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ΠΡΟΕΔΡΙΑ ΝΙΚΟΛΑΟΥ ΑΡΤΕΜΙΑΔΗ

ΦΥΣΙΚΗ. – **Semi-reducible Einstein[1] Spaces of dimension n ≥ 4,** by J.P. Constantopoulos*, διὰ τοῦ ἀκαδημαϊκοῦ Γεωργίου Κοντόπουλου.

ABSTRACT

Semi-reducible properties of Einstein Spaces are systematically examined for any dimension $n \geq 4$ and for arbitrary signature. In particular, the equations of semi-reducibility are introduced and then solved for many special cases. The results are presented in certain series of solutions. It is also shown that the aforementioned series are divided into two classes, each class displaying a different pattern of behavior. The essential features of each pattern are discussed in detail and new solutions for $n \geq 5$ are presented. The Newtonian limit of the S^+-subseries is also considered.

1. Introduction

The purpose of this paper is to present and classify as far as possible the locally semi-reducible[2] Einstein spaces for any number of dimensions $n \geq 4$. Spaces of this particular kind have been considered in the pioneering work of Brinkmann (1924, 1925) and they include Einstein spaces that can be conformally mapped on Einstein spaces. These solutions are non-trivial (i.e. not spaces of constant curvature) only for $n \geq 5$. De Vries (1954) has also suitably generalized these solutions. In the same line of thought the author has also presented solutions of the vacuum field equations for $n \geq 6$. These solutions are Einstein spaces which are also V(0)-spaces[3] and which satisfy the criterion of Kundt for gravitational waves (for the definition see Constantopoulos 1993).

In all the above cases an intimate relation between the dimensionality of the

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underlying manifold and the «type» of the solution is suggested. Further to this point the various constraints induced by the notion of semi-reducibility have never been considered seriously. In order to get better insight to the situation we recall that the (type-D) static solutions of the vacuum (subclass A and B) are of the semi-reducible type (Kramer et al, 1980), but they can not be related, in any particular way, to the above mentioned cases. Besides, we do not even know if these well known solutions can be extended for $n>4$. In exactly the same way, the Kasner's solutions are also semi-reducible, but we do not know how they behave as n incrases beyond 4. Questions of this particular type are also related to the possible values of the cosmological constant and they suggest a systematic examination of the semi-reducible Einstein spaces for any arbitrary dimension n ($n>4$).

Increasing the number of dimensions to $n>4$ implies that the standard classification tools can not be used any more. For example, type D, which implies two double roots[4], is meaningless now. On the other hand the new condition of semi-reducibility introduces different types of classification, as we shall see in the sequel. Furthermore the number of possible signatures increases rapidly with n. Thus, it is at least desirable, to work in a way which is independent of the particular signature although the physically interesting cases correspond to the Lorentz signature exclusively.

In the light of the above this paper is organized in the following way. In section 2, we consider the q-analysis of the semi-reducible spaces and in section 3, we reconsider a special class of semi-reducible spaces that are very important for our analysis, namely, the $V(K)$-spaces. In section 4, we present the differential equations which prescribe the semi-reducible Einstein spaces for any $n\geq 4$. To the best of our knowledge this set of equations has never been considered, but although they look tough they give us plenty of information about their solutions. In particular, our method reduces the original system of the vacuum field equations to a new set of equations that refer to a space V_p of a lower dimension. The rest of the original set reduces automatically to a subset of auxiliary conditions. Besides, the form of these conditions enable us to rediscover in a convenient way all the solutions that are $V(K)$-spaces. In particular, a new class of $V(K)$-spaces ($K\neq 0$), which are *not* included in the solutions given by De Vries is presented. These solutions have a threshold dimension $5p+4$ for any $p\geq 1$ and they are not equivalent to the solutions given by Brinkmann.

The net result of our analysis is that there are essentially two different classes of solutions, which follow different patterns of behavior for $n>4$. According to the first pattern, solutions exist, that can be constructed in a way, which is independent of the dimension of the space. In particular, these solutions depend only on some of the details of the q-analysis under consideration and not on the particular way the dimension of the space is partitioned among the various slices of the aforementioned analysis. Further to this point an unexpected new type of solutions, *for* $n>4$, is presented. These solutions are said to be of the N-type and they are essentially different from the $V(K)$-spaces considered previously. We conjecture that *semi-reducible* spaces of the N-type and the $V(K)$-spaces are the only solutions that exhibit the aforementioned properties.

Last but not least in section 6 we present extensions of the Kasner's solutions for $n>4$. In addition, extensions of the type-D static solutions are treated in detail in section 7. All these results are summarized, together with the known solutions of De Vries and Petrov in a particularly flexible classification scheme (see table I) which can be further elaborated in those regions, where solutions of unknown type may exist. The special feature, of our classification scheme, is that we sharply distinguish those sub-cases, where the Einstein space is also a $V(K)$-space, from those where this does not happen. The main reason for this discrimination is that Einstein spaces which are also $V(K)$-spaces are of less interest from the physical point of view[5]. A possible exception to this comment are the new $V(K)$-solutions mentioned previously the threshold dimension of which is $n=9$.

2. Semi-reducible Riemannian spaces

A Riemannian space will be called semi-reducible, if to a suitably chosen coordinate system, its metric ds^2 can be written in the form,

(1) $$ds^2 = ds_0^2 + \sigma_1^2\, ds_1^2 + \ldots + \sigma_q^2\, ds_q^2, \quad (\sigma_a \neq c\sigma_\beta,\ \alpha\neq\beta,\ c=\text{constant}),$$

where,

(1a) $$ds_a^2 = g_{i_a j_a}\, dx^{i_a} dx^{j_a}, \quad (a=0,1,\ldots,q)$$

Here, each one of the appended metrics ds_a^2, $(a=1, \ldots, q)$, depends only on the coordinates $\{x_a\}$, while the funtions σ_a depend only on the coordinates $\{x_0\}$ of the fundamental part ds_0^2 of the above metric.

