ΜΑΘΗΜΑΤΙΚΑ.— **Modular hyper-lattices**, *by Maria Konstantinidou-Serafimidou**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

In [**3**] paper the definition of a hyper-lattice has been given, which is the following:

Definition 1. We call *hyper-lattice* a set H on which a hyper-operation $a \vee b$ (union) and an operation $a \wedge b$ (intersection) have been defined, which satisfy the following axioms:

I. $a \in a \vee a \quad a \wedge a = a$

II. $a \vee b = b \vee a \quad a \wedge b = b \wedge a$

III. $(a \vee b) \vee c = a \vee (b \vee c), \quad (a \wedge b) \wedge c = a \wedge (b \wedge c)$

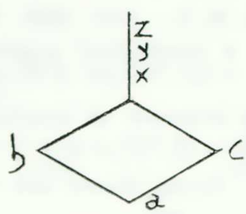
IV. $a \in [a \vee (a \wedge b)] \cap [a \wedge (a \vee b)]$

V. $a \in a \vee b \Longrightarrow b = b \wedge a$.

Examples 1. a) The set $H = \{0, 1\}$ with a hyper-operation $0 \vee 0 = 0$, $0 \vee 1 = 1 \vee 0 = 1$, $1 \vee 1 = H$ and an operation $0 \wedge 0 = 1 \wedge 0 = 0 \wedge 1 = 0$, $1 \wedge 1 = 1$ is a hyper-lattice. Respectively, for $0 \vee 0 = H$, $0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1$ and $0 \wedge 0 = 1 \wedge 0 = 0 \wedge 1 = 0$, $1 \wedge 1 = 1$.

b) The set which is given in the diagram

Fig. 1.



is obviously a $\wedge$-semi-lattice. If a hyper-operation $\vee$ is defined as $x \vee y = \{z \in H : y \leqslant Z\}$, for each pair $x, y \in H$ with $x \leqslant y$ and $b \vee c = H \ldots \{a, b, c\}$ then it is easy to verify that the structure $(H, \wedge, \vee)$ is a hyper-lattice. It is obvious that the sup(a, b) does not exist.

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* ΜΑΡΙΑΣ ΚΩΝΣΤΑΝΤΙΝΙΔΟΥ - ΣΕΡΑΦΕΙΜΙΔΟΥ, **Περὶ εἰδικῆς κατηγορίας ὑπερδικτυωτῶν.**

In the present paper I deal with the class of the modular hyper-lattices, which are defined as follows:

Definition 2. We call a hyper-lattice H a *modular* one, when in addition it satisfies the axiom

$$a \leqslant b \not\Rightarrow a \vee (c \wedge b) = (a \vee c) \wedge b$$

for any $c \in H$ (as in the case of lattices).

Remark 1. Obviously, in every modular hyper-lattice we have

$$a \vee (b \wedge a) = (a \vee b) \wedge a.$$

Examples 2. a) The ordered set $H = \{o, a, u\}$ with an obvious operation and a hyper-operation as $o \vee a = a \vee o = a$, $o \vee u = u \vee o = u$, $u \vee u = \{a, u\}$ is a modular hyper-lattice.

b) The ordered set H, which is given in the diagram below is verified that is a modular hyper-lattice when the operation is defined as

$$a \wedge a = a \wedge b = b \wedge a = a \wedge c = c \wedge a = b \wedge c = c \wedge b = a \wedge d = d \wedge a = a$$
$$b \wedge d = d \wedge b = b, \quad c \wedge d = d \wedge c = c, \quad d \wedge d = d$$

and the hyper-operation as in the following table

v	a	b	c	d
a	a	b	c	d
b	b	{a,b}	d	d
c	c	d	{a,c}	d
d	d	d	d	H

Fig. 2.

P r o p o s i t i o n 1. *If* a, b *elements of* H *such that* $a \leqslant b$, *then the relation*

$$a \wedge (c \vee b) \subseteq \{[(a \vee b) \wedge b] \vee c\} \wedge a$$

is held for any $c \in H$.

Proof. Indeed, because of $a \leqslant b$, that is $b \in a \vee b$, we obtain

$$b \wedge b = b \in (a \vee b) \wedge b \Rightarrow (b \vee b) \subseteq [(a \vee b) \wedge b] \vee c \Rightarrow a \wedge (b \vee c) \subseteq$$
$$\subseteq \{[(a \vee b) \wedge b] \vee c\} \wedge a.$$

P r o p o s i t i o n 2. *If* $a, b, c \in H$ *such that* $a \leqslant b$ *and* c *is comparable with either* a *or* b, symbolically $c \# a$ or $c \# b$, then we have $[a \vee (c \wedge b)] \cap [(a \vee c) \wedge b] \neq \varnothing$.

Proof. Suppose that $c \leqslant b$, thus $c \wedge b = c$, from which we obtain $a \vee (c \wedge b) = a \vee c$. Because it is also $a \leqslant b$, there will be an element $x \in a \vee c = a \vee (c \wedge b)$ such that $x \leqslant b$ [**3**]. Therefore $b \wedge x = x \in (a \vee c) \wedge b$, thus

$$[a \vee (c \wedge b)] \cap [(a \vee c) \wedge b] \neq \varnothing.$$

If $a \leqslant c$, it is $a \wedge b \leqslant c \wedge b$, which, because of $a \leqslant b$ becomes $a \leqslant c \wedge b$, that is $c \wedge b \in a \vee (c \wedge b)$. Also, from the fact that $a \leqslant c$, and thus from the fact that $c \in a \vee c$, we obtain $c \wedge b \in (a \vee b) \wedge b$. Therefore and in this case we have

$$[a \vee (c \wedge b)] \cap [(a \vee c) \wedge b] \neq \varnothing.$$

If we work in an analogous way for the case in which it is $b \leqslant c$, and for $c \leqslant a$ as well, we see that the proposition is also true.

