ μέσου καὶ μὲ τὸν ἄξονα σ_3 ὡς τὸν ἰσχυρὸν ἄξονα τοῦ μέσου εὑρίσκονται αἱ συνιστῶσαι τοῦ τανυστοῦ ἐνδόσεως S_{ijkl} ἐκπεφρασμέναι συναρτήσει τοῦ μέτρου καὶ τοῦ λόγου Poisson τοῦ ὑλικοῦ, καθὼς καὶ αἱ ἐκφράσεις τῶν ἰδιοτιμῶν λ_N τοῦ τανυστοῦ ἐνδόσεως, ἐκ τῶν ὁποίων δύο εἶναι διπλῆς πολλαπλότητος καὶ ἑπομένως ὁ ἀριθμὸς τῶν διαφόρων ἰδιοτιμῶν λ_N καὶ S_{ijkl} περιορίζεται εἰς τέσσαρας.

Περαιτέρω δίδεται τὸ σύνολον τῶν τανυστῶν $\{\mathbf{E}_N\}$ τοῦ τανυστοῦ $\mathbf{I}$ οἱ ὁποῖοι προφανῶς ἀνέρχονται πάλιν εἰς τέσσαρας καὶ ὁρίζονται ἀπὸ τοὺς συμμετρικοὺς τανυστὰς **a** καὶ **b** καθὼς καὶ ἀπὸ τοὺς τανυστὰς **f** καὶ **g** ἐξηρτημένους ἀπὸ τὰς συνιστώσας τοῦ **S**.

Κατ' αὐτὸν τὸν τρόπον ὁρίζεται πλήρως ἡ φασματικὴ ἀνάλυσις τοῦ τανυστοῦ ἐνδόσεως **S**. Τέλος, ὁρίζεται ἡ ἰδιογωνία **ω** ἐκφραζομένη συναρτήσει τῶν μηχανικῶν σταθερῶν τοῦ σώματος, συνάγεται δὲ τὸ συμπέρασμα ὅτι αἱ τέσσαρες ἰδιοτιμαὶ λ_N καὶ τὸ ὅρισμα ω ἀποτελοῦν τὴν ἁπλουστέραν πεντάδα μεγεθῶν τὰ ὁποῖα ὁρίζουν πλήρως τὴν μηχανικὴν συμπεριφορὰν παντὸς ἐγκαρσίως ἰσοτρόπου μέσου.

Ἐπὶ τῇ βάσει αὐτῶν τῶν τιμῶν ὁρίζονται αἱ τέσσαρες ἰδιοτανυσταὶ $\boldsymbol{\sigma}_1$, σ_2, $\boldsymbol{\sigma}_3$ καὶ $\boldsymbol{\sigma}_4$ συναρτήσει τῶν σ_1, σ_2, σ_3, σ_4 καὶ σ_5 συνιστωσῶν τῶν τάσεων καὶ τῆς γωνίας ω, ἀποδεικνύεται δὲ ὅτι οἱ ἰδιοτανυσταὶ $\boldsymbol{\sigma}_1$ καὶ $\boldsymbol{\sigma}_2$ εἶναι ἀποκλίνοντες καὶ παραμένουν οἱ αὐτοὶ δι' ὅλα τὰ ἐγκαρσίως ἰσότροπα μέσα, ἐν ἀντιθέσει πρὸς τοὺς ἰδιοτανυστὰς $\boldsymbol{\sigma}_3$ καὶ $\boldsymbol{\sigma}_4$, οἱ ὁποῖοι ἐξαρτῶνται ἐκ τῶν τιμῶν τῶν συνιστωσῶν τοῦ τανυστοῦ **S** καὶ τοῦ ὁρίσματος ω. Οἱ τανυσταὶ $\boldsymbol{\sigma}_1$ καὶ $\boldsymbol{\sigma}_2$ παριστοῦν διατμήσεις καὶ δὴ ὁ $\boldsymbol{\sigma}_2$ παριστᾶ ἁπλὴν διάτμησιν, ἐνῶ ὁ $\boldsymbol{\sigma}_1$ - τανυστὴς καθαρὰν διάτμησιν μὲ ὑπέρθεσιν ἁπλῆς διατμήσεως. Οἱ τανυσταὶ $\boldsymbol{\sigma}_3$ καὶ $\boldsymbol{\sigma}_4$ παριστοῦν τριαξονικὰς ὀρθὰς τάσεις ἀποτελουμένας ἀπὸ ὑδροστατικὸν ἐφελκυσμὸν κατὰ τὸ ἰσότροπον ἐπίπεδον σὲ ὑπέρθεσιν μὲ ἀξονικὸν ἐφελκυσμὸν κατὰ τὸν ἰσχυρὸν ἄξονα τοῦ μέσου διὰ τὸν $\boldsymbol{\sigma}_3$ - τανυστὴν καὶ ἀξονικὴν θλῖψιν κατὰ τὸν αὐτὸν ἄξονα διὰ τὸν $\boldsymbol{\sigma}_4$ - τανυστήν.

Κατὰ συνέπειαν ἡ ἀποσύνδεσις τοῦ τανυστοῦ $\boldsymbol{\sigma}$ τῶν τάσεων διὰ τὸ ἐγκαρσίως ἰσότροπον σῶμα δίδει δύο καταστάσεις συνδεομένας ἀποκλειστικῶς μὲ τὴν στροφικὴν ἐλαστικὴν ἐνέργειαν, ἐνῶ αἱ ὑπόλοιποι δύο παριστοῦν συνδυασμοὺς διογκωτικῆς καὶ στροφικῆς ἐνεργείας. Ἡ ἀποσύνδεσις αὐτὴ ἀποτελεῖ τὴν ἁπλουστάτην δυνατὴν διὰ τὸ ἐγκαρσίως ἰσότροπον σῶμα.