The above analysis, is called the *q-analysis* of the semireducible space V_n $(n\geq 3)$ and the terminology will be the same as in the case of the $V(K)$-spaces, which are defined in section 3 and consist a special class of semi-reducible spaces. Thus, the principal part of the above metric ds^2 is called the *kernel* of the q-analysis, while the additional metrics are called appended metrics or slices of the analysis under consideration.

In general the *q-analysis* considered above is not unique (Kruckovic 1957, 1961 and 1963). It also occurs that the very same metric allows more than one q-

analysis, each one with a different q. However in all cases the particular q-analysis implies a corresponding partition of n, l.e. of the dimension of the underlying manifold. In fact, we have the condition

$$n = p + \sum_{a=1}^{q} n_a, \tag{2}$$

which correlates the dimension of the space n and the dimension of the *kernel* p to the *lenght* $q(p)$ of the semireducible analysis. It is worth noticing that in our case trivial appended metrics (i.e. 1-dimensional slices of the the q-analysis) contribute to the total length of the analysis and that they are not absorbed in the kernel[6].

The above mentioned partitions prescribe the *type* of the q-analysis under consideration, which can be represented as a sequence of arbitrary integers with some multiplicity. For example, the sequence

$$\{p, k_1, 2(k_2),....\},$$

represents a space, the semi-reducible analysis of which has a kernel of dimension p and at least three appended metrics of dimension k_1, k_2 and k_2 respectively.

Although q may be not unique, our conventions (see footnote 2) enable us to define always a $q_{max} \leq n$-1, which represents an *invariant* of the underlying space. The q_{max}-analysis may not be unique but we can always prove[7] that in all these cases p can be chosen in such a way that $p = p_{min}$ or p_{max}. Thus, if $p_{min} \neq p_{max}$, the triplet (p_{min}, p_{max}, q_{max}) is invariant and it can be used in all cases, providing us with a convenient classification tool. On the other hand, if $p_{min} = p_{max}$, then the pair (p, q_{max}) is a sufficient tool for the characterization of the aforementioned analysis.

It is worth mentioning that the q-analysis (1) becomes trivial if and only if one, at least, of the functions σ_a is a constant. In this particular case, the underlying space is *reduciblei* and not semi-reducible. Einstein spaces that are reducible are well known for any dimension $n \geq 4$ (Fialkow 1938). They are trivial cases of our subsequent analysis.

3. *V(K)*-spaces

A special class of semi-reducible spaces are the *V(K)*-spaces. Many of these spaces appear as solutions of a certain differential equation (see De Vries 1954) but their geometric properties and their identification as a more or less special class of Riemannian spaces, is due to Solodovnikov (1956).

In a *V(K)*-space the kernel is always a space of constant curvature, but in addition the functions σ_a, are not arbitrary any more. In fact, they are determined by the conditions

(3a) $$\sigma_{\alpha,ij} = - K\sigma_\alpha g_{ij}$$

and

(3b) $$\Delta_1(\sigma_\alpha, \sigma_b) = -K\sigma_\alpha\sigma_b, \ (\alpha\neq b)$$

where K is an invariant constant, characteristic of the space under consideration. Here g_{ij} is the fundamental tensor of the kernel of analysis (1) and[8]

(3c) $$\Delta_1(\sigma_\alpha, \sigma_b) \equiv g^{ij}\partial_i\sigma_\alpha\partial_j\sigma_b.$$

The q-analysis of a $V(K)$-space which satisfies the above mentioned conditions is a K-analysis. However, it is worth noticing that although every K-analysis of a $V(K)$-space is necessarily a q-analysis, the converse is not in general true.

It is essential for our analysis to notice that the solutions of Brinkmann and De Vries, mentioned in our introduction, are $V(K)$-spaces. Furthermore, the solutions presented by the author (Constantopoulos 1993) are also $V(0)$-spaces[9]. However, the important point is that the aforementioned cases do *not* exhaust the $V(K)$-spaces, which are Einstein spaces too.

4. Semi-reducible Einstein spaces

We consider the general Einstein space E_n, which can be regarded as a solution of the vacuum field equations in the presence of a cosmological term. The condition that this particular solution is of the semi-reducible form (1) leads after some algebra to the following set of equations,

(4a) $$R^{(0)}_{ij} = \frac{R}{n} g^{(0)}_{ij} + \sum_{a=1}^{q} n_a\sigma_\alpha^{-1}\sigma_{\alpha;ij}$$

(4b) $$R^{(\alpha)}_{ij} = \frac{R^{(\alpha)}}{n_\alpha} g^{(\alpha)}_{ij}$$

(4c) $$\frac{R^{(\alpha)}}{n_\alpha} = \frac{R}{n}\sigma_\alpha^2 + (n_\alpha-1)\Delta_1\sigma_\alpha+\sigma_\alpha\Delta_2\sigma_\alpha+\sigma_\alpha\sum_{b\neq\alpha}^{q} n_b\sigma_b^{-1}\Delta_1(\sigma_\alpha, \sigma_b).$$

(4d) $$\Delta_2\sigma \equiv g^{ij}\sigma_{;ij}.$$

Here, $R^{(0)}_{ij}$ and $R^{(\alpha)}_{ij}$ represent the Ricci tensor of the kernel and of the various slices respectively while R/n is the cosmological constant of the total space and $R^{(\alpha)}$ are the scalar curvatures of the appended metrics which are, for the moment, quite arbitary.