Obviously, when a, b, c are elements of a hyper-lattice and fulfil the conditions of the proposition 2 we have

$$a \vee (c \wedge b) = (a \vee c) \wedge b.$$

In the continuity, H will present a modular hyper-lattice, and we have the propositions:

P r o p o s i t i o n 3. *If* a, b, c *are elements of* H, *then it is*

$$\left.\begin{array}{l} a \leqslant b \\ c \vee a = c \vee b \\ c \wedge a = c \wedge b \end{array}\right\} \Rightarrow a = b.$$

Proof. Indeed, provided that H is a modular hyper-lattice, from the fact that $a \leqslant b$ it follows that $a \vee (c \wedge b) = (a \vee c) \wedge b$, which because of the relations $c \vee a = c \vee b$ and $c \wedge a = c \wedge b$ becomes $a \vee (c \wedge a) = (b \vee c) \wedge b$. Thus $b \in a \vee (c \wedge a)$, because it is $b \in (b \vee c) \wedge b$. On the other hand, we have $a \vee (c \wedge a) = (a \vee c) \wedge a$ [rem. 1]. Therefore it will be $b \in (a \vee c) \wedge a$. From this relation it follows that there will be an element $x \in a \vee c$ such that $b = x \wedge a$, that is $b \leqslant a$. Because it is also $a \leqslant b$, by supposition, it will be $a = b$.

The conditions of the above proposition are not sufficient in order the hyper-lattice to be a modular one, as we see it in the example (1. b), where, while the conditions are held, the hyper-lattice is not a modular one.

Indeed, this hyper-lattice is not a modular one, because

$$b \leqslant x \Rightarrow b \vee (c \wedge x) = (b \vee c) \wedge x$$

since $\quad b \vee (c \wedge x) = b \vee c = H \,..\, \{a, b, c\}$

and $\quad (b \vee c) \wedge x = [H \,..\, \{a, b, c\}] \wedge x = \{H \,..\, \{a, b, c\}\} \,..\, \{z \in H : x < z\}$

although the relations

$$a' \leqslant b'$$
$$c' \vee a' = c' \vee b'$$
$$c' \wedge a' = c' \wedge b'$$

are held, as it is obvious, only if $a' = b'$.

In the lattices, as it is known, the above proposition is held and inversely.

P r o p os i t i o n 4. *For* a, b, c, d *of* H *we have*

$$\left.\begin{array}{l} a \leqslant c \\ b \leqslant d \end{array}\right\} \Rightarrow a \vee [d \wedge (c \vee b)] = [(a \vee d) \wedge c] \vee b.$$

Proof. Since H is a modular hyper-lattice, it will be

$$b \leqslant d \Rightarrow d \wedge (c \vee b) = b \vee (d \wedge c) \Rightarrow a \vee [d \wedge (c \vee b)] = (a \vee b) \vee (d \wedge c)$$

and

$$a \leqslant c \Rightarrow (a \vee d) \wedge c = a \vee (d \wedge c) \Rightarrow [(a \vee d) \wedge c] \vee b =$$
$$= [a \vee (d \wedge c)] \vee b = (a \vee b) \vee (d \wedge c).$$

Therefore $a \vee [d \wedge (c \vee b)] = [(a \vee d) \wedge c] \vee b$.

Likewise, as in the previous proposition, the above conditions are not sufficient for a hyper-lattice to be a modular one in contrary with the case of lattices, where, it is known, this happens.

We already know example (1. b) that for any $(a, b) \in H \times H$, the sup (a, b) generally does not exist. After that, we examine the upper bounds of a and b, and also their position with respect to the union

$a \vee b$, when, of course, they exist. The following proposition refer to such bounds of two or more elements, which, as we shall see, play an important role in the structure of the modular hyper-lattices.

Proposition 5. *If* a, b, d *are elements of* H, *we shall have* $a \leqslant d$ *and* $b \leqslant d$, *if and only if for every* $x \in a \vee b$ *it is* $x \leqslant d$.

Proof. Because H is a modular hyper-lattice the relation $a \leqslant d$ implies the relation $a \vee (b \wedge d) = (a \vee b) \wedge d$, which because of the relation $b \leqslant d$ becomes $a \vee b = (a \vee b) \wedge d$. Consequently, for each $x \in a \vee b$ there is an $x' \in a \vee b$, such that $x' \wedge d = x$. From this relation we conclude that for each $x \in a \vee b$ it is $x \leqslant d$.

The inverse of the proposition is not necessary to be proved, because, [**3**], it holds generally in every hyper-lattice.

Proposition 6. *If* $a_i \in H$ *for every* $i \in I = \{1, 2, \ldots n\}$, $n \in \mathbb{N}$ *and* $d \in H$ *we shall have* $a_i \leqslant d$ *for all* $i \in I$, *if and only if for every* $x \in a_1 \vee a_2 \vee \ldots \vee a_n$ *it is* $x \leqslant d$.

Proof. The proposition will be proved by induction on the natural number n.

