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ΜΗΧΑΝΙΚΗ.– **Failure Criteria for Engineering Materials Based on Anisotropic Hardening**, *by P.S. Theocaris**.

ABSTRACT

A review of the existing failure criteria was undertaken in this paper. A comparison between criteria based in quadrics, that is the conical and circular-paraboloid failure loci, was presented and the advantages and disadvantages of these criteria were outlined. The cylindrical failure locus comes out as a special case of both these criteria.

It was shown that the Mises-Hencky criterion, represented by a cylindrical failure locus, implies a compulsory condition for equal behaviour of the material in tension and compression. This constraint makes the Mises criterion only of a limited value suitable for highly ductile materials.

More important criteria are those based on a conical failure surface, which was introduced by Coulomb. This criterion yields excellent results in the compression-compression zone and it is most convenient for brittle materials because of its simplicity.

The difficulties encountered by the Coulomb-Mohr criterion to confront the failure behaviour, especially in the tension-tension zone of all materials, disappear as soon as the failure criterion, based on an circular-paraboloid locus, is used. Since this criterion is based on a two-parameter expression, it is a versatile one and presents a good agreement with the results derived from any previous experimental evidence.

It was also shown that this criterion is in complete agreement with newly established criteria, depending on the deviatoric and dilatational components of stresses which undergo rather moderate deformations and which are based on the theory of void formation.

All these criteria are convenient for initially isotropic materials and they are repre-

* Π. Σ. ΘΕΟΧΑΡΗ, **Τὰ κριτήρια ἀστοχίας καὶ θραύσεως δι' ὑλικὰ μὲ ἀνισότροπον κράτυνσιν.**

sented by surfaces of revolution about the space diagonal in the principal stress space. If one admits a rotation of their axis of symmetry about the origin, a translation of their initial position and some distortion of their shape he may encounter a really universal failure criterion, where anisotropy of the material could be also incorporated. These criteria constitute the so-called *tensorial failure criteria.*

Then, the rotating elliptic-paraboloid type of criterion may be considered as a *generalized failure criterion* for all engineering materials.

INTRODUCTION

It was as early as 1904 that Huber [**1**] has introduced his yield or brittle-fracture criterion, where he distinguished two cases, when the hydrostatic component of stresses applied to the specimen was tensile, or compressive. Huber's criterion was based on the distortional component of the elastic energy, for compression, whereas for tension the criterion depended on the total elastic energy.

Von Mises [**2**], and independently Schleicher [**3**], have introduced afterwards the notion of the *equivalent critical yielding stress,* instead of that of simple shear k, as an arbitrary function of the hydrostatic component of stresses. The criterion was convenient for materials, whose yielding depended on hydrostatic tension or compression, and therefore they presented different critical values for yielding under the different modes of loading.

A similar criterion than the Mises criterion was previously introduced by Tresca [**4**] in 1864 which was based on the maximum shear stress and proved to be convenient for mild steels.

Although from the early tests of failure of materials it has been realized that the yield stress in simple tension never coincided with the yield stress in simple compression, it was assumed, at least for the ductile metals, where this difference was not so important, that a complete symmetry of the yield locus exists in the tension and compression spaces. Than, Tresca's and Mises' yield conditions were accepted as describing universally the plastic behaviour of ductile substances [**5**].

In brittle materials, where the ratio of the yield stress in simple compression σ_{oc} was always much different to the yield stress in simple tension σ_{ot}, it was accepted that a *Mohr-Coulomb* type of yield locus described the plastic behaviour of these substances [**6**, **7**]. Although the *Mohr-Coulomb,* or *internal friction,* criterion fitted satisfactorily the results for nonmetallic geological materials, the agreement established between experimental results and the theory may be mainly attributed to the large scatter of experimental results and the lack of such results in the third and most critical quadrant of the principal stress space.

Multiaxial failure criteria were historically developed to characterize the biaxial failure of materials. They represent the maximum stress, maximum shear, maximum strain, maximum strain energy, and distortion energy theories of failure, and may be portrayed as failure loci in a principal stress plane. The simplest of the biaxial criteria represent polygonal failure loci. Generalization to three dimensions is straightforward, and leads, for the simpler criteria, to three-dimensional polygonal failure surfaces in the principal stress space [**38**].

Following Tschoegl [**8**], failure criteria will be presented in this review, which contain the three principal stresses in a symmetric mode. This restricts application of the criteria to isotropic bodies and refers the corresponding failure surfaces to the principal-stress space, as surfaces of revolution around the space diagonal. The space diagonal $\sigma_1=\sigma_2=\sigma_3$, has equal direction cosines $\xi_1=\xi_2=\xi_3=1/\sqrt{3}$ with the positive principal stress axes.