From the above equations combining (4b) and (4c) we get that the coefficients $R^{(a)}$ are necessarily constants. In fact contracting equation (4b) with the metric tensor $g^{ij(a)}$, we conclude that the scalar curvature of the slice V_a, is a function of the $\{x_a\}$. On the other hand, equation (4c) demonstrates that $R^{(a)}$ should be a function of the $\{x_0\}$ and this proves our previous assertion. Now, if $n_a>3$, the corresponding appended metric is that of an Einstein space. For $n_a=2$ and $n_a=3$ our previous proof implies that this metric is that of a space of constant curvature.

Summarizing our previous results we conclude with the following result.

Theorem 1 *If an Einstein space is semi-reducible then each non trivial (i.e. not 1-dimensional) appended metric is either that of a space of constant curvature or that of an Einstein space.*

The significance of this theorem is obvious. It reduces the problem of finding the semi-reducible Einstein spaces to that of finding the appropriate, in each case, kernel and then solving, by any available method, equation (4a). In all cases, equations (4c) can be regarded as the compatibility conditions, which restrict the possible solutions.

Using equation (2) we may recast equation (4a) in a more convenient form namely,

$$\{R_{ij}^{(0)} - (p-1)Kg_{ij}^{(0)}\} = \sum_{\alpha=1}^{q} n_\alpha \left(\frac{\sigma_{\alpha,ij}}{\sigma_\alpha} + Kg_{ij}^{(0)}\right), \tag{5a}$$

where we have replaced the cosmological constant R by the constant K through the substitution,

$$R=n(n-1)K. \tag{5b}$$

Equation (5a) suggests that solutions of the system of equations (4a,b) may exist which do *not* depend on the n_a and consequently *do not depend* on the dimension n of the space. In particular, an unexpected new class of such solutions can be discovered in the special case, where the kernel of our q-analysis is *not flat* and the σ_a are solutions of the equation[10]

$$\sigma_{\alpha,ij} = 0, \tag{6}$$

where again the indices i and j refer to the kernel. From equations (6) and because of our assumption that the kernel is not flat and not reducible, we conclude that it admits $r\leq[p/2]$ mutually orthogonal fields of parallel *null* vectors (Eisenhart 1938). This remark fixes the possible σ_α and q_{max}. In all these cases $q_{max} = [p/2]$, where [] indicates the integer part of $p/2$. Furthermore, combining our previous remarks with (4b) and (4a) we conclude that

(7) $$R^{(\alpha)} = R = R^{(0)} = 0,$$

in all such cases. Hence, both equation (5a) as well as the conditions (4b) are satisfied, which proves our assertion.

These solutions, although very similar to *V(0)*-spaces, are of an essentially different nature and they exist only for $p \geq 4$. They are said to be solutions of N-type and they form the N-series, each member of which is completely characterized by the values of p and q, where $q \leq q_{max}$. In principle N-series should exist for any $n \geq 5$. However, the proof of our assertion depends on the existence of kernels with the aforementioned properties, which actually exist. In particular, any four-dimensional solution of the vacuum field equations of general relativity which satisfies the criterion of Kundt can be used as a kernel, to generate spaces of the N-type through the solutions of equation (6) (e.g. the Peres solutions, the Takeno solution etc.). Further to this point, for any $p > 4$, solutions which satisfy our requirements exist (Walker 1950) and they can be used in principle, for the construction (see also Constantopoulos 1992). Hence, N-series of solutions exist for any $p \geq 4$.

The cases $p=1$ and $p=2$ are of a special interest. Thus, for $p=1$ the equations (4a) and (4b) reduce to a system of ordinary differential eqauations while the left-hand side of equation (5a) identically vanishes. The case $p=2$ is more interesting. In fact, in this particular case equation (4a), far from being trivial, is considerably simplified. The reason for this simplification is that every two-dimensional space is an Einstein space. Hence, instead of equation (4a), for p=2 we have an eqation of the form

(8) $$\sum_{\alpha=1}^{q} n_\alpha \sigma_\sigma^{-1} \sigma_{\alpha,ij} = \left(\frac{nR^{(0)}(x_0) - 2R}{2n}\right) g_{ij}^{(0)},$$

where the scalar curvature of the kernel is not a constant any more. Another special case of considerable interest results when the kernel is flat, but the space itself is not a *V(0)*-space. All these cases will be considered in detail in the sequel.

5. Solutions of the V-type

Our first task is to prove, that solutions of the set of equations (4), which are *V(K)*-spaces, exist for any p and any q. Since K in (3a) is arbitrary it is sufficient to replace R by K using (5b) in equations (4a, b). In this case, taking into consideration that the kernel is necessarily a space of constant curvature and using (3a), we conclude that (4a) is automatically satisfied for any p and q.

The compatibility conditions (4b) are more delicate. But (3a) admits a first integral of the form,

(9a) $$\Delta_1(\sigma_\alpha) + K\sigma_\alpha^2 = k_\alpha = \text{const.}$$

where k_a are artitrary constants[11]. Thus substituting (9a) in (4c) and using (3a) we conclude that

(9b) $$R^{(\alpha)} = n_\alpha(n_\alpha - 1)k_\alpha,$$

which proves our assertion. In fact, for any $K\neq 0$ and for any k_a associated with a given solution of the equations (3a), through the first integral (9a), we have a $V(K)$-space which is also a non-trivial Einstein space. The final step, namely, that of identifying the appended metric $ds_a{}^2$ is completely at our disposal the only restrictions being those implied by Theorem 1 and the conditions (9b). This means that the n_a involved in the construction are quite arbitrary, which can be also verified directly from equation (5a).