Obviously the proposition is held for $k = 2$ [pr. 5] and we suppose that it is held for $k = n - 2$. Thus, if for each $i \in I \, . \, . \{n\}$ it is $a_i \leqslant d$, then we shall have

$$a_1 \vee a_2 \vee \ldots \vee a_{n-1} = d \wedge (a_1 \vee a_2 \vee \ldots \vee a_{n-1}).$$

From this relation and if $a_n \in H$ such that $a_n \leqslant d$ we obtain

$$(a_1 \vee a_2 \vee \ldots \vee a_{n-1}) \vee a_n = [d \wedge (a_1 \vee a_2 \vee \ldots \vee a_{n-1})] \vee a_n \Longrightarrow a_1 \vee a_2 \vee \ldots \vee a_n =$$

$$= [\bigcup_{w \in a_1 \vee \ldots \vee a_{n-1}} (d \wedge w)] \vee a_n = \bigcup_{w \in a_1 \vee \ldots \vee a_{n-1}} [(a_n \vee w) \wedge d] =$$

$$= [\bigcup_{w \in a_1 \vee \ldots \vee a_{n-1}} (a_n \vee w)] \wedge d = (a_1 \vee a_2 \vee \ldots \vee a_n) \wedge d.$$

Therefore for each $x \in a_1 \vee a_2 \vee \ldots \vee a_n$ there exists an $x' \in a_1 \vee a_2 \vee \ldots \vee a_n$ such that $x = x' \wedge d$. Thus, it follows that for every $x \in a_1 \vee a_2 \vee \ldots \vee a_n$ we have $x \leqslant d$.

Inversely, if for each $x \in a_1 \vee a_2 \vee \ldots \vee a_n$ it is $x \leqslant d$ then, because of $a_i \leqslant x_{a_i}^{a_1 \vee a_2 \ldots \vee a_{i-1} \vee a_i \vee \ldots \vee a_n}$ (1) [**3**], it follows that $a_i \leqslant d$.

Corollary 1. If $a_i \in H$ *for every* $i \in I = \{1, 2, \ldots, n\}$ $d \in H$ *and* $a_i \leqslant d$, *then* d *is an upper* bound of the sub-hyper-lattice h, which is generated of the elements $a_1, a_2, \ldots, a_n$ (2).

Proposition 7. *If an upper bound* d *of two elements* a, b *of a modular hyper-lattice belongs to the union* $a \vee b$, *then it is unique and furthermore it is* $d = \sup(a, b) = \sup(a \vee b)$.

Proof. Because of $a \leqslant d$ and $b \leqslant d$, we have $x \leqslant d$ for each $x \in a \vee b$ [pr. 5]. If now there exists another element d′, such that it belongs to the union $a \vee b$ and $a \leqslant d'$, $b \leqslant d'$, then, for every $x \in a \vee b$ we shall have $x \leqslant d'$ and thus $d \leqslant d'$. Likewise it is proved that $d' \leqslant d$, and so it is $d = d'$. So, we conclude that a union $a \vee b$ possesses at most one upper bound of a and b. On the other hand, for any other upper bound d_1 of a and b, it will be $x \leqslant d_1$ for each $x \in a \vee b$ and thus $d \leqslant d_1$ [pr. 5]. Therefore it is $d = \sup(a, b)$. Similarly, if d^* is an upper bound of the union $a \vee b$, it will be [pr. 5] an upper bound of a and b and thus $d \leqslant d^*$. Thus $d = \sup(a \vee b)$.

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(1) In [**3**] has been prooved that for each pair $(a, b) \in H \times H$ there exist elements $x_a^{a \vee b}$, $x_b^{a \vee b} \in a \vee b$, which are called *distinguished elements of the pair* (a, b), such that $a \leqslant x_a^{a \vee b}$, $b \leqslant x_b^{a \vee b}$. So, we have the *distinguished sets* of the pair (a, b) $\mathscr{X}_a^{a \vee b} = \{x_a^{a \vee b} \in a \vee b : a \leqslant x_b^{a \vee b}\}$, $\mathscr{X}_b^{a \vee b} = \{x_b^{a \vee b} \in a \vee b : b \leqslant x_b^{a \vee b}\}$.

More generally, for $a_1, a_2, \ldots, a_n \in H$ there exist elements

$$x_{a_i}^{a_1 \vee a_2 \vee \ldots a_{i-1} \vee a_{i+1} \ldots \vee a_n}$$

for each $i \in I = \{1, 2, \ldots, n\}$ which are called distinguished elements of the n-ple $(a_1, a_2, \ldots, a_n)$, such that $a_i \leqslant x_{a_i}^{a_1 \vee a_i \vee \ldots \vee a_{i-1} \vee a_{i+1} \vee \ldots \vee a_n}$.

(2) A subset $h \neq \varnothing$ of a hyper-lattice H is its *sub-hyper-lattice* if it is a hyper-lattice with respect to the operation $\wedge$ and the hyper-operation $\vee$ of H and it is prooved that $h \wedge h \subseteq h$ and $h \vee h \subseteq h$.

The sub-hyper-lattice $h \subseteq H$ which is generated of a subset $h' \subseteq H$ is the union of the subsets of H, which are created when we apply the operation $\wedge$ and the hyper-operation $\vee$ on the elements of h′ in any way and in a finite number of times.

Corollary 2. If $a \leqslant b$ *then* $b = \sup(a \vee b)$. *In the same way the following proposition is proved.*

Proposition 8. *If an upper bound* d *of* n *elements* $a_1, a_2, \ldots, a_n$ *of* H *belongs to the union* $a_1 \vee a_2 \vee \ldots \vee a_n$ *then it is unique and*

$$d = \sup\{a_1, a_2, \ldots, a_n\} = \sup(a_1 \vee a_2 \vee \ldots \vee a_n).$$

Corollary 3. If $d \in H$ *satisfies the conditions of the proposition* 8, *then it is the supremum of the sub-hyper-lattice which is generated of the elements* $a_1, a_2, \ldots, a_n$ *and furthermore it is its maximum.*

Proof. Indeed, if h is the sub-hyper-lattice which is generated of the elements $a_1, a_2, \ldots, a_n$, then [corol. 1] d is an upper bound of the elements of h. However, provided that d satisfies the conditions of the proposition 8 and because of the way of the construction of h we shall have $d = \sup h$. On the other hand $d \in h$, because

$$d \in a_1 \vee a_2 \vee \ldots \vee a_n \subseteq h,$$

so d is the maximum element of h as well.