The requirement that the surface should be open in the compressive octant, because hydrostatic compression at moderate pressures cannot lead to failure, restricts the choice of surfaces. We shall consider only *quadrics*. This restriction leaves the cone and the circular paraboloid as the only failure surfaces with the cylinder as special case of both these surfaces. For the two parameters defining a *quadric surface* we can conveniently select the failure stress in simple tension, σ_{ot}, and in simple compression, σ_{oc}.

THE QUADRIC FAILURE SURFACES

i) The Conical failure locus: This failure criterion corresponds to the well-known Coulomb criterion [**6**, **8**]:

$$|\tau| + \mu\sigma = \tau_0 , \tag{1}$$

where σ and τ are the normal and shear stresses across the plane of failure, τ_0 is the failure stress, and μ is the so-called *coefficient of internal friction.*

The Coulomb criterion is usually applied to biaxial stressing and τ is then taken as the maximum shear stress. In a triaxial state of stress the simplest modification, which involves the three principal stresses symmetrically, is obtained by considering the τ-stress as the octahedral shear stress σ_{ns} and the σ-stress as the mean-normal stress σ_{nn}. The criterion can then be stated as

$$\sigma_{ns} + \mu\sigma_{nn} = \sigma_o, \tag{2}$$

where σ_0 is a material constant. Eq. (2) is also known as the Drucker criterion [**9**]. This criterion is frequently expressed in the form:

$$J_2^{1/2} + \beta I_1 = \sigma_y \tag{3}$$

$$\text{where } \beta = \mu/\sqrt{\sigma} \text{ and } \sigma y = \sqrt{3/2}\ \sigma_0.$$

The failure surface in principal stress space corresponding to this criterion is a cone, coaxial with the space diagonal, whose semi-angle, α, is given by $\tan\alpha = \mu$. The intersection of the cone with the $\sigma_1 = \sigma_3$ plane is shown in Fig. 1a. The apex of the cone has coordinates $\sigma_A = \sigma_0/\mu$ on the principal stress axes. The cone intersects the positive and negative axes at σ_{ot} and σ_{oc}, respectively, where σ_{oc} is the failure stress in simple compression and it is generally larger than σ_{ot}. For $\sigma_{oc} = \sigma_{ot}$ the cone reduces to a cylinder and Eq. (2) may therefore be regarded as a modification of the Huber-Mises criterion, introducing into it the mean normal stress, weighted by μ.

It is worthwhile noting that Eq. (2) represents a linear combination of the octahedral shear and normal stresses. Expressing μ and σ_0 in terms of the angle 2α and the coordinates $\sigma_A = \sigma_A = \sigma_A$ of the apex of cone in the principal space stress Eq. (2) takes the form:

$$\sigma_{ns}\cot\alpha + \sigma_{nn} = \sigma_A \tag{4}$$

Inserting Eqs. (2) and (4) we find

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - (\sigma_1 + \sigma_2 + \sigma_3 - 3\sigma_A)^2 \tan^2\alpha = 0.$$

This relation represents a cone coaxial with the space diagonal. It can be readily shown that the coordinates σ_A of the apex of the cone and its angle are expressed by:

$$\sigma_A = \frac{2}{3}\ \frac{\sigma_{oc}\sigma_{ot}}{\sigma_{oc} - \sigma_{ot}} \tag{5}$$

and

$$\tan\alpha=\sqrt{2}\,\frac{\sigma_{oc}-\sigma_{ot}}{\sigma_{oc}+\sigma_{ot}} \tag{6}$$

Then, introducing these relations into the previous one we derive for the failure criterion the expression:

$$\frac{1}{\sqrt{2}}(\sigma_{oc}+\sigma_{ot})\sigma_{ns}+(\sigma_{oc}-\sigma_{ot})\,\sigma_{nn}=\frac{2}{3}\sigma_{oc}\,\sigma_{ot} \tag{7}$$

which may be also written as follows:

$$\frac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}-\left\{3p\frac{\sigma_{oc}-\sigma_{ot}}{\sigma_{oc}+\sigma_{ot}}-\frac{2\sigma_{oc}\sigma_{ot}}{\sigma_{oc}+\sigma_{ot}}\right\}^2=0 \tag{8}$$