Our proof is independent of the values of p, q and q_{max}. In fact the value of q_{max} is an intrinsic property of the $V(K)$-space under consideration. In some cases $q_{max} = q_{max}(p)$, while in other cases q_{max} is independent of p and in addition unbounded (compare Kruckovic 1961, Constantopoulos 1993). The question about the possible values of q_{max} is still open and it is closely related to the existence or not of solutions of the equation (3a) which are *non-trivial* $V(K)$-spaces. However, the only fact that is important for our analysis is that for $K\neq 0$ and $q_{max}=p+1$, $V(K)$-spaces exist which are not solutions of the equation (3a) (Kruchkovic 1967). Combining this remark with our Theorem 1, we conclude that there are two classes of Einstein spaces which are also $V(K)$-spaces namely, the spaces of the V1-type which are solutions of equation (3a) and those of the V2-type which are not. In the second case, from equation (2) using the fact that $q_{max}=p+1$, we find that the threshold dimension for the spaces of the V2-type is $n_{thr}=5p+4$ $(p\geq 1)$. These solutions are essentially different from those considered by Brinkmann and De Vries. They form a new class of Einstein spaces, which can not be mapped conformally on other Einstein spaces (see footnote 3). The first representative of this class, for $n=9$, will be given explicitly in the Appendix.

The net result of this section is: two different series of solutions of the fundamental equation (4a) and (4b). In the V1-series, each solution is characterized by p and q. In this case the well-known solutions of Brinkmann and De Vries are rediscovered. The solutions of the V2-series depend only on p and they are essentially of a new type, having different properties. In both cases the distribution of the n_a in (1) and consequently the total dimension of the resulting semi-redicible space are quite arbitrary[12].

6. Solutions for $p=1$

The only solutions possible for $p=1$ and $q=1$, are those prescribed by Brinkmann. In our formalism these can be rediscovered immediately from equation (3), which is now reduced to an ordinary differential equation of the second order namely, the harmonic oscillator equation. The only conditions required because of (3a) and (3c) are, (5b) and (9b), where now $n_1=n-1$. The case where $p=1$ and $q=2$ is much more delicate because now we have to consider simultaneously both the linearly independent solutions of the aforementioned equation. The resulting two solutions (depending on the sign of K) are the first of the V2-series to be considered and they introduce a threshold dimension of $n=9$ as we have already explained. The details of these solutions are indicated in the Appendix.

For $p=1$, $R^{(0)} = 0$, the particular sub-case prescribed by the rest of the conditions (7) is immediately solvable. In this case (4a) and (4b) can be written in the form

$$0 = \sum_{\alpha=1}^{q} n_\alpha \sigma_\alpha^{-1} \ddot{\sigma}_\alpha \tag{10a}$$

$$\dot{\sigma}_\alpha^2 = \sigma_\alpha \ddot{\sigma}_\alpha + \sigma_\alpha \dot{\sigma}_\alpha \sum_{b=1}^{q} n_b \sigma_b^{-1} \dot{\sigma}_b \tag{10b}$$

where, in general, some of he n_a are not equal to one[13]. Now substituting in the above set of equations the expressions,

$$\sigma_\alpha = C_\alpha t^{p_\alpha}, \tag{11}$$

we reduce the system of differential equations (10a) and (10b) into an algebraic set of conditions namely,

$$\sum_{1}^{q} n_\alpha p_\alpha^{\,2} = \sum_{1}^{q} n_\alpha p_\alpha = 1, \tag{12}$$

where again the n_a are completely arbitrary.

If $n_a=1$ ($a=1,\ldots, q$) our results are a generalization of the well-known Kasner solution, for any arbitrary n (see also Petrov 1946). However our analysis for an arbitrary set of n_a, some of which are different from 1, reveals new and unexpected features. For $q=2$ and $n_1+n_2\geq 2$, the equations (12) have either a unique non-trivial solution, or two distinct solutions which, combined with our Theorem 1, prescribe the K1, K2-series of special Einstein spaces ($R=0$). For $q\geq 3$, the situation is essentially different because now the resulting K3-series may include members which are reducible (e.g. if $q=5$ and $p_4=0$). If we exlude this possibility, e.g. requiring that $p_a\neq 0$ for all $a=1,..., q$, the resultint K3-series involve, for each possible combination of the n_a, an infinity of solutions. If $n_a<4$, the corresponding slice V_a is necessarily flat, hence the appended metric is unique. In all other cases this metric is that of a special[14] but otherwise arbitrary Einstein space.