Remark 2. It is evident of the previous that for every $a, b \in H$ we have

$$a \notin \mathscr{X}_a^{a \vee b} \Longrightarrow \mathscr{X}_a^{a \vee b} \cap (a \vee b) \neq \varnothing.$$

Proposition 9. *If* a, b, c *are elements of* H, *then*

$$\left.\begin{array}{c} c \leqslant a, \quad c \leqslant b \\ a \vee c = b \vee c \end{array}\right\} \Longrightarrow a = b.$$

Proof. From $c \leqslant a$, that is $a \in a \vee c$, it follows that $a = \sup(a, c)$. Thus, (pr. 7) it will be $a = \sup(a \vee c) = \sup(b \vee c)$, because $a \vee c = b \vee c$. Consequently, it follows that $b \leqslant a$. Similarly, from $c \leqslant b$ we obtain $a \leqslant b$. Thus $a = b$.

With respect to the existence of the supremum of two elements of a modular hyper-lattice, we have the following propositions:

Proposition 10. *If for* $a, b \in H$ *there exists a couple of elements* $\left(x_a^{a \vee b}, x_b^{a \vee b}\right) \in \mathscr{X}_a^{a \vee b} \times \mathscr{X}_b^{a \vee b}$ *comparable, that is* $x_a^{a \vee b} \nparallel x_b^{a \vee b}$ *then* $\sup(a, b) = \max\left(x_a^{a \vee b}, x_b^{a \vee b}\right)$.

Proof. Obviously, if $d = \max\left(x_a^{a\vee b}, x_b^{a\vee b}\right)$ we have $a \leqslant x_a^{a\vee b} \leqslant d$ and $b \leqslant x_b^{a\vee b} \leqslant d$, that is the element d is an upper bound of a, b and furthermore it belongs to the union $a \vee b$. Therefore we have $d = \sup(a, b)$ [pr. 7].

Proposition 11. *If* $\mathscr{X}_a^{a\vee b} = a\vee b$, *then there exists the* $\sup(a, b)$ *and it is* $\sup(a, b) \in a\vee b$.

Proof. It is already known, [**3**], that there exists the element $x_b^{a\vee b} \in a\vee b = \mathscr{X}_b^{a\vee b}$ for which we have $b \leqslant x_b^{a\vee b}$. Therefore, there exists $x_a^{a\vee b} \in a\vee b = \mathscr{X}_a^{a\vee b}$ such that $x_a^b = x_b^a = d$, thus $d = \sup(a, b)$ and moreover $\sup(a, b) \in a\vee b$.

Corollary 4. If $\mathscr{X}_a^{a\vee b} = a\vee b$ then card $\mathscr{X}_b^{a\vee b} = 1$.

Proof. Indeed, if there exist elements $x_b^{a\vee b}, x'^{a\vee b}_b \in \mathscr{X}_b^{a\vee b}$, it follows that $x_b^{a\vee b} = x'^{a\vee b}_b = \sup(a, b)$ [pr. 11].

Proposition 12. *If* $a, b \in H$ *and* $a\vee(a\wedge b) = a$, *then there exists the* $\sup(a, b)$ *and* $\sup(a, b) \in a\vee b$.

Proof. In fact, we shall have [rem. 1] $a = a\vee(a\wedge b) = a\wedge(a\vee b)$, from which it follows that for each $x \in a\vee b$ it is $a \leqslant x$, thus $\mathscr{X}_a^{a\vee b} = a\vee b$. Therefore there exists the $\sup(a, b)$ and we have $\sup(a, b) \in a\vee b$ [pr. 11].

With respect to the proposition 11 and the corollary 3 we have and the following proposition:

Proposition 13. *In every modular hyper-lattice the union* $a\vee b$ *is a set of one element if and only if* $a\vee b = \mathscr{X}_a^{a\vee b} = \mathscr{X}_b^{a\vee b}$.

Proof. In fact, it is evident that if $a \vee b = c$ we shall have $\mathscr{X}_a^{a\vee b} = \mathscr{X}_b^{a\vee b} = a\vee b$. Inversely, if $a\vee b = \mathscr{X}_a^{a\vee b} = \mathscr{X}_b^{a\vee b}$ we shall have $\text{card}(a\vee b) = \text{card}\, \mathscr{X}_a^{a\vee b} = \text{card}\, \mathscr{X}_b^{a\vee b} = 1$ [cor. 3], thus, the union $a\vee b$ is a set of one element.

The following proposition gives us the conditions under which, with the presupposition the existence of the $\sup(a, b)$, we have $\sup(a, b) \in a\vee b$.

P r o p o s i t i o n 14. *For* $a, b, d \in H$ *we have*

i) *If* $a \prec d$, $b \prec d$ *and* $a \neq b$, *then* $\sup(a, b) \in a \vee b$

ii) *If* $a \prec d$, $b \leqslant d$ *and* $b \nleqq a$, *then there exists the* $\sup(a, b)$ *and it is* $\sup(a, b) = d \in a \vee b$.