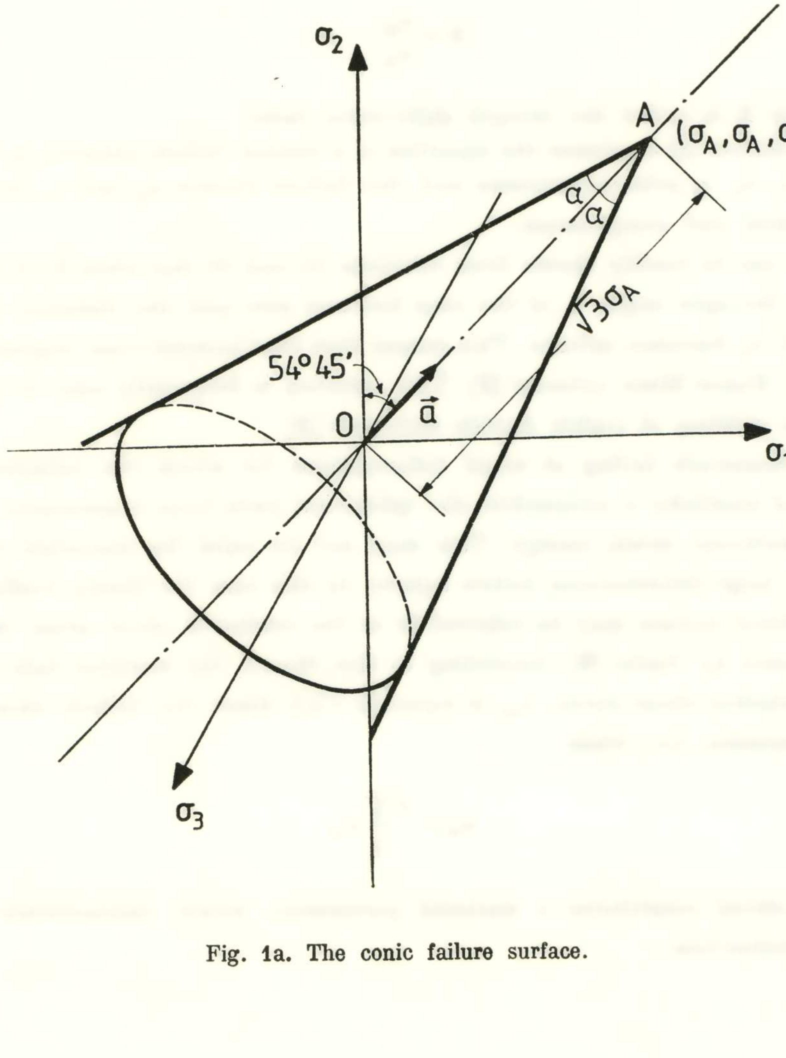

Fig. 1a. The conic failure surface.

where $p=\sigma_{nn}$ expresses the hydrostatic stress and it is equal to the octahedral normal stress σ_{nn} given by:

$$p = \sigma_{nn} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} \tag{9}$$

Similarly the octahedral shear stress, σ_{ns}, is expressed by [**5**]:

$$\sigma^2_{ns} = \frac{1}{9}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{10}$$

Moreover, the ratio of the yield or failure stresses in simple compression σ_{oc} and in simple tension σ_{ot} is expressed by:

$$R=\frac{\sigma_{oc}}{\sigma_{ot}} \tag{11}$$

Quantity R is called the *strength differential factor.*

Relation (8) expresses the equation of a conical failure criterion in terms of the σ_1, σ_2, σ_3 principal stresses and the failure stresses σ_{oc} and σ_{ot} for simple tension and compression.

It can be readily shown from relations (5) and (6) that when $R=1.0$ and $\sigma_{oc}=\sigma_{ot}$ the apex angle, α, of the cone becomes zero and the distance of the apex $\sqrt{3}\,\sigma_A$ becomes infinite. This means that the Coulomb-cone degenerates into the Huber-Mises cylinder [**2**]. This criterion is frequently used to represent the yielding of highly ductile materials [**3**].

In materials failing at small deformations for which the infinitesimal theory of elasticity is admissible, the *cylindrical* yield locus is associated with the distortional strain energy. This may not be valid for materials which undergo large deformations before failure. In this case the theory leading to a cylindrical surface may be referred to as the *octahedral shear stress theory,* as proposed by Nadai [**8**]. According to this theory, the material fails when the octahedral shear stress, σ_{ns}, is equal to $\sqrt{2/3}$ times the failure stress in simple tension, *i.e.,* when

$$\sigma_{ns}= \frac{\sqrt{2}}{3}\,\sigma_{ot} \tag{12}$$

The σ_{ot}-stress constitutes a material parameter, which characterizes the failure behaviour.

Introducing the octahedral shear stress, the Huber-Mises criterion may be stated as

$$(\sigma_1\text{-}\sigma_2)^2 + (\sigma_2\text{-}\sigma_3)^2 + (\sigma_3\text{-}\sigma_1)^2 = 2\sigma^2_{ot} \ , \tag{13}$$

Equation (13) represents a cylinder of radius $\sqrt{2/3}\ \sigma_{ot}$, coaxial with the stress-space diagonal. Figure 1b presents the intersection of the cylinder with the plane of symmetry containing the σ_2-axis and the space diagonal. The cylinder cuts the principal stress axes at $\pm\sigma_{ot}$ and the criterion assumes that the failure stress in uniaxial tension and compression are equal. Eq. (12) also expresses the fact that failure depends on the octahedral shear stress alone and the octahedral normal stress σ_{nn} is without influence.