Examples of each case can be easily obtained. Fixing $n_a=1$, we have a special K1-series of solutions prescribed by equation (10), where

$$p_1 = \frac{3-n}{n-1},\ p_2 = \frac{2}{n-1}\ (n\geq 4). \tag{13a}$$

Another interesting K1-series results for $n_1=n_2=p$, where $p>1$. The threshold dimension here is $n=5$ and each one of the solutions corresponds to an odd integer ($n=2p+1$). Again our solution is prescribed by (11), where

$$p_{1,2} = \frac{1\pm\sqrt{2p-1}}{2p}\ . \tag{13b}$$

So far we have considered only special Einstein spaces. However, the situation changes dramatically for a nonvanishing cosmological constant ($R\neq 0$). In fact solutions of the form (11) are no more possible. Following Petrov (Petrov 1946, 1969) we introduce the solutions

$$\sigma_\alpha(t) = \sin^\lambda(t)\tan^{\mu_\alpha}\left(\frac{t}{2}\right), \tag{14}$$

where λ and μ_a are arbitrary constants. Substituting in (4a) and (4c) the above solutions we conclude with a set of algebraic conditions which are the analogous of equations (11) in the case of a non vanishing cosmological constant namely,

$$R = n(n-1)\lambda^2 \tag{15a}$$

$$\sum_{\alpha=1}^{q} n_\alpha\mu_\alpha = 0$$

$$\sum_{\alpha=1}^{q} n_\alpha\mu_\alpha^2 = (n-1)\lambda(1-\lambda) \tag{15c}$$

$$\lambda = \frac{1}{n-1} \tag{15d}$$

where n is the dimension of the space. These conditions generalize the results of Petrov (Petrov 1946) in our case and they introduce the P-series in complete analogy with the K-series considered previously. The crucial point here is that although we have a non vaninshing cosmological constant $R>0$, the scalar curvatures of the appended metrics systematically vanish. The case $R<0$, is exactly the same as before but *sin* and *tan* are now replaced by their hyperbolic analogues.

7. Solutions for p=2

The case $p=2$ is rather special as we have already mentioned (see equation 5). In particular, for $p=2$ and $q=1$ our original set of equations (4a) and (4c) reduces to the equations

$$\text{(16a)} \qquad \frac{R^{(1)}}{(n-2)(n-3)} = \frac{(n-4)R+nR^{(0)}}{n(n-2)(n-3)}\sigma^2+\Delta_1\sigma,$$

$$\text{(16b)} \qquad \sigma_{,ij} = \frac{nR^{(0)}(x)-2R}{2n(n-2)}\sigma g^{(0)}_{ij}$$

where again the covariant differentiation and the indices l, j refer only to the two-dimensional kernel. The essential point here is that $R^{(0)}$ is *not* a constant any more. Equation (16a) has been studied in many cases and by many authors. However, from De Vries (1954) we know, that in the above case there is only one solution, if and only if $R^{(0)} = R^{(0)}(\sigma)$. Further to this point it can be easily proved that equations (16a) and (16b) are compatible if and only if, $R^{(0)}(\sigma)$ satisfies the ordinary differential equation

$$\text{(17)} \qquad \dot{R}^{(0)}(\sigma)+(n-1)R^{(0)}(\sigma)\sigma^{-1} = \frac{2}{n}R\sigma^{-1}$$

where the dot indicates derivatives with respect to the unknown function σ. Equation (17) can be immediately solved, the solution being of the form,

$$\text{(18)} \qquad R^{(0)}(\sigma) = \alpha\sigma^{1-n} + \frac{2R}{n(n-1)}$$

where a is an arbitrary integration constant. Since we now know precisely the form of the function $R^{(0)}(\sigma)$, we can easily integrate our original equation (16a). In particular, we have

$$\text{(19a)} \qquad ds_0^2 = g_{11}dt^2 - \frac{1}{g_{11}}dr^2,$$

where

(19b) $$g_{11} = \frac{\alpha}{(n-2)(n3)} r^{3-n} + \frac{R}{n(n-1)} r^2 + C,$$

and

(19c) $$\sigma(r) = r.$$

Here, *C* is an integration constant which is very important for our analysis. In fact using equation (16b), we conclude that *C* prescribes the scalar curvature of the appended metric namely

(19d) $$R^{(1)} = (n-2)(n-3)C,$$

where $n>3$ is the total dimension of the underlying space. For $n=4$, $C=1$ and $a=-2m$ we have the Schwarzschild solution as this has been generalized by Kottler in the presence of an arbitrary cosmological constant. Further to this point C can be always normalized to the values ± 1 or 0,. fixing the scalar curvature of the corresponding slice to positive, negative or zero. Thus the S-series of solutions, induced by the equations (19a) to (19d) include naturally, for $n=4$, all the static solutions of the vacuum of type D (Kramer et al, 1980). In order to complete the proof of our assertion it is sufficient to notice that in (19a) the choice of the signature was quite arbitrary. Thus readjusting the signature in all possible ways and computing g_{11} for each case separately we recover ($n=4$) all the aforementioned solutions. Thus, we conclude with three distinct sub-series of the S-type, S^+, S^- and S^0 corresponding to the three possible values of the normalized constant *C*.

Clearly, our previous analysis does not exhaust the various possibilities even for $p=2$. In particular, special solutions for $p=2r$ ($r\geq 1$) always exist, assuming only that the kernel is flat and requiring the appropriate[15] signature. In this special case the flatness of the kernel guarantees that equation (4a) is considerably simplified to the form of a differential equation with partial derivatives which can be easily satisfied for $q\geq 2$. Further to this point, choosing the flat kernel and the σ_α in the form

(20) $$ds^2_0 = 2dx^1dx^{r+1} + \ldots + 2dx^r dx^{2r}$$

(21a) $$\sigma_b\,(x^{r+1},\ldots, x^{2r}) = \cos\Big(\sum_1^r \omega_{bs}x^{r+s} + c_b\Big),\ (b=1,\ldots,k)$$

(21b) $$\sigma_b\,(x^{r+1},\ldots, x^{2r}) = \cosh\Big(\sum_{s=1}^r \omega_{bs}x^{r+s}+c_b\Big),\ (b=k+1,\ldots,q)$$

we have the conditions (4c) automatically satisfied while the equation (4a) reduces to the equations

(22) $$R_{r+s_1,\, r+s_2} = 0,$$

which, because of the particular choice of the σ-functions, degenerate into algebraic conditions among the w namely

(23) $$\sum_{s=1}^{k} n_b \omega_{bs_1} \omega_{bs_2} - \sum_{b=k+1}^{q} n_b \omega_{bs_1} \omega_{bs_2} = 0,$$

where $s_1, s_2 = 1,\ldots,r$.