Proof. i) It is known [**3**] that, when in a hyper-lattice the relations $a \prec d$, $b \prec d$ and $a \neq b$ are held, then there exists the $\sup(a, b)$ and it is $\sup(a, b) = d$. Now, in order to prove that $d \in a \vee b$, it is sufficient to prove that $d \in \mathscr{X}_a^{a \vee b}$. We suppose that $d \notin \mathscr{X}_a^{a \vee b}$. From the relations $a \prec d$, $b \prec d$, which are special cases of the relations $a < d$ and $b < d$ respectively, it follows that for each $w \in a \vee b$ we shall have $w < d$ [pr. 5]. Thus, it will be $a \leqslant x_a^{a \vee b} < d$, which is opposite to the supposition. Therefore $d \in \mathscr{X}_a^{a \vee b} \subseteq a \vee b$.

ii) At first, we shall prove the uniqueness of the element d. Let us suppose that there exists another element d' such that $a \prec d'$ and $b \leqslant d'$. If $d \# d'$, then we shall have $a < d < d'$ or $a < d' < d$, that is absurd. If $d // d'$, because of $a \prec d$ and, $a \prec d'$, from the theory of lattices we have $d \wedge d' = a$. On the other hand, since H is a modular one $b \leqslant d' \Rightarrow (b \vee d) \wedge d' = b \vee (d' \wedge d)$, from which, because of $b \leqslant d$, that is $d \in b \vee d$, and $d \wedge d' = a$, it follows that $d \wedge d' = a \in b \vee a$. Thus $b \leqslant a$, which is also absurd.

With respect to the existence of $\sup(a, b)$, it is $\sup(a, b) = d$, because d is the smallest of the upper bounds of a and b. In fact, if there existed a d_1 such that $a \leqslant d_1$, $b \leqslant d_1$ and $d_1 < d$, then we should have $a \leqslant d_1 < d$, in contrary with the supposition. Therefore $\sup(a, b) = d$. Finally, we have that $\sup(a, b) \in a \vee b$, that is $d \in a \vee b$, and it is proved as the part (i) of the proposition.

Corollary 5. If $a, b \in H$ *and* $d \prec a$, *then there exists not an element* $x \in a \vee b$, *different from* a *and greater or equal to* b, *and it is obvious that* $a = \sup(a \vee b)$.

P r o p o s i t i o n 15. *If* $a, b, d \in H$ *such that* $a \prec d$, $b \leqslant d$ *and* $b \nleqq a$ *then* $a \wedge b \prec b$.

Proof. We suppose that $a \wedge b \nprec b$. Then, there will exist an element $x \in H$, for which we shall have $a \wedge b < x < b$. From this relation we obtain $a\wedge b \leqslant x\wedge a \leqslant b\wedge a$, that is $a\wedge b = x\wedge a$. Consequently we have

$$x \vee (a \wedge b) = x \vee (a \wedge x) = (x \vee a) \wedge x \quad \text{[rem. 1]}.$$

Also, because of the relation $x \leqslant b$ we have

$$x \vee (a \wedge b) = (x \vee a) \wedge b,$$

since H is a modular one. From the last two equalities, it follows that

$$(x \vee a) \wedge b = (x \vee a) \wedge x.$$

Moreover, it will be $x \nleqslant a$, because if $x \leqslant a$, then $x \wedge a = x$ and according to above we shall have $x = a \wedge b$, fact which does not happen. From the relations $a \prec d$, $x < d$ and $x \nleqslant a$, we have [pr. 14] $d \in a \vee x$, from which we obtain

$$d \wedge b = b \in (a \vee x) \wedge b = (x \vee a) \wedge b = (x \vee a) \wedge x.$$

So, it follows that there will be an element $z \in x\vee a$, such that $b = z\wedge x$, that is $b \leqslant x$. However since and $x < b$, it will be $x = b$.

Corollary 6. If $a, b, d \in H$ *such that* $a \prec d$, $b \prec d$ *and* $a \neq b$, *then* $a \wedge b \prec b$ *and* $a \wedge b \prec a$.

Proposition 16. *If for* $a, b \in H$ *it is* $a \wedge b \prec a$ *and there exists a* $d \in a \vee b$ *such that* $a \leqslant d$, $b \leqslant d$, *then* $b \prec d$.

Proof. Suppose that $b \not\prec d$. Then there will exist an element $x \in H$ for which the inequality $b < x < d$ will be held. From that we obtain

$$b \wedge a \leqslant x \wedge a \leqslant d \wedge a = a.$$

However, since $a\wedge b \prec a$, it will be either $a = a\wedge x$ or $x\wedge a = b\wedge a$. So, we have two cases.

i) Suppose that $a = a \wedge x$, that is $a \leqslant x$. In this case we have two sub-cases: 1) $x \in a \vee b$. Then x as an upper bound of a and b and since it belongs to the union $a \vee b$, it will be equal to d, because of the uniqueness of the latter, fact which is absurd. 2) $x \notin a \vee b$. Then x as an upper bound of a and b and since it does not belong to the union $a \vee b$, it will satisfy the relation $d < x$, also absurd.

ii) If $a \wedge x = b \wedge x$, from the relation $d \in a \vee b$ we obtain

$$d \wedge x = x \in (a \vee b) \wedge x = b \vee (a \wedge x) = b \vee (a \wedge b) = (b \vee a) \wedge b.$$

Therefore, there will be an element $z \in a \vee b$, such that $x = z \wedge b$, that is $x \leqslant b$. Thus it will be $x = b$, which is also absurd. Consequently, it is $b \prec d$.

Corollary 7. If $a, b \in H$ *and the relations* $a \wedge b \prec a$, $a \wedge b \prec b$ *are satisfied, and if there exists* a $d \in a \vee b$, *such that* $a \leqslant d$, $b \leqslant d$, *then it will be* $a \prec d$, $b \prec d$.