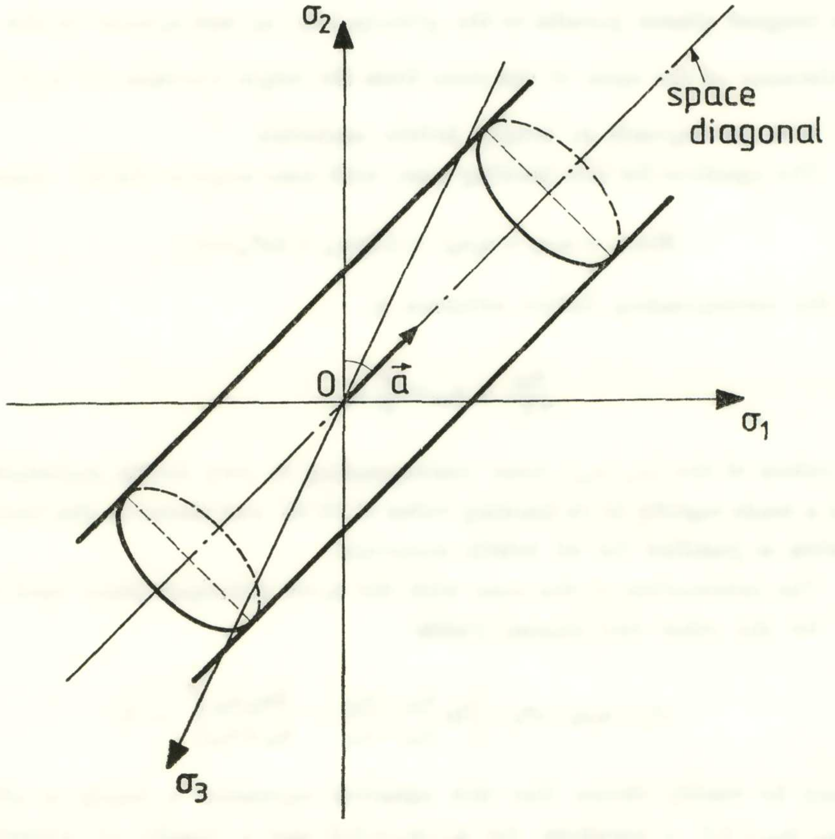

Fig. 1b. The cylindrical failure surface.

The intersection of the failure surface with anyone of the coordinate planes yields a failure curve representing, in the plane of the two principal stresses, the associated failure criterion in biaxial loading. The intersection of the cylinder with the $\sigma_3=0$ plane is an ellipse, whose major axis lies along the projection of the space diagonal on the $\sigma_3=0$ plane.

Moreover, when $\sigma_{oc}=3\sigma_{ot}$ and therefore $\sigma_A=\sigma_{ot}$, we derive that tanα $1/\sqrt{2}$, which yields $\alpha_{ot}=35°15'$. This angle α_0 is the complementary angle to the angle of the direction cosines of the space diagonal, defining the direction of the octahedral normal stress. For this critical angle the cone disposes three tangent planes parallel to the coordinate planes $\sigma_1=\sigma_2$, $\sigma_2=\sigma_3$ and $\sigma_3=\sigma_1$.

On the other hand, for $\frac{\sigma_{oc}}{\sigma_{ot}}\rightarrow\infty$ the apex angle becomes $\alpha=54°45'$ which is the complementary to the α_0 and, in this case, the cone disposes three tangent planes parallel to the principal σ_1-, σ_2- and σ_3-axes. In this case the distance of the apex of the cone from the origin becomes $\sqrt{3}\,\sigma_A=\frac{2\,\sigma_{ot}}{\sqrt{3}}$. This cone corresponds to totally brittle materials.

The equation for this limiting cone, with semi-angle $\alpha=54°45'$, becomes:

$$3(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) - 12p\sigma_{ot}+4\sigma^2_{ot}=0\ , \qquad (14)$$

and the corresponding failure criterion is

$$\frac{\sigma_{ns}}{\sqrt{2}}+\sigma_{nn}=\frac{2}{3}\,\sigma_{ot} \qquad (15)$$

For values of the $(\sigma_{oc}/\sigma_{ct})$ - ratio corresponding to very brittle materials the angle α tends rapidly to its limiting value of 54°45′ and therefore the Coulomb criterion is justified for all brittle materials.

