ΜΗΧΑΝΙΚΗ.— **Gauging the Singularity in General Stress Fields by Optical Means,** by the Academician *P. S. Theocaris* *.

ABSTRACT

The optical method of caustics was used up-to-now for the evaluation of the stress intensity at the vicinity of singular areas of stress fields. A combination of the properties of caustics and pseudocaustics created by the singularity and a small circular gauge, traced in its vicinity, allows the complementary evaluation of the order of singularity and thus yields the complete information about the state of stress at the singular areas of stress fields. The method is based on the quasi-conformal mapping of a small circular gauge, tangent to the initial curve of the caustic and lying outside it, which then is expressed by a pseudoanalytic function and therefore maps into an ellipse. By measuring the dimensions and orientation of the elliptic pseudocaustic and combining with data from the caustic the order of singularity can be readily and accurately evaluated. The method was applied with satisfactory results to evaluate singularities in cracked or indented plexiglas or polycarbonate plates.

INTRODUCTION

The definition of the order of a stress singularity, eventually existing in elastic fields is of fundamental importance in elastomechanics. The only analytical techniques, which have been used up-to-now for the evaluation of stress singularities are the *eigenfunction-expansion theory* and the *application of Mellin transform.* The first attempt to study stress singularities was made by Williams [**1**], who succeeded to derive their characteristic equations for wedges, by making the assumption that the order of singularity is a real number. The same assumption has been accepted by Kalandiia [**2**], for evaluating the orders of singularities at corners.

Moreover, Bogy [**3**], and Dundurs [**4**] have derived solutions for stress singularities in wedges, by using a straightforward application of the *Mellin transform.* Rice and Sih [**5**], later on, have treated the plane extension problem of two bonded dissimilar media with cracks existing along their common interface and found the stress intensity factors, by using the eigenfunction-expansion technique for complex-variable integration. Hein and Erdogan [**6**] have used the Mellin transform and the theory of residues to determine the stress singularities in a two-material wedge.

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On the other hand, England [**7**] used the method of complex variables to examine a group of boundary-value problems, and has found that the wedge geometry affects the value of singularity, whereas the imposed boundary conditions determine its type. Theocaris [**8**] has evaluated the stress singularity at the corner of a multiwedge. He extended the analysis made by England for the cases of arbitrary complex singularities and studied the order of singularity at wedges with different contact conditions prevailing at the interfaces between adjacent wedges in the multiwedge.

However, until now, there does not exist any experimental technique for the evaluation of the order of singularities developed in stress fields. In the present paper the theory *of pseudoanalytic functions* and *quasi-conformal mappings* were successfully used, in combination with the method of *caustics* and *pseudocaustics*, for the evaluation of the order of stress singularities in elastic or plastic stress fields. In the theory, only the dominant term in the Laurent-series expansion of the complex potential, describing the stress field, was considered for the evaluation of the stresses. Infinitesimal circles were drawn in plane specimens at the vicinity of the singular points, which created elliptic pseudocaustics. A single measurement of the rotation of the major axis of this ellipse, together with data from the caustics gave the order of the stress singularity.

THE CONTINUITY LAWS BETWEEN CAUSTICS AND PSEUDOCAUSTICS

In order to define the continuity laws between a caustic and a pseudocaustic, whose initial curves possess a common point, we consider both curves as composite functions, whose we shall determine the tangent vectors at their common points. For this reason we consider the mapping:

$$\begin{aligned} f &: R^2 \rightarrow R^2 \\ f &: (x,y) \rightarrow (u,v) \end{aligned} \tag{1}$$

where:

$$\begin{aligned} u &= W_x = \lambda_m x + CA(x,y) \\ v &= W_y = \lambda_m y - CB(x,y) \end{aligned} \tag{2}$$

and:

$$\mathbf{W} = W_x i + W_y i \tag{3}$$

with $A(x,y) = ReF'(z)$ and $B(x,y) = ImF'(z)$ and $F(z)$ the complex potential function, expressing the elastic field of the plate. The coefficient C denotes a multiplication factor, relying on the experimental set-up, which depends on the optical properties and the geometry of the plate and, also at the distance z_0 between the middle plane of the plate and the reference screen, where the caustics are formed [**9, 10**]. Relation (3) expresses the position of the projection P′ of a generic point P of the loaded plate, referred to the Oxy-system, on a reference screen, where the deviation of the reflected light vector, **W**, is given in the Cartesian system O′x′y′ indicated in Fig. 1.

The singular points of this mapping satisfy the algebraic equation:

$$J(x,y) = u_{,x}v_{,y} - u_{,y}v_{,x} = 0 \tag{4}$$

where commas before indices mean differentiation with respect to the index. This equation defines another mapping, h_1, in the following manner:

$$h_1 : R \rightarrow R^2$$
$$h_1 : t \rightarrow (t, y(t)) \tag{5}$$

under the condition that $J(t,y(t)) = 0$. This condition constitutes a necessary and sufficient condition for the creation of a singular curve, the caustic, on the screen, and it is expressed by:

$$J = \frac{(W_x, W_y)}{(x, y)} = W_{x,x}W_{y,y} - W_{x,y}W_{y,x} = 0 \tag{6}$$

which is, of course, the same as relation (4). Then, it is easy to show that the compound function $f(h_1)$ defines the equations of the caustic curve.

We consider now a generic curve in the elastic field, for instance, the boundary of its domain, which can be expressed by:

$$q(x,y) = 0 \tag{7}$$

If this curve can be represented by the mapping:

$$h : R \rightarrow R'$$
$$h' : t \rightarrow (t, g(t)) \tag{8}$$

under the condition that $q(t,q(t)) = 0$, it can be readily derived that the compound function $f(h_2)$ defines the equations of the *pseudocaustic curve*,