The resulting F-series depend on $r \geq 1$ and on the particular distribution of the n_a. Solutions of this particular type exist for any $n \geq 4$ but only the first member of this F-series is known for $n=4$ and $r=1$ (Petrov 1967). In our case and for $r=1$ an unbounded number of solutions can be generated by the following prescription,

(24) $$q=2k,\ \omega_b = \omega_{b+k},\ n_b = n_{b+k},\ (b=1,\ldots,k)$$

$$c_\alpha \neq c_\beta \ (\beta \neq \alpha).$$

This particular class of solutions forms the F_{2k}-series the properties of which, in an abuse of language, imitate the properties of the V-series.

7. Conclusions

Our analysis demonstrates explicitly, that the solutions of the Einstein equation in the vacuum, which have the extra property of being semi-reducible, display two different patterns of behavior for any $n \geq 4$. However, this difference becomes transparent only for dimensions $n \geq 5$. Thus, solutions of the V-type and the N-type can be constructed in a way, which is entirely independent of any distribution of the n_a and which is compatible with the dimension n. We have also shown that there are sub-cases where this phenomenon is realized only for dimensions $n \geq 9$. Further to this point equation (5a) suggests that the V-series and the N-series are the only cases where the aforementioned phenomenon occurs. Although arguments based on the equation (5a) are very strong, a rigorous proof of this conjecture is desirable. The rest of the solution given in this paper is classified into series of solutions and the members of all these series depend explicitly on the distribution of the n_a. The solutions presented here, which belong to the same series, have certain properties in common. However, the essential point is that they all have a representative, or certain (degenerate) representatives at dimension 4. For both patterns and for sufficiently large dimensions the structure of the slices is more or less irrelevant assuming only that it satisfies the requirements of our Theorem 1. The essentials of each type are summarized, in a compact form in table I, where the first column indicates the value of the cosmological constant possible, for the particular type under consideration.

Solutions with a non-flat kernel for $p=2$ and $q>1$ exist. This can be proved indirectly in the following way. We may start with a 6-dimensional member of the S^+-series. Then we may replace the arbitrary so far slice of this space by the Kottler solution (this particular choice is permitted by Theorem I). Now, rearranging the various terms we end with a new q-analysis which is of the type {2, 2(1), 2} and this proves our assertion. However, this is nothing more than a special S^+-solution written in a different form. Thus the possibility of new series of solution for $p=2$ and $q>1$ is still open and this is indicated by the black region of table I. In the same way the shaded area indicates the range of our ignorance as far as the values of q_{max} are concerned.

Among the extensions presented here, the extension of the Schwarzschild solution has some interest. According to Theorem 1 from $n=4$ to $n=5$ the extension is unique[16] but this additional dimension, however curved it may be, leads to an unphysical Newtonian limit. This is derived from equation (19b) for $R=0$, where now $a=-2m$. In fact, the Newtonian potential derived from this expression behaves like r^{3-n}. For $n\geq 5$ this behavior is quite unphysical as we have already mentioned. If $n\geq 6$ the addition of any *spatial* dimension has the same effect, destroying the Newtonian limit of the theory. Now however, the extension is not any more unique and our previous argument may not be valid any more.

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Appendix

We present the first member of the V2-series (I.e. $p=1$, $q_{max}=2$ and $n=9$). This particular solution is of the form

(A1) $$ds^2 = \varepsilon dx^2 + \sigma_1^2 ds_1^2 + s_2^2 ds_2^2$$

where $\varepsilon=\pm 1$ and x is a general variable that may be an angle, a cartesian coordinate or even time. Here, σ_1 and σ_2 are the solutions of (3a) subject to the condition (3b). In our case ($p=1$) equation (3a) is reduced to

(A2) $$\ddot{\sigma} + \varepsilon K \sigma = 0$$

There are two distinct cases for the solutions of (A2) namely,
I) $K=+\varepsilon\lambda^2$, $x=\theta$, $\sigma_1=\cos(\lambda\theta)$, $\sigma_2=\sin(\lambda\theta)$, $k_1=k_2=+\varepsilon\lambda^2$
II) $K = -\varepsilon\lambda^2$, $x=u$, $\sigma_1=\mathrm{ch}(\lambda u)$, $\sigma_2=\mathrm{sh}(\lambda u)$, $k_1=-\varepsilon\lambda^2$, $k_2=\varepsilon\lambda^2$,
where k_1 and k_2 are the constants that appear in equation (9a). The appended metrics are of the general form

(A3) $$ds_\alpha^2 = g^{(\alpha)}_{i_\alpha j_\alpha} dx^{i_\alpha} dx^{j_\alpha}, \ (\alpha=1,2), \ (i_\alpha, j_\alpha = 1,\dots,4).$$

Now, according to Theorem 1, *both* the appended metrics should be non-trivial Einstein spaces, otherwise the solution (A1) degenerates either to one of the solutions prescribed by De Vries or to a space of constant curvature. The cosmological constants of the aforementioned metrics are given by equation (9b). Now, the minimal choice, as indicated by (A3), is $n_1=n_2=4$, which introduces $n=9$ as a threshold dimension. It is worth noticing that this is *not* a unique construction, because of the arbitrariness implicit in the choice of the appended metrics. This particular feature is characteristic of the V2-series.