The proposition 16 can be also stated as follows:

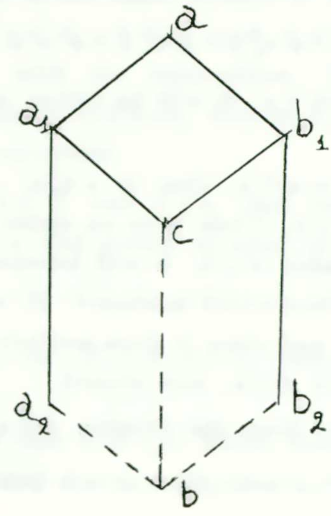
Proposition 17. *If* x, a, b *are elements of* H *for which the relations* $x \prec a$, $x \leqslant b$ *and* $a \not\equiv b$ *are held and if there exists* a $d \in a \vee b$, *such that* $a \leqslant d$, $b \leqslant d$, *then it will be* $b \prec d$.

Proof. In fact, from the relations $x \prec a$, $x \leqslant b$ and $a \not\equiv b$ it follows that $x = a \wedge b$. So we have the proposition 16.

Now we shall study the modular hyper-lattices within which all the bounded chains have a finite length. Relatively, we have the following proposition, which, as it is known, is held in the theory of the modular lattices as well.

Proposition 18. *A modular hyper-lattice within which all the bounded chains have a finite length, satisfies the condition of Jordan-Dedekind, that is, all the maximal chains which have the same ends have the same length.*

Proof. If $b \prec a$ the length of the maximal chain between a and b is obviously 1. Let us consider the diagram below, in which as we can see it is $b \nprec a$.



Σχ. 3.

We shall prove the proposition by induction on the length of a maximal chain between two elements.

Let us suppose that the proposition is true if between two elements there exists a maximal chain of length $n-1$. Suppose now that between $a, b \in H$ there exists a maximal chain of length n, that is

$$b = a_n \prec \ldots a_2 \prec a_1 \prec a_0 = a$$

and another maximal chain of length l, that is

$$b = b_l \prec \ldots b_2 \prec b_1 \prec a_0 = a.$$

If $a_1 = b_1$, it is evident that the proposition comes true. If $a_1 \neq b_1$, because of the relations $a_1 \prec a$ and $b_1 \prec a$, we shall have

$$a = \sup(a_1, b_1) \in a_1 \vee b_1 \quad [\text{pr. } 14].$$

Therefore, by the proposition 15 it will be $c \prec a_1$ and $c \prec b_1$, where $c = a_1 \wedge b_1$. On the other hand the maximal chains between a_1 and b have a length $n-1$, thus those ones between b and c will have a length of $n-2$, since $c \prec a_1$. Consequently, since $c \prec b_1$, the maximal chains between b_1 and b will have a length $n-1$. Therefore, we have $l-1 = n-1$, that is $l = n$.

With respect to the modular hyper-lattices, we have the following properties:

Property 1. Every sub-hyper-lattice of a modular hyper-lattice is a modular one.

Proof. It is easy to be verified.

Property 2. In every modular hyper-lattice H *with a zero element, the interval* $I = [o, a]$ *for every* $a \in H$ *is a sub-hyper-lattice of* H.

Proof. In fact, if $x_1, x_2 \in [o, a]$ we have $o \leqslant x_1 \wedge x_2 \leqslant a$, that is $x_1 \wedge x_2 \in I$. On the other hand, for each $w \in x_1 \vee x_2$ it will be $o \leqslant w \leqslant a$ [pr. 5], thus $x_1 \vee x_2 \subseteq I$. Therefore I is a sub-hyper-lattice of H.

Property 3. The homeomorphic image of H, $f(H)$ (1), *is a modular hyper-lattice.*

---

(1) Let H_1, H_2 hyper-lattices. A mapping $f : H_1 \rightarrow H_2$ is an *homomorphism* if for each couple of elements $(a, b) \in H_1 \times H_1$ we have $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ and it has been proved that the homeomorphic image $f(H_1)$ is a sub-hyper-lattice of H_2 and also, f is an isotone mapping between H_1 and H_2.

Proof. It is already known that f(H) is a hyper-lattice. If $a, b, c \in H$ and $a \leqslant b$, by the definition of a modular hyper-lattice we should have

$$a \vee (c \wedge b) = (a \vee c) \wedge b.$$

Also, because of the isotone of the homomorphism $a \leqslant b \Rightarrow f(a) \leqslant f(b)$. So, if we get the left side of the previous equality, we have

$$f[a \vee (c \wedge b)] = f(a) \vee [f(c \wedge b)] = f(a) \vee [f(c) \wedge f(b)].$$

On the other hand, of the right side we take

$$f[(a \vee c) \wedge b] = f[\bigcup_{w \in a \vee c} (w \wedge b)] = \bigcup_{w \in a \vee c} f(w \wedge b) = \bigcup_{w \in a \vee c} [f(w) \wedge f(b)] =$$

$$= f(a \vee c) \wedge f(b) = [f(a) \vee f(c)] \wedge f(b).$$

Thus, $\quad f(a) \leqslant f(b) \Rightarrow f(a) \vee [f(c) \wedge f(b)] = [f(a) \vee f(c)] \wedge f(b).$

Therefore, the hyper-lattice f(H) is a modular one.