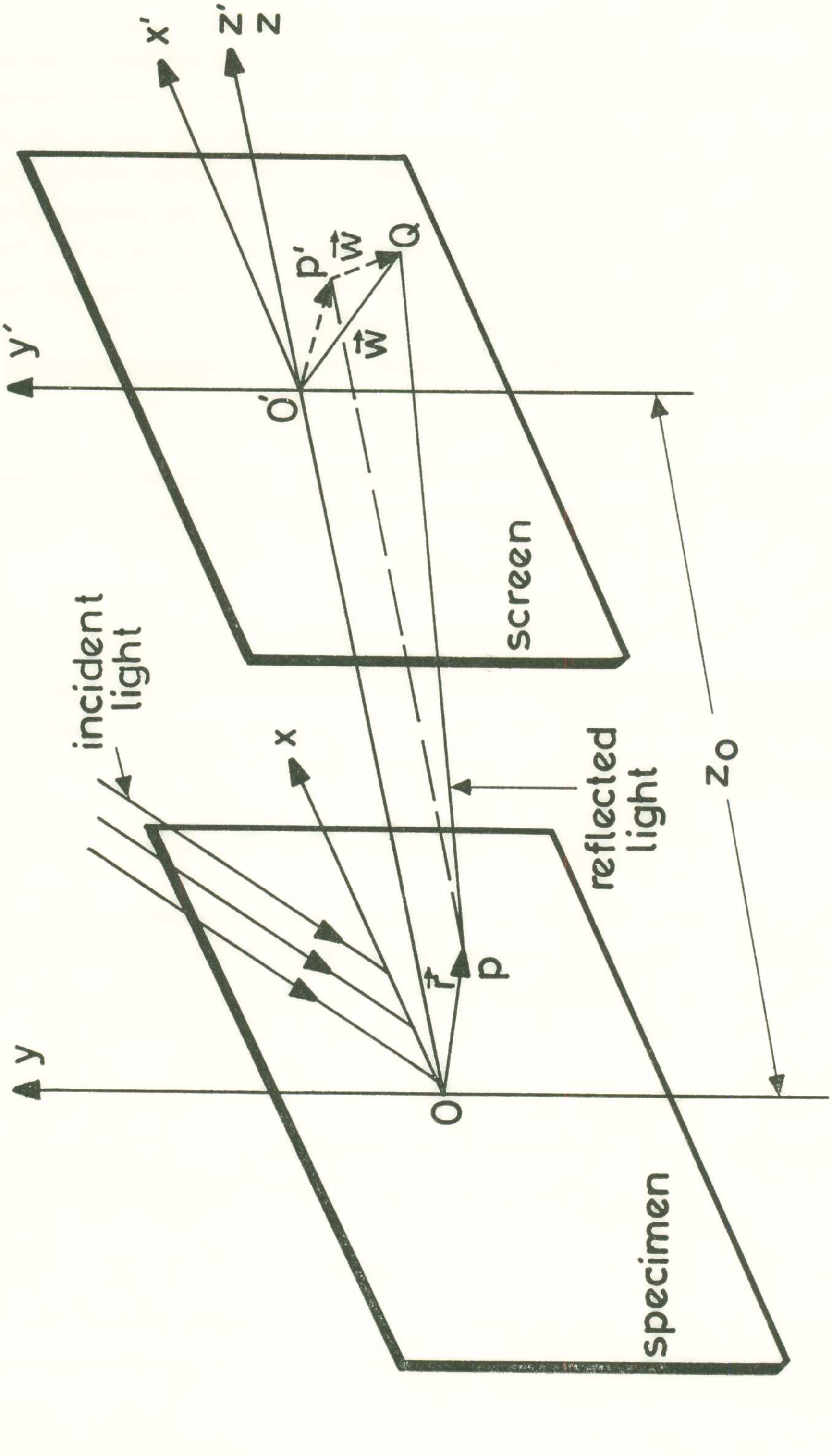


Fig. 1: Schematic of the optical arrangement.

which is generated from this boundary. The tangent vectors of the above curves are, respectively:

$$\nabla_J f = \mathbf{J}\,\nabla f = (-J_{,y}, J_{,x})\,((u_{,x}, v_{,x}), (u_{,y}, v_{,y})) =$$
$$= (-J_{,y}u_{,x} + J_{,x}u_{,y}, -J_{,y}v_{,x} + J_{,x}v_{,y}) \tag{9}$$

$$\nabla_t f = \mathbf{t}\,\nabla \mathbf{f} = (-q_{,y}, q_{,x})\,((u_{,x}, v_{,x}), (u_{,y}, v_{,y})) =$$
$$= (-q_{,y}u_{,x} + q_{,x}u_{,y}, -q_{,y}v_{,x} + q_{,x}v_{,y}) \tag{10}$$

The vectors **J** and **t** are the tangent vectors for the curves, which are expressed by relations (4) and (7) respectively.

We construct now their vector-product:

$$\nabla_j f \times \nabla_t f = -\mathbf{k}[J_{,x}q_{,y}\,(u_{,y}v_{,x}-u_{,x}v_{,y})-J_{,x}q_{,y}\,(u_{,x}v_{,y}-u_{,y}v_{,x})] \tag{11}$$

Taking into account relation (4) it is easy to derive for the common points of the caustics and pseudocaustics that:

$$\nabla_j f \times \nabla_t f = 0 \tag{12}$$

Relation (12) proves that the caustics and pseudocaustics have common tangential directions with the same or opposite sense, at their common points (Fig. 2). However, it is worthwhile indicating that the form of contact shown in Fig. 2a is a cusplike one, but it is not a real cusp point, because the necessary condition for the existence of a cusp point is given by the relation [**11**]:

$$\nabla_j f = 0 \tag{13}$$

which, in this case, does not hold. Indeed, these points are in reality fold points, whereas the respective common points in Fig. 2b possess a common tangent, since at these points the caustic and the pseudocaustic have tangents at their common point, which lie on the same straight line.

In order to define the form of the contact at the common points of the caustics and pseudocaustics we construct the scalar product:

$$Q = \nabla_j f \cdot \nabla_t f = a_1u^2_{,x} + a_2u^2_{,y} + b_1v^2_{,x} + 2c_1u_{,x}v_{,y} + 2c_2u_{,y}v_{,x} \tag{14}$$

where we have inserted the following coefficients:

$$a_1 = b_1 = J_{,y}\,q_{,y}$$
$$a_2 = b_2 = J_{,x}\,q_{,x} \tag{15}$$
$$c_1 = c_2 = -\frac{1}{2}\,(J_{,x}\,q_{,y} + J_{,y}\,q_{,x})$$

