ΔΙΑΦΟΡΙΚΗ ΓΕΩΜΕΤΡΙΑ.— **On homothetic mappings of Riemann spaces,** *by P. Bozonis and Th. Hasanis**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ὄθ. Πυλαρινοῦ.

The purpose of this note is to generalise some results concerning the homothetic mappings of Riemann spaces obtained in [2] and to give some others related mainly to the same mappings.

1. Let M be a Riemannian manifold with fundamental metric $g_{ij}(p)$. Let X be a vector field on M and L the symbol of Lie derivative with respect to X. It is known ([3], [4]) that X defines a motion, an homothetic transformation, a conformal transformation or an affine collineation if

$$Lg_{ij} = 0, \qquad Lg_{ij} = 2cg_{ij},$$
$$Lg_{ij} = 2pg_{ij}, \qquad L\Gamma^i_{jk} = 0$$

respectively, where c is a constant function, p a function on M and Γ^i_{jk} are Christoffel symbols. When c vanishes, an homothetic transformation reduces to a motion. Thus we call an homothetic transformation proper if $c \neq 0$.

From the formulas (cf [3]).

$$L\Gamma^i_{jk} = \frac{1}{2} g^{il} \left\{(Lg_{jl})_{;k} + (Lg_{lk})_{;k} - (Lg_{jk})_{;l}\right\}$$
$$(L\Gamma^i_{jk})_{;m} - (L\Gamma^i_{jm})_{;k} = LR^i_{jkm}$$

it is easily seen that a motion and an homothetic transformation are both affine collineations and that an affine collineation preserves the curvature tensor.

2. Let X be an homothetic transformation on M. Then it is an affine collineation and hence it preserves the curvature tensor. Moreover, X preserves the Ricci tensor, that is

$$LR_{ij} = 0. \tag{2.1}$$

From

$$R = g^{ij} R_{ij},$$

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where R is the scalar curvature of M, we have

$$\begin{aligned} LR &= Lg^{ij} R_{ij} + g^{ij} LR_{ij} \\ &= Lg^{ij} R_{ij} \\ &= -2cg^{ij} R_{ij} \\ &= -2cR \end{aligned}$$

or

$$LR = -2cR. \qquad (2.2)$$

From (2.2) and $LR = XR$ we have that R is an eigenfunction of X with corresponding eigenvalue $-2c$.

Then from a well known result it holds (cf [**1**], proposition 8.5.2)

$$R(\gamma(t)) = R(\gamma(0))\, e^{-2ct}, \qquad (2.3)$$

where $\gamma : R \rightarrow M$ is an integral curve of X.

If $c = 0$ we get from (2.3) that R is constant on the range of every integral curve of X.

So we obtain the following

Proposition 2.1 If the vector field X on M is a motion, then the scalar curvature of M is constant on the range of every integral curve of X.

In the case where the manifold is compact, then all eigenvalues of X are zero (cf [**1**], proposition 8.5.3) and so from (2.2) we have that $c = 0$. So every homothetic transformation on a compact manifold is a motion.

Let M be a Riemannian manifold and X a vector field on M which is an homothetic transformation. If the scalar curvature of M is a non zero constant on the range of an integral curve, then from (2.2) we have that $c = 0$. Thus we obtain the following

Theorem 2.1 Let M be a Riemannian manifold, which admits an homothetic transformation X. If the scalar curvature of M is non zero constant on the range of an integral curve of X, then X is a motion.

This generalizes Theorem 2 of [**2**].

Let X be a proper homothetic transformation (that is $c \neq 0$). If the scalar curvature R of M is constant on the range of an integral curve of X, then $R = 0$ on the range of this integral curve. Moreover, if R is constant on the range of every integral curve of X, then the manifold is of constant zero curvature.

So we obtain the following

2. Theorem 2.2 *Let M be a Riemannian manifold which admits a proper homothetic transformation X, such that the scalar curvature is constant on the range of every integral curve of X, then M is of constant zero scalar curvature.*

This generalizes theorem 3 of [**2**].