Property 4. For any elements $a, b, c \in H$ *we have*

$$\{[a \vee (b \wedge c)] \wedge [b \vee (c \wedge a)]\} \cap \{(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)\} \neq \varnothing.$$

Proof. In fact, it will be

$$[a \vee (b \wedge c)] \wedge [b \vee (c \wedge a)] \supseteq [a \vee (b \wedge c)] \wedge \mathscr{X}_b^{b \vee (c \wedge a)}.$$

Also, since $c \wedge a \leqslant a$, by the definition of the modular hyper-lattices, we shall have

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (b \wedge c) \vee [(a \wedge b) \vee (c \wedge a)] =$$

$$= (b \wedge c) \vee [[(c \wedge a) \vee b] \wedge a] \supseteq (b \wedge c) \vee [\mathscr{X}_b^{(c \wedge a) \vee b} \wedge a] =$$

$$= (b \wedge c) \vee \{x_b^{(c \wedge a) \vee b} \wedge a : x_b^{(c \wedge a) \vee b} \in \mathscr{X}_b^{(c \wedge a) \vee b}\} = \bigcup_{x_b^{(c \wedge a) \vee b} \in \mathscr{X}_b^{(c \wedge a) \vee b}} [(b \wedge c) \vee (x_b^{(c \wedge a) \vee b} \wedge a)]$$

which, because of the relation $b \wedge c \leqslant b \leqslant x_b^{c \wedge a}$ and the modularity becomes equal to

$$\bigcup_{x_b^{(c \wedge a) \vee b} \in \mathscr{X}_b^{(c \wedge a) \vee b}} [[(b \wedge c) \vee a] \wedge x_b^{(c \wedge a) \vee b}] = [(b \wedge c) \vee a] \wedge \mathscr{X}_b^{(c \wedge a) \vee b}.$$

As consequence of these we have that

$$[[a \vee (b \wedge c)] \wedge [b \vee (c \wedge a)]] \cap [(a \wedge b) \vee (b \wedge c) \vee (c \wedge a)] \supseteq [(b \wedge c) \vee a] \wedge \mathscr{X}_b^{(c \wedge a) \vee b} \neq \varnothing.$$

This property, when H is a lattice, is obviously the identity

$$[a \vee (b \wedge c)] \wedge [b \vee (c \wedge a)] = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a).$$

Property 5. In every modular hyper-lattice H, *for any* $a, b, c \in H$ *the relation* $a \wedge b \in [(a \wedge b) \vee (a \wedge c)] \wedge [(a \wedge b) \vee (b \wedge c)]$ *is held.*

Proof. In fact, from the relations $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$ we obtain respectively

$$(a \wedge b) \vee (a \wedge c) = [(a \wedge b) \vee c] \wedge a \quad \text{and} \quad (a \wedge b) \vee (b \wedge c) = [(a \wedge b) \vee c] \wedge b$$

from which it follows that

$$[(a \wedge b) \vee (a \wedge c)] \wedge [(a \wedge b) \vee (b \wedge c)] = [[(a \wedge b) \vee c] \wedge a] \wedge [[(a \wedge b) \vee c] \wedge b] =$$
$$= [[(a \wedge b) \vee c] \wedge (a \wedge b)] \wedge [(a \wedge b) \vee [(a \wedge b) \vee c]] \ni (a \wedge b) \wedge (a \wedge b) = a \wedge b.$$

Property 6. The product (1) H *of a family of modular hyper-lattices* $\{H_i\}_{i \in A}$ *is a modular hyper-lattice, and inversely, if the product* H *of a family of hyper-lattices* $\{H_i\}_{i \in A}$ *is a modular one, then all* H_i *for any* $i \in A$ *are modular ones.*

Proof. It is already known that H is a hyper-lattice. If now for $a, b \in H$, it is $a \leqslant b$, this is equivalent with $a_i \leqslant b_i$ for any $i \in A$, by the definition of the product of the hyper-lattices. Also, since all H_i are modular ones, it follows that for $a_i, b_i, c_i \in H_i$, because of the relation $a_i \leqslant b_i$ we have

$$a_i \vee (b_i \wedge c_i) = (a_i \vee c_i) \wedge b_i \quad \text{for each } i \in A.$$

So, if $a \leqslant b$, we have

$$a \vee (b \wedge c) = \{a_i\}_{i \in A} \vee \{b_i \wedge c_i\}_{i \in A} = \{\{w_i\}_{i \in A} : w_i \in a_i \vee (b_i \wedge c_i)\} =$$
$$= \{\{w_i\}_{i \in A} : w_i \in (a_i \vee b_i) \wedge c_i\} = \{a_i \vee b_i\}_{i \in A} \wedge \{c_i\}_{i \in A} = (a \vee b) \wedge c.$$

---

(1) The product H of a family of hyper-lattices $\{H_i\}_{i \in A}$, which is defined as the product of sets, can be ordered by the relation R as follows

$$aRb \Longleftrightarrow a_i \wedge b_i = a_i \text{ or equivalently } aRb \Longleftrightarrow b_i \in a_i \vee b_i \text{ for each } i \in A.$$

Also, it has been proved that H is a hyper-lattice when the operation and the hyper-operation $\vee$ are defined as follows:

$$a \wedge b = \{a_i \wedge b_i\}_{i \in A} \quad \text{and} \quad a \vee b = \{\{w_i\}_{i \in A} : w_i \in a_i \vee b_i\}.$$

Inversely, suppose that H is a modular hyper-lattice. If $a_i \leqslant b_i$ for every $i \in A$, it is equivalent with $a \leqslant b$, which gives

$$a \vee (b \wedge c) = (a \vee c) \wedge b \Longrightarrow \{a_i\}_{i \in A} \vee \{b_i \wedge c_i\}_{i \in A} =$$
$$= [\{a_i\}_{i \in A} \vee \{c_i\}_{i \in A}] \wedge \{b_i\}_{i \in A} \Longrightarrow \{\{w_i\}_{i \in A} : w_i \in a_i \vee (b_i \wedge c_i)\} =$$
$$= \{\{z_i\}_{i \in A} : z_i \in (a_i \vee c_i) \wedge b_i\} \Longrightarrow a_i \vee (b_i \wedge c_i) = (a_i \vee c_i) \wedge b_i$$

for every $i \in A$. Consequently, the hyper-lattices H_i are modular ones.