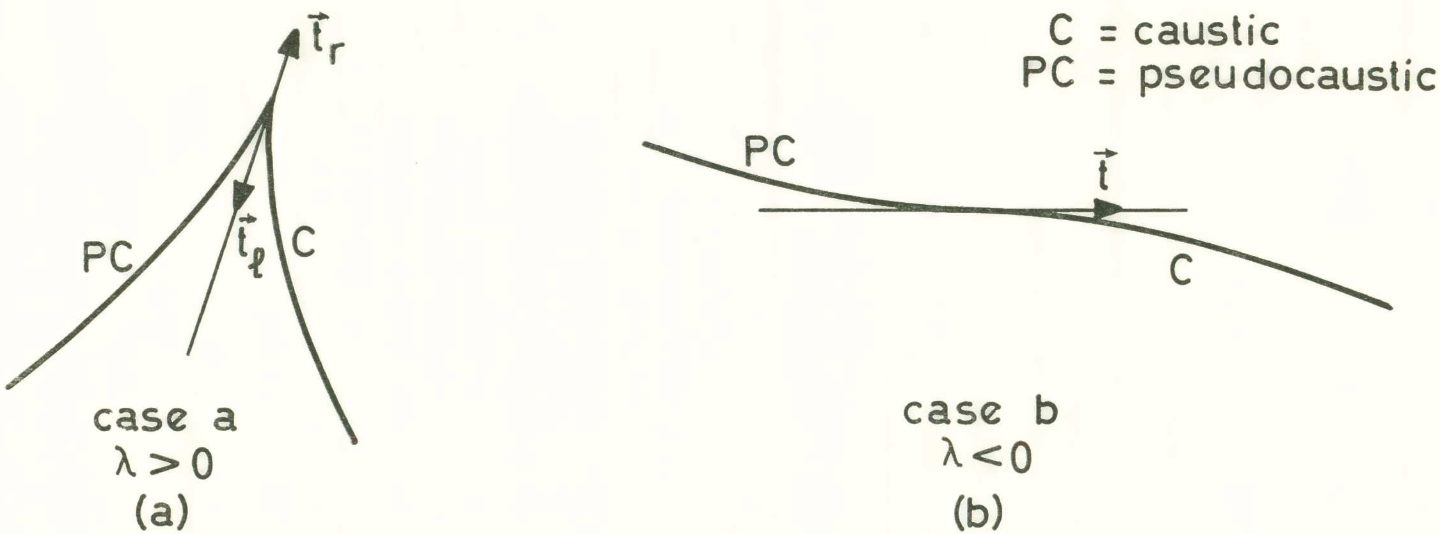


Fig. 2: Contact forms between caustics and pseudocaustics.

The quadratic form (14) can be written in the following matrix-form:

$$\mathbf{Q} = \mathbf{X^T A X} \tag{16}$$

where:

$$\mathbf{X}^{\mathrm{T}} = (u_{,x}\ v_{,y}\ u_{,y}\ v_{,y})$$

and A is the matrix of the coefficients:

$$A = \begin{matrix} a_1 & c_1 & 0 & 0 \\ c_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & c_2 \\ 0 & 0 & c_2 & b_2 \end{matrix} \tag{17}$$

The matrix A is symmetric and in general non-singular, because of the existence of the relationship:

$$|A| = (a_1 a_2 - c^2{}_1)^2 = \frac{1}{16} (J_{,x} q_{,y} - J_{,y} q_{,x})^2 \neq 0. \tag{18}$$

We consider now the eigen-equation of the matrix A and we take into account that at the common points of the caustics and pseudocaustics the following relation holds:

$$\mathbf{J} \cdot \mathbf{t} = \mathbf{0} \tag{19}$$

Then, whenever the singular point, which creates the caustic, lies across the curve it is valid that $q = 0$. So we have:

$$|A - \lambda I| = (\lambda^2 - \frac{1}{4} (J_{,x} q_{,y} - J_{,y} q_{,x})^2)^2 = 0 \tag{20}$$

which has as double roots the expressions:

$$\lambda_{1,2} = \pm \frac{1}{2} (J_{,x} q_{,y} - J_{,y} q_{,x}) \tag{21}$$

which generally are nonzero numbers.

Relation (21) shows that the quadratic form **0** changes sign. So, each of the contact forms, either the form of Fig. 2a, or the form of Fig. 2b, is possible for some value of λ_m.

It is evident from this figure that, *when the equations for the caustics and pseudocaustics satisfy the relationship* $\lambda > 0$, *they possess a common point, but not a common tangent (case 2a). On the contrary, when the caustics and pseudocaustics satisfy the relationship* $\lambda < 0$, *they possess in addition a common tangent, and either curve lies on either side of this tangent (case 2b).*

The above analysis is valid also for every curve, which is parallel to the curve considered, satisfying the condition $q = 0$, and therefore cuts the initial curve of the caustic. Therefore any curve which is expressed by Eq. (7) creates similar forms of contact at the common points of the caustic and the respective pseudocaustic created by this curve.

DEFINITION OF THE ORDER OF SINGULARITY IN THE GENERAL ELASTIC OR ELASTOPLASTIC STRESS FIELD

Consider the following system of partial differential equations:

$$\begin{aligned} Mu_{,x} + Nu_{,y} &= v_{,y} \\ Mu_{,y} - Nu_{,x} &= -v_{,x} \end{aligned} \tag{22}$$

If we solve this for M and N, we have:

$$M = \frac{u_{,x} v_{,y} - u_{,y} v_{,x}}{u_{,x}^2 + u_{,y}^2}, \qquad N = \frac{u_{,x} v_{,x} + u_{,y} v_{,y}}{u_{,x}^2 + u_{,y}^2} \tag{23}$$

It is evident from these relations that the sign of the function M depends on the sign of the function $J = u_{,x} v_{,y} - u_{,y} v_{,x}$. It is already indicated that in any elastic field with singular points the curve $J = 0$ represents *the initial curve* corresponding to the singular region. Then, the quantity M remains positive when $J > 0$, that is for all points which lie outside the initial curve.

Then, the condition:

$$M = \frac{u_{,x} v_{,y} - u_{,y} v_{,x}}{u_{,x}^2 + u_{,y}^2} > 0 \tag{24}$$

implies that the points of the stress-field satisfying this condition lie outside the initial curve of the caustic.

At these points *the differential system (22) is of the elliptic type* and the complex-valued function:

$$W(z) = u + iv \tag{25}$$

is a pseudoanalytic function of the second kind [**12**].

Then, the following rule is valid: A complex-valued function is quasi-conformal, if and only if a real number δ can be found, with $\delta > 1$, for which the following inequality is valid:

$$(u_{,x}^2 + u_{,y}^2 + v_{,x}^2 + v_{,y}^2) \leqq 2\delta\, (u_{,x}\, v_{,y} - u_{,y}\, v_{,x}) \qquad (26)$$

where u and v are the real and imaginary parts of the function respectively.

Since every pseudocaustic curve, which is generated by an arbitrary curve and which lies outside the initial curve of the caustic surrounding the singularity, defines a pseudoanalytic function, because it satisfies the conditions (22) and (24), then the same curve defines also a quasi-conformal mapping.

However, the inequality (26) together with relations (5), can be written in the form:

$$\frac{\delta + 1}{\delta - 1} \,|\, \Phi''\,(z)\,|^2 \leqq (\lambda_m / C)^2 \qquad (27)$$

On the other hand, it has been shown [**9**] that the equation: $|\, \Phi''(z)\,|^2 = (\lambda_m / C)^2$ describes the initial curve of the caustic. Then, all points outside the initial curve are given by the inequality:

$$|\Phi''\,(z)|^2 < (\lambda_m / C)^2$$

It is easy to define a real number $\delta(\lambda_m, C)$, $\delta > 1$, such that the inequality (27) be valid.

A fundamental property of the quasi-conformal mappings is, that they map infinitesimal circles onto infinitesimal ellipses with uniformly bounded eccentricities, contrariwise to conformal mappings, which map circles onto circles.