Now, we show the following

Proposition 2.2 *If M is a Riemannian manifold (of dimension greater than 1) and X an homothetic transformation on M with a dense integral curve, then M is a space of constant curvature and X is a motion.*

Proof. By a well known result (cf [**1**], proposition 8.5.5) we have that all eigenfunctions of X are constant and all eigenvalues are zero. Thus M is of constant curvature and X is a motion.

The above proposition gives the following

Corollary 2.1 *Let M be a Riemannian manifold. If M admits a proper homothetic transformation X, then there exists no dense integral curve of X.*

Since an homothetic transformation on a compact Riemannian manifold is a motion, we obtain the following

Corollary 2.2 *Let M be a compact Riemannian manifold of non constant curvature. Then every motion on M has no dense integral curves.*

3. Let W be a tensor field with components given by

$$w^i_{jkl} = R^i_{jkl} - \frac{1}{n-2}\left(R_{jk}\delta^i_l - R_{jl}\delta^i_k + g_{jk}R^i_j - g_{jl}R^i_k\right) + \frac{1}{(n-1)(n-2)}\left(g_{jk}\delta^i_l - g_{jl}\delta^i_k\right)$$

where R^{l}_{jkl}, R_{jk} and g_{jk} are the components of the curvature tensor, Ricci tensor and metric tensor respectively. The tensor field W is called Weyl's tensor or conformal tensor curvature, since it is invariant under any conformal transformation of the metric.

R e m a r k. We shall denote by Ω the length of the tensor W.

Then it is well known the following (cf [**5**])

T h e o r e m. A necessary and sufficient condition for a Riemannian manifold of dimension greater than 3 to be conformally that is $W = 0$.

It is known that a Riemannian manifold (M, g) is called conformally flat if there exists a metric $\bar{g}$ conformal to g such that the manifold (M, $\bar{g}$) is locally Euclidean.

Now, we prove the following

P r o p o s i t i o n 3. 1 Let M be a Riemannian manifold. If the length of the tensor W is non-zero constant, then every infinitesimal conformal transformation X on M is a motion.

P r o o f. Let X be a infinitesimal conformal transformation on M, then we have that

$$Lg_{ij} = 2pg_{ij}. \tag{3.1}$$

From

$$\Omega^2 = W^{i}_{jkl} W_{i}^{jkl} \tag{3.2}$$

we obtain that

$$2\Omega L\Omega = -2p\Omega^2$$

or

$$2\Omega (L\Omega + p\Omega) = 0 \tag{3.3}$$

or

$$L\Omega + p\Omega = 0. \tag{3.4}$$

Since $L\Omega = X\Omega$ and Ω is non-zero constant we have that $p = 0$. This proves the proposition.

Π Ε Ρ Ι Λ Η Ψ Ι Σ

Ἓν διανυσματικὸν πεδίον X ἐπὶ ἑνὸς χώρου Riemann M ὁρίζει, ὡς γνωστόν, ἕνα μετασχηματισμὸν τοῦ χώρου τούτου. Τὸ πεδίον X, ὅταν ὁ ὑπ' αὐτοῦ ὁριζόμενος μετασχηματισμὸς τοῦ χώρου M εἶναι ἰσομετρικός, ὁμοθετικός, σύμμορφος ἢ ὁμοπαραλληλικός, λέγεται ἰσομετρικόν, ὁμοθετικόν, σύμμορφον ἢ ὁμοπαραλληλικὸν ἀντιστοίχως, ἐὰν δὲ L εἶναι τὸ σύμβολον τῆς παραγωγίσεως Lie ὡς πρὸς τὸ πεδίον X, τὸ πεδίον τοῦτο εἶναι ἰσομετρικόν, ὁμοθετικόν, σύμμορφον ἢ ὁμοπαραλληλικὸν καθ' ὅσον ἀντιστοίχως εἶναι:

$$Lg_{ij} = 0, \quad Lg_{ij} = 2cg_{ij}, \quad Lg_{ij} = 2pg_{ij}, \quad L\Gamma^{i}_{jk} = 0,$$