Finishing with the modular hyper-lattices, we prove the following lemma which is useful in order to prove the proposition which follows.

Lemma 1. If $f(H)$ *is the homomorphic image of* H *and if* $f(a) = a^* \in f(H)$ *is the image of the element* $a \in H$, *then for every* $b^* \in f(H)$ *which covers* a^*, *there exists a* $b \in H$ *such that* $a < b$ *and* $f(b) = b^*$.

Proof. The relation $a^* \prec b^*$ implies $b^* \in a^* \vee b^*$ and by the corollary 4 we shall have that b^* is the unique element of the union $a^* \vee b^*$ for which the relation $a^* < b^*$ is held. If now $c \in H$ and $f(c) = b^*$, then

$$f(a \vee c) = f(a) \vee f(c) = a^* \vee b^*.$$

However, in the union $a \vee c$ the set $\mathscr{X}_a^{a \vee c}$ is included for which we have

$$f(\mathscr{X}_a^{a \vee c}) \subseteq f(a \vee c) = a^* \vee b^*,$$

that is, the images of the distinguished elements of the pair (a, c) will bellong to the union $a^* \vee b^*$. On the other hand, since $a \leqslant x_a^{a \vee c}$, we have, because of the isotone of the homomorphism $f(a) \leqslant f(x_a^{a \vee c})$ that is $a^* \leqslant f(x_a^c)$. Combining this relation with the above, we have $f(x_a^{a \vee c}) = b^*$ for each $x_a^{a \vee c} \in \mathscr{X}_a^{a \vee c}$. Consequently, there exists a $b = x_a^{a \vee c}$, which is not necessarily the unique which satisfies the requirements of the lemma.

Proposition 19. *If* $f(H)$ *is an homeomorphic image of a modular hyper-lattice* H, *within which every ascending chain is stationary, then every maximal chain of* $f(H)$ *is also stationary.*

Proof. Suppose that in $f(H)$ there exists a maximal chain, which is not stationary one

$$a_1^* \prec a_2^* \prec \ldots \prec a_n^* \ldots$$

If now $a_1 \in H$, such that $f(a_1) = \overset{*}{a_1}$, by the previous lemma, there will exist an $a_2 \in H$ such that $f(a_2) = a_2^*$ and $a_1 < a_2$. Consequently, if the chain $a_1^* \prec a_2^* \prec \ldots \prec a_n^* \ldots$ is not a stationary one, then and the chain $a_1 < a_2 < \ldots < a_n < \ldots$ because of its construction is not a stationary one, fact which is absurd.

ΠΕΡΙΛΗΨΗ

Στὴν παροῦσα ἐργασία πραγματεύομαι μιὰ εἰδικὴ κατηγορία ὑπερδικτυωτῶν, τὰ τροπικὰ ὑπερδικτυωτά, τὰ ὁποῖα ὁρίζονται κατὰ τρόπο ἀνάλογο πρὸς τὰ τροπικὰ δικτυωτά, πληροῦν δηλαδὴ ἐπὶ πλέον καὶ τὸ ἀξίωμα

$$a \leqslant c \Longrightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

γιὰ ὁποιαδήποτε στοιχεῖα τους a, b, c.

Γενικὰ ἀποδεικνύονται προτάσεις ποὺ εἶναι γενίκευση τῶν ἀντιστοίχων τῆς κλασσικῆς θεωρίας ὅπως ἐπίσης καὶ ἄλλες καινούργιες ποὺ ὀφείλονται ἀκριβῶς στὴν ἰδιορρυθμία τῆς ἑνώσεως σὰν ὑπερπράξεως [**6**]. (Ἂν H τὸ ὑπερδικτυωτὸ καὶ $a, b \in H$ εἶναι $a \vee b \subseteq H$).

Σ' ἕνα τυχὸν ὑπερδικτυωτὸ τὸ supremum δυὸ στοιχείων του a, b ἀντίθετα πρὸς τὰ δικτυωτὰ γενικῶς δὲν ὑπάρχει [**3**]. Στὰ τροπικὰ ὑπερδικτυωτὰ τὸ ἐπὶ πλέον ἀξίωμα ποὺ τὰ χαρακτηρίζει βοηθάει στὴν εὕρεση συνθηκῶν ποὺ πρέπει νὰ πληροῖ ἕνα ἄνω φράγμα δυὸ στοιχείων γιὰ νὰ εἶναι supremum αὐτῶν καθὼς καὶ ἄλλων συνθηκῶν, ἔτσι ὥστε, στὴν περίπτωση ποὺ τὸ ἄνω πέρας δυὸ στοιχείων ὑπάρχει, ν' ἀνήκει στὴν ἕνωσή τους.

Στὴν ὅλη θεωρία ἰδιαίτερη σημασία ἔχουν τὰ λεγόμενα διακεκριμένα στοιχεῖα καὶ διακεκριμένα σύνολα δυὸ ἢ καὶ περισσοτέρων στοιχείων τοῦ ὑπερδικτυωτοῦ. Ὅπως δὲ στὴν κλασσικὴ θεωρία ἔτσι καὶ στὴν παροῦσα τὰ ὑπερδικτυωτὰ τῆς κλάσεως αὐτῆς παρουσιάζουν μεγάλο ἐνδιαφέρον διότι ἐπαληθεύουν πολλὲς καὶ ἀξιοσημείωτες ἰδιότητες.

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