Applying this property, we study now an elastic field with a real singularity at the point $z = 0$, of order p. If we approximate the series-expansion of the complex potential with its dominant term, we have:

$$F(z) = Kz^{p+1} \qquad (28)$$

We consider now an infinitesimal circle centered at the point z_χ, with radius ε, $\varepsilon \to 0$, which is defined by:

$$z = z_\chi + \varepsilon e^{i\theta} \qquad (29)$$

We suppose further that this circle lies outside the area of the initial curve, so that its pseudocaustic obeys a quasi-conformal mapping. This circle is mapped on the screen, by using relations (2) and (3). Then, we have:

$$W\,(z) - \lambda_m\, z_\chi = r\,(z) = \lambda_m\, \varepsilon e^{i\theta} + CK\,(p + 1)\, R^p e^{-ip\varphi} \qquad (30)$$

where:

$$R = | z_{\varkappa} + \varepsilon e^{i\theta} | \qquad \varphi = \arg (z_{\varkappa} + \varepsilon e^{i\theta}) \tag{31}$$

From relation (30) it can be readily deduced that, as $\varepsilon \to 0$, the following relation holds:

$$\omega = \arg(r(z)) = \frac{r_y}{r_x} = -\tan(p\varphi_{\varkappa}) \tag{32}$$

where:

$$r_y = Re(r(z)) \qquad r_x = Im(r(z)) \tag{33}$$

and:

$$\varphi_{\varkappa} = \arg z_{\varkappa}$$

Since an infinitesimal circle lying outside the initial curve of the caustic may be mapped into an infinitesimal ellipse, this quasi-conformal mapping, as $\varepsilon \to 0$, defines a single direction on the screen, along the major axis of the ellipse, which allows the evaluation of the order of singularity by means of relation (32).

EVALUATION OF THE ORDER OF SINGULARITY FROM THE CAUSTIC AND THE PSEUDOCAUSTIC

In order to achieve a high accuracy for the evaluation of the order of stress singularity, we take into account that the radius of the circle must be a quantity which can be measured, and therefore it is not negligible. So, if we consider the polar radius of the ellipse r(z) as a function of the angle θ, this radius, as indicated in Fig. 3, is expressed in terms of the radius ε by:

$$r(\theta) = (\lambda_m \varepsilon)^2 + \gamma^2 R^{2p} + 2\lambda_m \varepsilon \gamma R^p \cos(\theta + p\varphi) \tag{34}$$

where:

$$R = ((x_{\varkappa} + \varepsilon\cos\theta)^2 + (y_{\varkappa} + \varepsilon\sin\theta)^2)^{1/2} \tag{35}$$

and:

$$\gamma = C(p+1)\varkappa \tag{36}$$

We define now the angle θ in the plane of the mapped ellipse O′x′y′ corresponding to the major-axis of the ellipse, that is we search for $r(\theta) = r_{max}$.

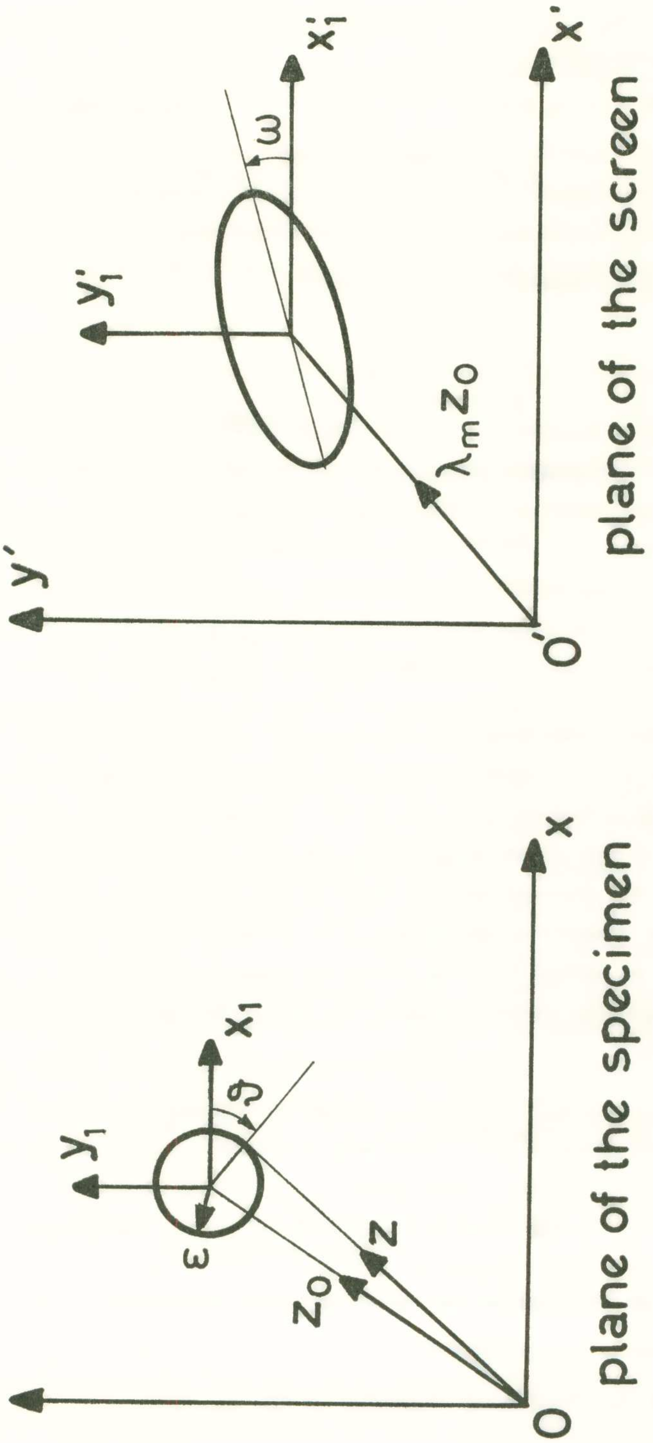


Fig. 3: Coordinates systems in the plane of the specimen and the plane of the screen.

This is equivalent to satisfy the condition of extremum, that is $dr/d\theta = 0$. This condition yields:

$$p\gamma^2 R^{2(p-1)} + \lambda_m \varepsilon\gamma R^{(p-2)} - \lambda_m \gamma R^{(p-2)} \{R^2 + p[(x_\varkappa \cos\theta + y_\varkappa \sin\theta) + \varepsilon^2]\} \frac{\sin(\theta + p\varphi)}{y_\varkappa \cos\theta - x_\varkappa \sin\theta} = 0 \quad (37)$$

with:

$$R^2 = (x_\varkappa + \varepsilon\cos\theta)^2 + (y_\varkappa + \varepsilon\sin\theta)^2 \quad (38)$$

and

$$\varphi = \arctan\left(\frac{y_\varkappa + \varepsilon\sin\theta}{x_\varkappa + \varepsilon\cos\theta}\right) \quad (39)$$

The slope of the generic polar radius of the ellipse is given by:

$$\tan\omega = \frac{\lambda_m \varepsilon\sin\theta - \gamma R^p \sin(p\varphi)}{\lambda_m \varepsilon\cos\theta + \gamma R^p \cos(p\varphi)} \quad (40)$$

where:

$$\varphi = \arctan\left(\frac{y_\varkappa + \varepsilon\sin\theta}{x_\varkappa + \varepsilon\cos\theta}\right) \quad (41)$$