The intersection of the cone with the $\sigma_3=0$ principal plane (and similarly for the other two planes) yields:

$$\sigma^2_1-\sigma_1\sigma_2+\sigma^2_2-\left\{3p\,\frac{\sigma_{oc}-\sigma_{ot}}{\sigma_{oc}+\sigma_{ot}}-\frac{2\sigma_{oc}\sigma_{ot}}{\sigma_{oc}+\sigma_{ot}}\right\}^2=0 \qquad (16)$$

It may be readily shown that this equation represents a family of ellipses for $\sigma_{oc}/\sigma_{ot}<3.0$, a parabola for $\sigma_{oc}/\sigma_{ot}=3.0$ and a family of hyperbolas for $\sigma_{oc}/\sigma_{ot}>3.0$.

The conic sections of this criterion for parametric values of the ratio σ_{oc}/σ_{ot} are shown in Fig. 2.

The limiting case for which $\sigma_{oc}/\sigma_{ot}=\infty$ yields a hyperbola, whose equation is given by:

$$\sigma_1\sigma_2-4p\sigma_{ot}+\frac{4}{3}\sigma_{ot}=0 \qquad (17)$$

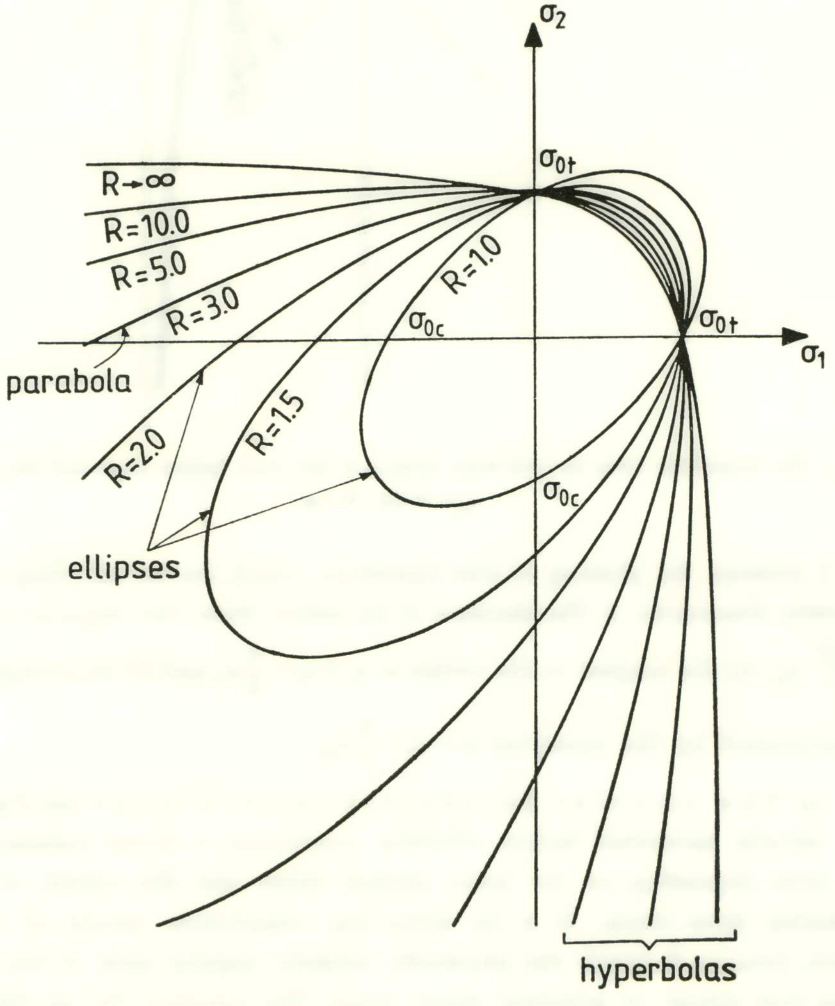

Fig. 2. Conic sections for the conic failure criterion for parametric values of R.