ἔνθα εἶναι (g_{ij}) ὁ θεμελιώδης μετρικὸς τανυστὴς τοῦ χώρου M, Γ^{i}_{jk} τὰ ἀντίστοιχα σύμβολα τοῦ Christoffel, c σταθερὰ καὶ p συνάρτησις ὡρισμένη ἐπὶ τοῦ χώρου M. Εἰς τὴν περίπτωσιν καθ' ἣν τὸ X εἶναι ὁμοθετικὸν διανυσματικὸν πεδίον, λέγεται τοῦτο γνήσιον ὁμοθετικὸν διανυσματικὸν πεδίον, ὅταν ἡ εἰς τὰς πρὸς τοῦτο πληρωτέας συνθήκας ὑπεισερχομένη σταθερὰ c εἶναι $\neq 0$.

Εἰς τὴν ἀνωτέρω ἐργασίαν, ἀναφερομένην εἰς τοὺς ὁμοθετικοὺς κυρίως μετασχηματισμοὺς τῶν χώρων Riemann, ἀποδεικνύονται ἐν πρώτοις τὰ θεωρήματα:

1. Ἐὰν κατὰ μῆκος μιᾶς ὁλοκληρωτικῆς καμπύλης ὁμοθετικοῦ διανυσματικοῦ πεδίου X ἐπὶ ἑνὸς χώρου Riemann M ἡ ἀριθμητικὴ καμπυλότης τοῦ χώρου M εἶναι σταθερά, τὸ X εἶναι κατ' ἀνάγκην ἰσομετρικὸν διανυσματικὸν πεδίον.

2. Ἐὰν ἡ ἀριθμητικὴ καμπυλότης ἑνὸς χώρου Riemann M εἶναι σταθερὰ κατὰ μῆκος οἱασδήποτε ὁλοκληρωτικῆς καμπύλης γνησίου ὁμοθετικοῦ διανυσματικοῦ πεδίου X ἐπὶ τοῦ χώρου M, ὁ χῶρος M εἶναι κατ' ἀνάγκην σταθερᾶς μηδενικῆς ἀριθμητικῆς καμπυλότητος.

Τὰ δύο ταῦτα θεωρήματα εἶναι γενικεύσεις γνωστῶν θεωρημάτων, δειχθέντων ὅμως ὑπὸ τὴν προϋπόθεσιν ὅτι ὁ χῶρος Riemann, εἰς τὸν ὁποῖον ἀναφέρονται, εἶναι σταθερᾶς ἀριθμητικῆς καμπυλότητος.

Ἐν συνεχείᾳ δίδονται βοηθητικαί τινες προτάσεις διὰ τῶν ὁποίων ἀποδεικνύονται τὰ θεωρήματα:

3. Ἡ ἀριθμητικὴ καμπυλότης ἑνὸς χώρου Riemann δεχομένου ὁμοθετικὸν διανυσματικὸν πεδίον X ἔχον μίαν ὁλοκληρωτικὴν καμπύλην πυκνὴν εἶναι κατ' ἀνάγκην σταθερὰ τὸ δὲ X εἶναι κατ' ἀνάγκην ἰσομετρικὸν διανυσματικὸν πεδίον.

4. Ἓν ὁμοθετικὸν διανυσματικὸν πεδίον ἐπὶ ἑνὸς χώρου Riemann, ἐὰν εἶναι γνήσιον, δὲν δύναται νὰ ἔχῃ ὁλοκληρωτικὴν καμπύλην πυκνήν.

Ἐν τέλει θεωροῦνται χῶροι Riemann M εἰς τοὺς ὁποίους ὁ σύμμορφος τανυστὴς καμπυλότητος (τανυστὴς τοῦ Weyl) εἶναι μέτρου μὴ μηδενικοῦ καὶ ἀποδεικνύεται τὸ θεώρημα:

5. Ἐὰν εἰς χῶρον Riemann M τὸ μέτρον τοῦ τανυστοῦ Weyl εἶναι σταθερὸν $\neq 0$, πᾶν σύμμορφον διανυσματικὸν πεδίον ἐπὶ τοῦ M εἶναι κατ' ἀνάγκην ἰσομετρικόν.

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