The above relations and the fact that the stress-intensity of the singularity K can be evaluated by means of the corresponding caustic, if the order of singularity is known, give rise to the following iteration procedure, for the evaluation of the stress singularity. *In this procedure the only experimental data, which are required are the slope of the major axis of the ellipse and the characteristic diameter of the caustic.* The iteration procedure follows the steps:

i) Define an initial value of the p by means of Eq. (32).

ii) Evaluate the stress intensity K using the following equations of the caustics:

$$D_{max} = \lambda_m \left(\frac{\lambda_m^2}{C^2 p(p+1)K}\right)^{\frac{1}{p-1}} \cos\theta_{max} + CK(p+1)\left(\frac{\lambda_m^2}{C^2 p(p+1)K}\right)^{\frac{p}{p-1}} \cos(p\theta_{max}) \quad (42)$$

where θ_{max} is the root of the transcendental equation:

$$C\sin\theta + \lambda_m \sin(p\theta) = 0, \qquad \text{for } -\pi \leqq \theta \leqq \pi \quad (43)$$

and D_{max} is the maximum diameter of the caustic.

iii) Define tne quantity γ from relation (36). Using now relation (37) the angle θ, which corresponds to the maximum polar radius, r_{max}, of the ellipse, is then evaluated.

iv) Solve relation (40) for the successive value of p. If this value is in good agreement with its previous value, the iterative process may be stopped, otherwise it must be continued, using for the next step (i) the new value of p, until it is succeeded to obtain the required approximation.

EXPERIMENTAL EVIDENCE

In order to verify the effectiveness of the abovementioned method, a series of experiments was executed concerning the evaluation of the order of singularity. In a companion paper [**13**] the method has been applied successfully to two particular problems, where the elastic fields contained two different types of singilarities. The first problem was the evaluation of the stress singularity at the position of a concentrated load applied on a half plane, whereas the second was referred to the evaluation of the singilarity at the tip of an edge crack existing in an infinite plate under conditions of generalized plane stress. Both problems were applied to two different materials. The one was polymethyl-methacrylate (PMMA), which behaved as ideally elastic material almost up to fracture. The second was polycarbonate (PCBA) and this material was behaving as an elastic-perfectly plastic material.

In this paper we have replaced the permanently scribed infinitesimal circle in the vicinity of the crack tip by either the projection of a single circle, printed on a glass plate interposed in the light beam of the optical set-up, or by a circular grating of a line-density of 20 lines per milimeter, printed on a glass plate. In the first case the distorted circle at the vicinity of the initial curve of the caustic around the crack tip appears in Fig. 4.

This circle compares well with the series of distorted figures of the scribed circles, indicated in Fig. 5. In these tests a fine circle with a small radius was scribed near the position of the singular point at various angular positions relatively to the expected position of singularity for each specimen. In order to define the relative position of the gauge-circle to the position of the initial curve of the caustic formed by the singularity, the known portions and shapes of the caustics and pseudocaustics for the case of edge cracks were used [**9**]. So we have succeeded to trace the reference circles always

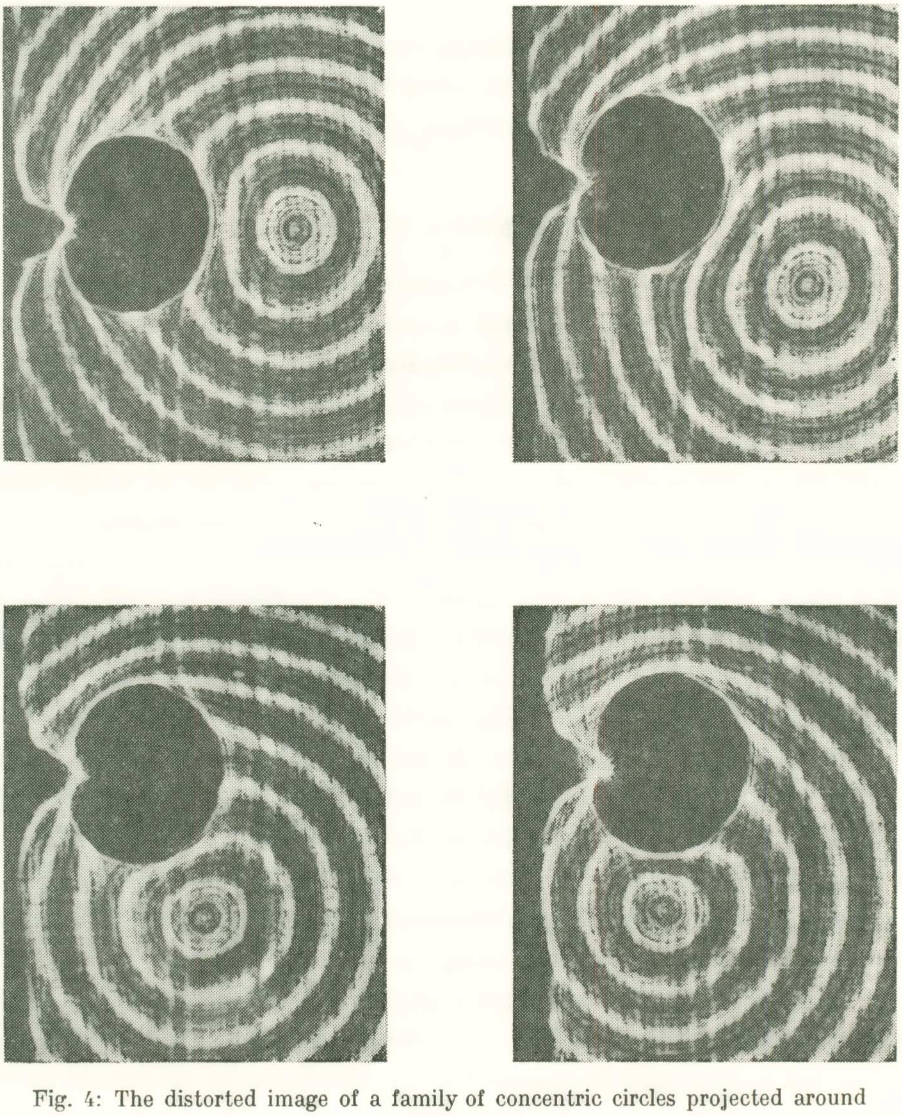
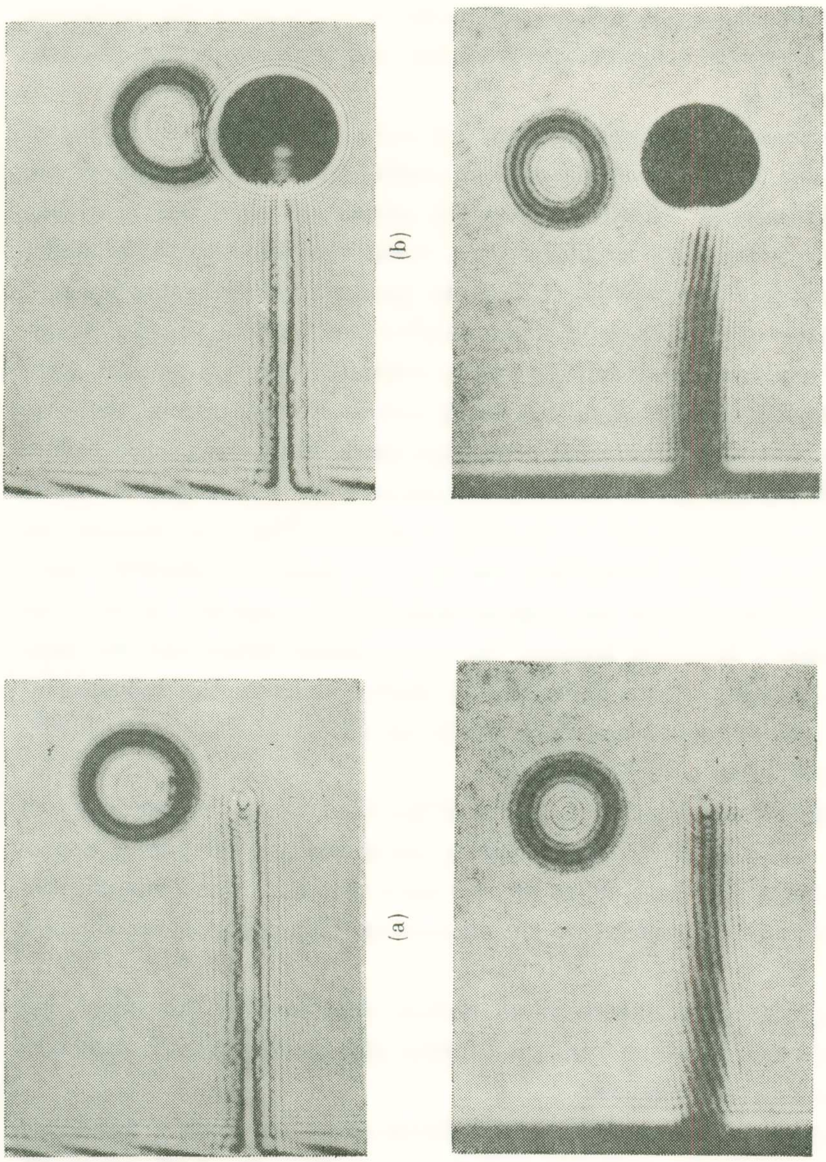