Table I

C.C	Kernel [p]	Length [q]	q_{max}	Type/Series	Comments
R	*1*	*1*	*2*	**V1**	Brinkmann Solutions
R	*1*	*2*	*2*	**V2**	New Solutions
0	*1*	≥*2*	∞	**K1, K2**	**Extension of Kasner's** Solutions
0	*1*	≥*3*	∞	**K3**	**Extension of Kasner's** General Solution
R	*1*	≥*2*	∞	**P1, P2**	**Extension of Petrov's** Solutions
R	*1*	≥*3*	∞	**P3**	**Extension of Petrov's** Solutions
R	*2*	*1*	*1*	**S^+** **S^-, S^0**	**Extension of the Type D** Static Solutions
R	*2*	≥*2*	≥*2*	[redacted]	
R	≥*2*			**V1**	**De Vries Solutions**
R	≥*2*			**V2**	**New Solutions**
0	*2r*	≥*2*	∞	**F**	**New Solutions**
0	≥*4*	$q \leq q_{max}$	[p/2]	**N**	**New Solutions**

List of the various solutions

ΠΕΡΙΛΗΨΗ

Ἡμιαναγώγιμοι χῶροι Einstein μὲ $n \geq 4$ διαστάσεις

Οἱ ἰδιότητες ἡμιαναγωγιμότητας τῶν χώρων Einstein ἐξετάζονται συστηματικὰ γιὰ κάθε διάσταση $n \geq 4$ καὶ γιὰ ὁποιοδήποτε εἶδος signature. Εἰδικότερα εἰσάγονται οἱ ἐξισώσεις ἡμιαναγωγιμότητας καὶ στὴ συνέχεια ἐπιλύονται σὲ διάφορες εἰδικὲς περιπτώσεις. Τὰ ἀποτελέσματα ποὺ προκύπτουν παρουσιάζονται σὰν ἀκολουθίες λύσεων. Ἀποδεικνύεται ὅτι ὅλες οἱ λύσεις μποροῦν νὰ χωρισθοῦν σὲ δύο διαφορετικὲς κατηγορίες, ἡ κάθε μία ἀπὸ τὶς ὁποῖες ἀποτελεῖ ἕνα διαφορετικὸ «ὑπόδειγμα συμπεριφορᾶς». Τὰ οὐσιώδη χαρα-

κτηριστικὰ αὐτῶν τῶν δύο κατηγοριῶν συζητοῦνται διεξοδικὰ καὶ παρουσιάζονται νέες λύσεις γιὰ $n \geq 5$. Ἐξετάζεται ἐπίσης καὶ τὸ Νευτώνειο ὅριο τῶν λύσεων ποὺ ἀνήκουν στὴν S^+ ὑποακολουθία λύσεων.

Λαβὼν τὸν λόγον ὁ Ἀκαδημαϊκὸς κ. **Γεώργιος Κοντόπουλος**, εἶπε τὰ ἑξῆς:

Οἱ χῶροι τοὺς ὁποίους μελετᾶ ὁ κ. Κωνσταντόπουλος εἶναι γενικεύσεις γνωστῶν χώρων, ὅπως τοῦ χώρου Minkowski τῆς Εἰδικῆς Σχετικότητος, ὅπου τὸ στοιχειῶδες μῆκος ds δίνεται ἀπὸ τὸν τύπο

$$ds^2 = dx^2+dy^2+dz^2-c^2dt^2 \tag{1}$$

Ἐδῶ dx, dy, dz εἶναι μικρὲς ἀποστάσεις τοῦ συνήθους χώρου 3 διαστάσεων καὶ dt εἶναι μικρὸ διάστημα χρόνου (4η διάσταση). Ὁ τύπος ποὺ δίνει τὸ στοιχειῶδες μῆκος ds^2 εἶναι γενίκευση σὲ 4 διαστάσεις τοῦ Πυθαγορείου θεωρήματος καὶ ὀνομάζεται «μετρικὴ» τοῦ χώρου.

Στὴν Γενικὴ Σχετικότητα ἡ μετρικὴ γίνεται

$$ds^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} g_{ij} dx_i dx_j \tag{2}$$

ὅπου g_{ij} εἶναι συναρτήσεις τῶν x_1, x_2, x_3, x_4 καὶ ἀποτελοῦν τὸ «μετρικὸ τανυστή». Οἱ g_{ij} εἶναι λύσεις τῶν διαφορικῶν ἐξισώσεων τοῦ Einstein

$$G_{ij}+\lambda g_{ij}= \varkappa T_{ij} \tag{3}$$

ὅπου ὁ G_{ij} εἶναι ὁ τανυστὴς τοῦ Einstein, ποὺ περιέχει τὰ g_{ij} καὶ πρῶτες καὶ δεύτερες παραγώγους τῶν g_{ij}. Ἡ ποσότης λ καλεῖται «κοσμολογικὴ σταθερὰ» καὶ τὸ T_{ij} ἀποτελεῖ τὸν τανυστὴ «ἐνεργείας-ὁρμῆς» καὶ χαρακτηρίζει τὴν κατανομὴ τῆς ὕλης καὶ ἐνεργείας στὸ χῶρο. Ἡ ϰ εἶναι μιὰ σταθερὰ ἀναλογίας.