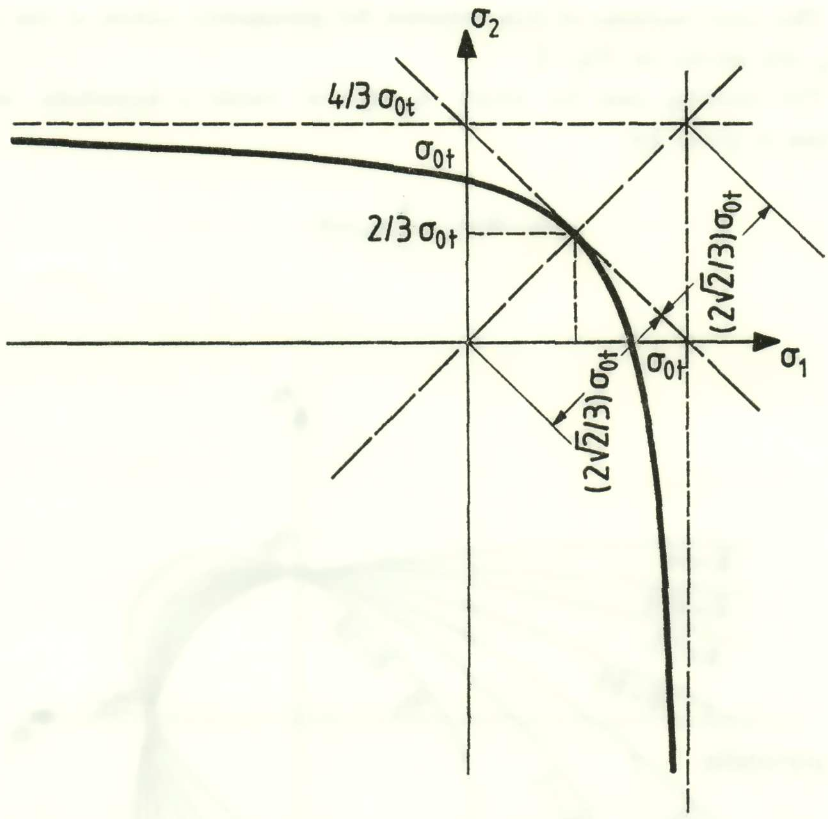

Fig. 3. The hyperbolic locus derived from section of the conic failure locus and the plane $\sigma_3 = 0$ for $R \to \infty$

Fig. 3 presents the plotting of this hyperbola, which has the following characteristic dimensions: i) The distance of its vertex from the origin is $\sqrt{3}\,\sigma_A = \frac{2\sqrt{2}}{3}\,\sigma_{ot}$, ii) the tangent to the vertex is $\sigma_1 + \sigma_2 = \frac{4}{3}\sigma_{ot}$ and iii) its asymptotes are expressed by the equations $\sigma_1 = \sigma_2 = \frac{4}{3}\sigma_{ot}$.

ii) T h e c i r c u l a r p a r a b o l o i d f a i l u r e l o c u s (see Fig. 4): The circular paraboloid failure criterion constitutes a linear combination of a term depending on the mean normal stress and the square of the octahedral shear stress. It is an attracting compromise among all other criteria because it avoids the physically unlikely angular apex of the cone in the first octant of principal stress space. The equation for an circular paraboloid, which is coaxial with the space diagonal is

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 12fd-4f\sqrt{3}\,(\sigma_1+\sigma_2+\sigma_3), \tag{18}$$

where f is the distance between the focus of the paraboloid and its vertex, and d is the distance of the vertex from the origin (see Fig. 5).

It may be readily shown from the properties of the parabolas that the distance f between any point of the parabola and its focus, which is equal to the distance of its vertex from its directrix, is expressed by:

$$f=\frac{\sigma_{oc}-\sigma_{ot}}{2\sqrt{3}} \tag{19}$$

whereas the distance d between the vertex from the origin of the coordinates in the stress space is given by:

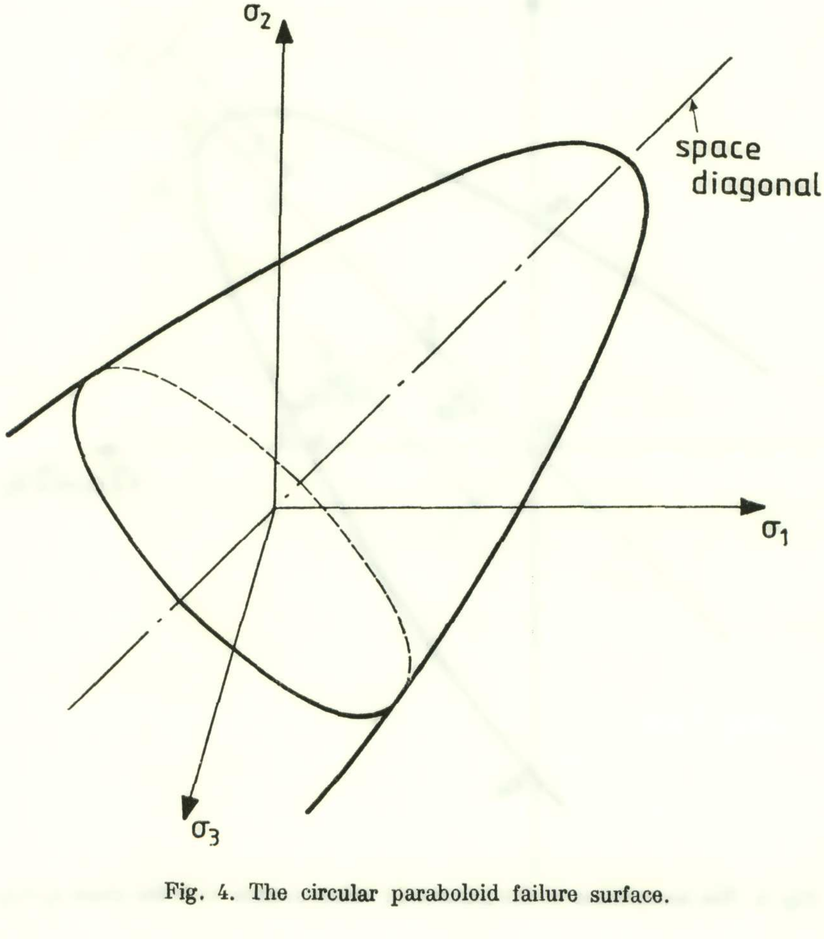

Fig. 4. The circular paraboloid failure surface.

$$d = \frac{\sigma_{oc}\sigma_{ot}}{\sqrt{3}\,(\sigma_{oc}-\sigma_{ot})} \tag{20}$$

Then, introducing these values into Eq. (18) we derive for the equation of the paraboloid the equation:

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2+6p(\sigma_{oc}-\sigma_{ot}) -2\sigma_{oc}\sigma_{ot} = 0 \tag{21}$$

whereas the corresponding failure criterion takes the form:

$$\sigma^2_{ns}+\frac{2}{3}(\sigma_{oc}-\sigma_{ot})\ \sigma_{nn} - \frac{2}{9}\ \sigma_{oc}\sigma_{ot} = 0 \tag{22}$$

This criterion in terms of the J_2 - and I_1 - invariants is expressed by:

$$3J_2+(\sigma_{oc}-\sigma_{ot})I_1 = \sigma_{oc}\sigma_{ot} \tag{23}$$

Fig. 5 presents the intersection of the paraboloid with the $\sigma_1=\sigma_3$ plane. For

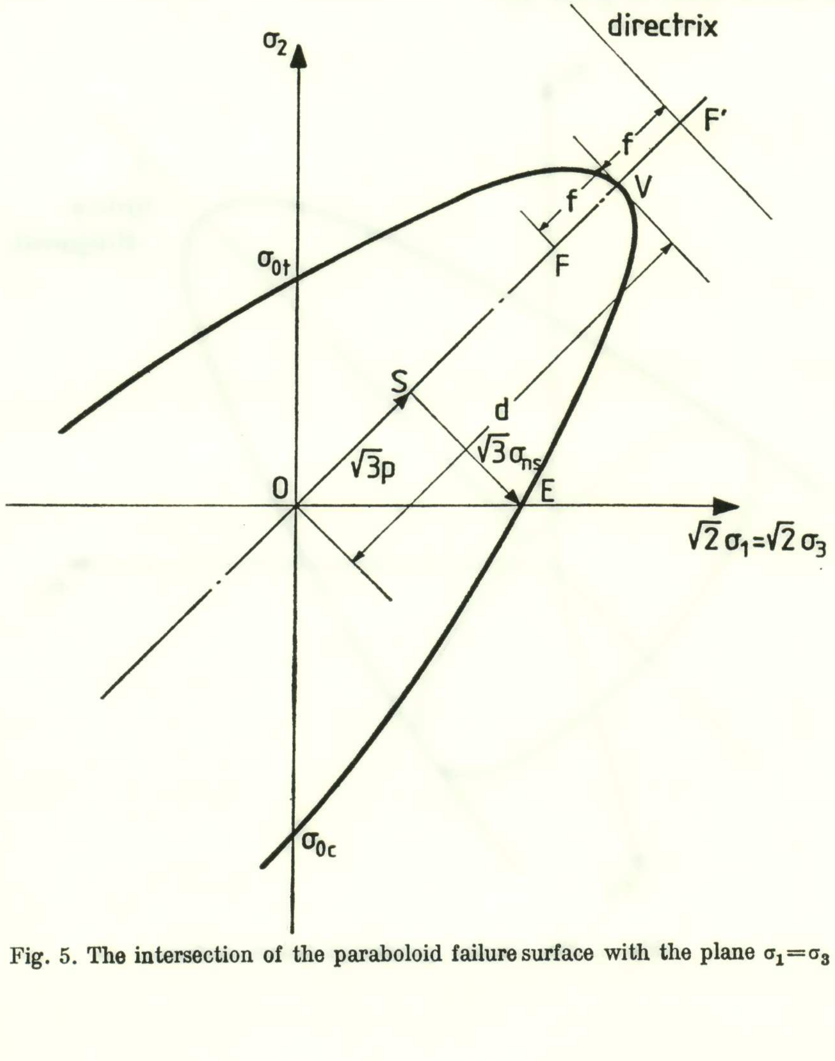

Fig. 5. The intersection of the paraboloid failure surface with the plane $\sigma_1=\sigma_3$

$\sigma_{oc}=\sigma_{ot}$ the focus is carried into infinity and the paraboloid, just as the cone, transforms into the cylinder.