Fig. 4: The distorted image of a family of concentric circles projected around the stress singularity at the tip of a cracked polycarbonate plate.



(a) (b) (c) (d)

Fig. 5: The pseudocaustics formed around the stress singularity at the tip of a cracked plate, made of polycarbonate when the circle possesses a common point with the initial curve of the caustic at the unloading state (a) and at the loading state (b). Figs 5 (c and d) represent the same cases but for non-contact between the two curves.

outside the initial curves, by changing the magnification factor, λ_m, and checking, if the contact-forms of Fig. 2a or Fig. 2b were created.

However, the projected-circle case presents a higher flexibility than the scribed circle technique, since the experimenter is free to move the projection of the circle everywhere around the crack tip and thus to optimize the evaluation of angle ω.

Furthermore, Fig. 6 presents the +1 and —1 orders of diffraction of the caustic, due to diffraction phenomena created by the interposition of the dense circular grating. All three diffraction orders are enveloped in a cone with apex the centre of the circular grating. Again, the distortion of the higher order diffractions of the caustic yield all the data needed for evaluating p.

The values of the orders of stress singularities for both types of singularities, as they have been experimentally evaluated, were in general good agreement with their respective theoretical values. However, from these results it is evident that, in the case of an edge crack, the experimental values for the PMMA-plates are in excellent agreement with their respective theoretical values, the errors not overpassing 2 percent. This was because the material behaved as an elastic material and the theory of elasticity gave satisfactory results even for the singular zone. On the contrary, in the case of PCBA-plates a deviation between the experimental values and the theoretical ones appears. These deviations in reality indicate the influence of plasticity in the singular zone, where the linear theory of elasticity is inadequate to predict the actual behaviour.

This deviation between predictions of the ideal theory of elasticity and reality with materials of the praxis proves the necessity of disposing appropriate experimental tools to define the real state of stress around singularities when plasticity or viscoelasticity phenomena are influencing these singular zones.

In such cases the experimental method developed in this paper and based on information taken from the caustics and pseudocaustics developed in these zones is very efficient.

Concerning the order of elastic and plastic-stress singularities developed near concentrated loads applied to half-planes, or other forms of specimens, the theory of elasticity evaluates these singularities to be equal to $p = -1.0$. The deviations found between theory and experiment for PMMA may be mainly referred to the fact that the application of a concentrated-point load