Ἂν $T_{ij}=0$ παντοῦ, ἐκτὸς ἀπὸ ἕνα σημεῖο, μία λύση τῆς ἐξισώσεως (3) (μὲ λ=0) εἶναι ἡ λύση τοῦ Schwarzschild

$$ds^2 = \frac{dr^2}{1-\frac{a}{r}} + r^2 (d\theta^2+\sin^2\theta \, d\varphi^2) - \left(1-\frac{a}{r}\right) dt^2 \tag{4}$$

ποὺ παριστάνει σὲ πολικὲς συντεταγμένες μιὰ «μελανὴ ὀπή». Ἡ ποσότης a εἶναι ἡ ἀκτίνα τῆς μελανῆς ὀπῆς.

Ὁ κ. Κωνσταντόπουλος, χρησιμοποιώντας κριτήρια ἡμιαναγωγιμότητας, βρῆκε δύο ἀκόμη λύσεις τῆς ἐξισώσεως (3) μὲ $T_{ij}=0$:

$$ds^2 = \frac{dr^2}{\frac{a}{r}-1} + r^2(d\theta^2+\sinh^2\theta \, d\varphi^2) - \left(\frac{a}{r}-1\right) dt^2 \tag{5}$$

καὶ
$$ds^2 = r dr^2 + r^2 (d\theta^2 + d\varphi^2) - \frac{1}{r} dt^2 \tag{6}$$

Ὅμως ἀργότερα διαπίστωσε ὅτι οἱ λύσεις αὐτὲς ἦσαν ἤδη γνωστές. Ἔτσι προχώρησε σὲ λύσεις μὲ περισσότερες διαστάσεις. Οἱ χῶροι (δηλαδὴ οἱ τύποι ποὺ δίνουν τὸ ds^2) ποὺ προκύπτουν ἀπὸ τὴν ἐξίσωση (3) ὅταν $T_{ij} = 0$, λέγονται «χῶροι Einstein».

Οἱ χῶροι Einstein τοὺς ὁποίους ἐξετάζει ὁ κ. Κωνσταντόπουλος εἶναι τῆς μορφῆς

$$ds^2 = ds_0^2 + \sigma_1^2 ds_1^2 + \ldots + \sigma_q^2 ds_q^2 \tag{7}$$

ὅπου ds_0^2 εἶναι τὸ «θεμελιῶδες τμῆμα τῆς μετρικῆς» καὶ εἶναι p διαστάσεων, ἐνῶ τὰ $ds_1^2 \ldots ds_q^2$ εἶναι «προσαρτημένες μετρικὲς» διαστάσεων $n_1, \ldots n_q$. Ὅταν οἱ συναρτήσεις $\sigma_1 \ldots \sigma_q$ ἐξαρτῶνται μόνο ἀπὸ τὶς p συντεταγμένες τοῦ θεμελιώδους τμήματος τῆς μετρικῆς, τότε ὁ χῶρος (7) λέγεται «ἡμιαναγώγιμος» ἐνῶ ὅταν $\sigma_1 \ldots \sigma_q$ εἶναι σταθερὲς ὁ χῶρος αὐτὸς λέγεται «ἀναγώγιμος».

Σὰν παράδειγμα ὁ χῶρος Schwarzschild εἶναι ἡμιαναγώγιμος χῶρος μὲ θεμελιῶδες τμῆμα

$$ds_0^2 = \frac{dr^2}{1 - \frac{a}{r}} \tag{8}$$

μὲ p=1 (δηλαδὴ μία μόνο θεμελιώδη μεταβλητή, τὴν r), ἐνῶ οἱ χῶροι

$$ds_1^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 \quad \text{καὶ} \quad ds_2^2 = dt^2 \tag{9}$$

εἶναι προσαρτημένοι χῶροι διαστάσεων $n_1=2$ καὶ $n_2=1$ ἀντιστοίχως. Ἐδῶ εἶναι

$$\sigma_1^2 = r^2 \, , \; \sigma_2^2 = -\left(1 - \frac{a}{r}\right) \tag{10}$$

δηλαδὴ τὰ σ_1, σ_2 εἶναι συναρτήσεις μόνον τῆς βασικῆς μεταβλητῆς r.

Ὁ κ. Κωνσταντόπουλος βρῆκε πολλὲς νέες ἡμιαναγώγιμες λύσεις τῶν ἐξισώσεων Einstein, ποὺ εἴτε ἀποτελοῦν γενικεύσεις γνωστῶν λύσεων (π.χ. τῶν λύσεων Kasner, Petrov κλπ.) ἢ εἶναι ἐντελῶς νέες λύσεις.

Στὸ τέλος τῆς ἐργασίας του κάνει μιὰ συστηματικὴ ταξινόμηση ὅλων τῶν λύσεων, γνωστῶν καὶ νέων.

Οἱ λύσεις τοῦ κ. Κωνσταντόπουλου δὲν ἔχουν ἄμεση ἐφαρμογὴ σὲ φυσικὰ προβλήματα, δὲν ἀποκλείεται ὅμως νὰ βρεθοῦν τέτοιες ἐφαρμογὲς εἰς τὸ μέλλον καὶ μάλιστα στὴν θεωρία τῶν ὑπερχορδῶν. Εἰδικότερα ὑπάρχουν ἐνδείξεις ὅτι ὡρισμένες λύσεις τοῦ κ. Κωνσταντόπουλου εἶναι χῶροι Cartan, ποὺ εἶναι σημαντικοὶ στὴν θεωρητικὴ Φυσική. Κατὰ συνέπεια μιὰ λεπτομερέστερη μελέτη τῶν λύσεων ποὺ ἀναφέραμε θὰ ἦταν ἰδιαίτερα χρήσιμη.