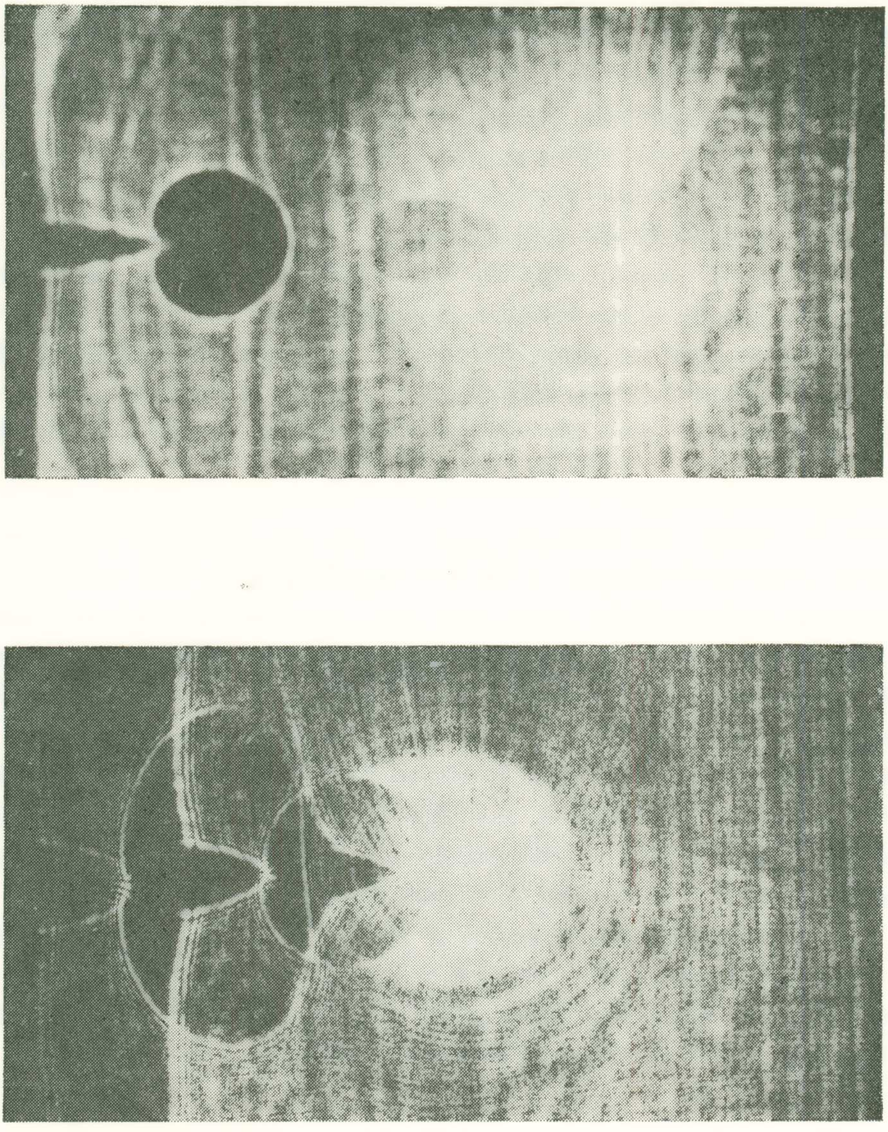


Fig. 6: The distorted images of the zero and the ± 1 orders of diffraction of a dense circular grating (20 lines per mm) as projected around the singularity of the tip of the cracked polycarbonate plate.

can only be achieved by a blunt indenter of some infinitesimal radius r = a. Then, the stress distribution around the indenter, which only at the early beginning of loading could be elastic, develops some kind of constrain, depending on the shape of the indenter, which changes drastically the state of stress near the singularity because of the constraining plastic zone. This plastic zone, which for PMMA may be very limited, becomes a large one in the case of PCBA, which presents an elastic-plastic behaviour, and this influences more severely the state of stress at the vicinity of the singularity.

Taking into account that the order of stress singularity represents in reality the velocity of the variation of stresses in the neighbourhood of the singular point, and the theoretical results developed by Hutchinson [**14**], about the form of the stress field near crack tip in the case of development of plastic deformation there, the order of singularity may be expressed by [**14**],

$$p = -1/(N+1) \tag{44}$$

where N denotes the hardening power-coefficient, evaluated by a simple tension stress-strain diagram of the material of the plate. The values of the hardening exponent N for PCBA were evaluated from the respective stress-strain diagrams of the material, when a series of typical tension specimens was submitted to simple tension. If the instantaneous values of the hardening exponent N are evaluated from a stress-strain diagram of the material for each loading step of the cracked or indented plate and the respective plastic stress singularity is evaluated, and this value is compared with the value derived experimentally, a good coincidence of results may be established, the deviations not overpassing in all cases the 1 to 4 percent [**13**].

The experimental and theoretical results are incorporated in Table I for singularities appearing at the tips of cracked plates, or at the points of application of concentrated loads at the straight boundaries of semi-infinite plates, made either of elastic (PMMA) or elastic--plastic materials (PC).

CONCLUSIONS

The experimental method based on caustics and pseudocaustics for the evaluation of the order of singularities developed in elastic, as well as in plastic fields provides a direct evaluation for the order of singularity, by a simple experimental procedure. The accuracy of the method was grown up by the introduction of a simple iterative procedure using data from the caustic curve.

T A B L E I

Experimental values of the orders of singularities in the case of a concentrated normal load applied at a straight boundary of a half-plane, and the crack tip of an edge crack existing in an infinite plate. In both cases the materials of the plates were either polymethyl-methacrylate (a brittle material), or polycarbonate of bisphenol A (an elastic quasi-perfectly plastic material). Experimental values were confronted with theoretical values, where the real stress situation was taken into account.

ELASTIC FIELD

polymethylmethacrylate (PMMA)	Cracked plate	Theory	Experimental Values				
	order of sing.	—0.500	—0.495	—0.510	—0.490	—0.490	—0.500
	error (%)	—1	—1.0	0	—2.0	—2.0	0
	concentrated load	Theory	—0.910	—0.870	—0.870	—0.870	—0.870
		Exp.	—0.920	—0.860	—0.860	—0.860	—0.860
		error %	1.0	1.0	1.0	1.0	1.0

PLASTIC FIELD

polycarbonate (PCBA)							
	Cracked	Theory	—0.435	—0.426	—0.414	—0.402	—0.392
		Exp.	—0.420	—0.420	—0.400	—0.390	—0.380
		error %	—3.40	—1.40	—3.30	—2.90	—3.00
	Concentrated load	Theory			—0.720		
		Experim	—0.730	—0.730	—0.720	—0.720	—0.710
		error %	1.30	1.30	0	0	—1.30

Thus, the method of caustics, which has been up-to-now exclusively used for the evaluation of the stress intensity factors existing at the tips of cracks, is now proved to be a versatile method for the evaluation also of the order of stress singularities in elastic and plastic fields with real singularities.

The method, with its realistic evaluation of singularities in real cases of elastic and plastic distributions, constitutes a potential method for the study of concrete problems with singular domains.

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ΠΕΡΙΛΗΨΙΣ

Ἡ ὀπτικὴ μέθοδος τῶν καυστικῶν ἐφηρμόσθη εὐρύτατα μέχρι σήμερον διὰ τὸν πειραματικὸν προσδιορισμὸν τῆς ἐντάσεως τῶν τάσεων εἰς ἰδιομόρφους περιοχὰς καὶ σημεῖα πεδίων τάσεων. Ὁ καθορισμὸς τοῦ συντελεστοῦ ἐντάσεως τάσεων εἰς τὰ ἰδιόμορφα σημεῖα τῶν πεδίων ἐπιτυγχάνεται μετὰ μεγάλης ἀκριβείας εἰς ἐντατικὰ πεδία εἴτε τῆς ἐλαστικῆς εἴτε καὶ τῆς πλαστικῆς καὶ ἰξοελαστικῆς καταστάσεως. Ἐν τούτοις μέχρι σήμερον δὲν ὑπῆρχε πειραματικὴ μέθοδος προσδιορισμοῦ τῆς τάξεως τῆς ἰδιομορφίας τοῦ πεδίου, ἡ ὁποία κατὰ γενικὸν κανόνα ἐλαμβάνετο εἴτε ἴση πρὸς $p = 1/2$ διὰ ρωγμὰς καὶ ἄλλας γεωμετρικὰς ἀσυνεχείας εἰς ἐπιπέδους ἢ καμπύλας πλάκας ἀπείρων διαστάσεων, εἴτε ἴση πρὸς $p = 1$ διὰ συγκεντρωμένα φορτία ἐπιβαλλόμενα εἰς τὰ σύνορα ἐλαστικῶν πεδίων ἀπείρων σχετικῶς διαστάσεων.

Ἐν τούτοις ὅμως κατέστη γνωστόν, ὅτι διὰ πεπερασμένα πεδία τάσεων ὅταν ἡ ἰδιομορφία παρουσιάζεται πλησίον συνόρου διαφόρων ἐλαστικῶν ὑλικῶν ἢ ὅταν ἡ ἰδιομορφία ὑπάρχει εἰς πεδίον πλαστικῶν παραμορφώσεων ἡ τάξις τῆς ἰδιομορφίας παύει νὰ παραμένῃ ἴση πρὸς τὰς ἀνωτέρω ἀναφερθείσας τιμάς.

Τὸ πρόβλημα καθορισμοῦ τῆς τάξεως τῆς τασικῆς ἰδιομορφίας εἰς ἐλαστικὰ πεδία ἀντεμετωπίσθη μέχρι σήμερον μὲ τὴν βοήθειαν ἀναλυτικῶν μεθόδων βασιζομένων εἰς τὴν θεωρίαν ἀναπτύξεως τῶν χαρακτηριστικῶν ἢ ἰδιοσυναρτήσεων καὶ εἰς ἐφαρμογὰς τοῦ μετασχηματισμοῦ Mellin. Μόλις τὸ 1952 ὁ ἀμερικανὸς καθηγητὴς Williams [**1**] κατώρθωσε νὰ συναγάγῃ τὴν χαρακτηριστικὴν ἐξίσωσιν διὰ τὸν προσδιορισμὸν τῆς ἰδιομορφίας τῶν τάσεων εἰς σφηνοειδῆ ἐλαστικὰ πεδία, βασισθεὶς εἰς τὴν παραδοχὴν ὅτι ἡ τάξις τῆς ἰδιομορφίας εἶναι πραγματικὸς ἀριθμός. Ἐπὶ τῶν

αὐτῶν παραδοχῶν ἐβασίσθη σειρὰ ὅλη ἐπιστημόνων διὰ τὸν καθορισμὸν διαφόρων ἰδιομορφιῶν τασικῶν πεδίων, οἱ ὁποῖοι ἐχρησιμοποίησαν πολλάκις καὶ μεθόδους ἀποτελούσας ἐφαρμογὰς τοῦ μετασχηματισμοῦ Mellin [**1** - **7**].

Ἡ τάξις ἰδιομορφίας εἰς πολύσφηνον ἀποτελουμένη ἀπὸ σειρὰν τομέων ἐλαστικῶν ὑλικῶν καθωρίσθη ὑπὸ τοῦ Θεοχάρη [**8**] διὰ περιπτώσεις τυχαίων μιγαδικῶν ἰδιομορφιῶν.

Ἐν τούτοις μέχρι σήμερον δὲν ἔχει ἐπινοηθῆ οὐδεμία πειραματικὴ τεχνικὴ διὰ τὸν καθορισμὸν τῆς τάξεως τῆς ἰδιομορφίας τασικῶν πεδίων. Εἰς τὴν παροῦσαν ἀνακοίνωσιν θὰ παρουσιάσωμεν μεικτὴν θεωρητικὴν καὶ πειραματικὴν μέθοδον διὰ τὸν ἀκριβῆ καθορισμὸν τῆς τάξεως τῆς ἰδιομορφίας τασικῶν πεδίων. Ἡ μέθοδος αὐτὴ στηρίζεται εἰς τὴν θεωρίαν τῶν ψευδοαναλυτικῶν συναρτήσεων καὶ τῆς ὁμοσυμμόρφου ἀπεικονήσεως, ἡ ὁποία, συνδυαζομένη μὲ τὴν θεωρίαν τῶν ψευδοκαυστικῶν, ἐπιτρέπει τὸν προσδιορισμὸν τῆς τάξεως ἰδιομορφίας εἰς ἐντατικὰ πεδία ἐλαστικῶν καὶ πλαστικῶν παραμορφώσεων.

Ἡ μέθοδος αὐτὴ δέχεται τὴν γενικῶς παραδεδεγμένην ὑπόθεσιν ὅτι μόνον ὁ κυριαρχῶν ὅρος τοῦ ἀναπτύγματος εἰς σειρὰν Laurent τοῦ μιγαδικοῦ δυναμικοῦ τοῦ περιγράφοντος τὸ πεδίον τῶν τάσεων καὶ παραμορφώσεων εἰς τὸ θεωρούμενον πρόβλημα λαμβάνεται ὑπ' ὄψιν διὰ τὸν ὑπολογισμὸν τῶν συνιστωσῶν τῶν τάσεων.

Ἀποδεικνύεται ὅτι πᾶσα γραμμὴ καὶ εἰδικῶς κύκλος κείμενος ἐκτὸς τῆς περιοχῆς τοῦ πυρῆνος πέριξ τῆς ἰδιομορφίας, ὁ ὁποῖος ὁρίζεται ὑπὸ τῆς ἀρχικῆς καμπύλης τῆς ἀντιστοίχου καυστικῆς, παράγει ψευδοκαυστικήν, ἡ ὁποία ὑπακούει εἰς τὰς συνθήκας ὁμοσυμμόρφου ἀπεικονίσεως. Κατ' αὐτὴν κύκλοι εὑρισκόμενοι εἰς τὴν γειτονίαν τῆς ἀρχικῆς καμπύλης ἀπεικονίζονται εἰς τὸ ἐπίπεδον προβολῆς εἰς ἐλλείψεις, οἱ μέγιστοι ἄξονες τῶν ὁποίων ὁρίζουν γωνίαν μετὰ τοῦ ἄξονος τῆς ρωγμῆς, ἡ ὁποία δύναται νὰ μετρηθῆ ἀκριβῶς καὶ εὐκόλως ἐπὶ τοῦ ἐπιπέδου ἀναφορᾶς. Ἡ ἐφαπτομένη τῆς γωνίας αὐτῆς συνδέεται ἀπ' εὐθείας καὶ δι' ἁπλῆς σχέσεως μὲ τὴν τάξιν τῆς ἰδιομορφίας τοῦ πεδίου, ἡ ὁποία κατ' αὐτὸν τὸν τρόπον μπορεῖ νὰ ὑπολογισθῆ εὐκόλως.

Εἰς τὴν ἀνακοίνωσιν αὐτὴν εἰσάγεται διαδικασία προσεγγιστικοῦ ὑπολογισμοῦ τῆς τάξεως τῆς ἰδιομορφίας διὰ συνδέσεως στοιχείων ἐκ τῶν ψευδοκαυστικῶν καὶ τῆς ὑπαρχούσης καυστικῆς. Ἐφαρμογαὶ τῆς μεθόδου διὰ τὸν ἀκριβῆ ὑπολογισμὸν τῆς ἀληθοῦς τιμῆς τῆς ἰδιομορφίας ρηγματωμένων πλακῶν καὶ πλακῶν δεχομένων συγκεντρωμένα φορτία καταδεικνύουν τὰς δυνατότητας τῆς μεθόδου.

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