On the other hand, the paraboloid degenerates into the principal octahedral plane for $\sigma_{oc}/\sigma_{ot}\to\infty$. Its equation becomes:

$$\sigma_1+\sigma_2+\sigma_3=\sigma_{ot} \quad , \tag{24}$$

and the failure criterion takes the form:

$$\sigma_{nn}=\frac{\sigma_{ot}}{3} \tag{25}$$

Relation (25) states that failure occurs when the stress normal to the octahedral plane is equal to one-third of the failure stress in uniaxial tension. In small strain theory the plane can be associated with the dilatational strain energy and is often referred to as *the dilatational plane.* When dealing with large deformations it is best referred to as the *principal octahedral plane.*

The intersection of the paraboloid with the $\sigma_3=0$ plane yields an ellipse whose equation is given by:

$$\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2+3p(\sigma_{oc}-\sigma_{ot})=\sigma_{oc}\sigma_{ot} \tag{26}$$

This ellipse degenerates into a straight line for $\sigma_{oc}/\sigma_{ot}\to\infty$. In this case we have:

$$(\sigma_1+\sigma_2)=\sigma_{ot} \tag{27}$$

Fig. 6 presents the intersections of the elliptic paraboloids with the $\sigma_3=0$ plane for parametric valus of the σ_{oc}/σ_{ot}-ratio varying between unity and infinity.

It is worthwhile now giving the distance $\sqrt{3}\,\sigma_F$ of the focus of the paraboloid from the origin. This quantity is given by:

$$\sqrt{3}\,\sigma_F=\frac{1}{2\sqrt{3}}\left\{\frac{4\sigma_{oc}\sigma_{ot}-\sigma^2_{oc}-\sigma^2_{ot}}{(\sigma_{oc}-\sigma_{ot})}\right\} \tag{28}$$

Finally, the following remarks concerning the applicabitity of these three types of criteria seems to be in order:

i) The Mises-Hencky failure criterion depends only on the octahedral shear

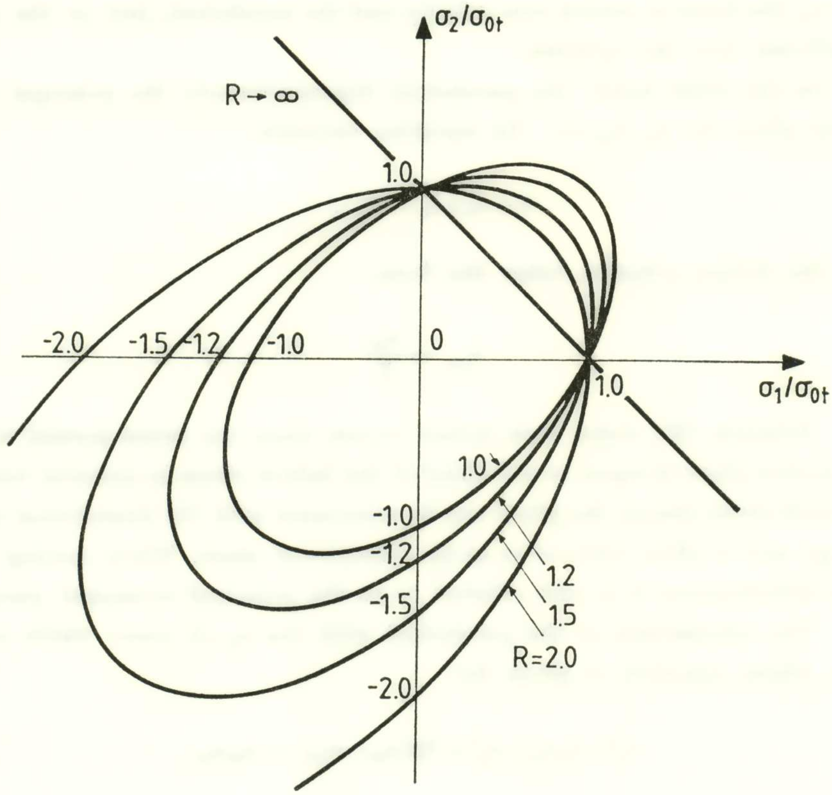

Fig. 6. Intersections of the circular paraboloid failure surface with the $\sigma_3=0$ plane for parametric values of R.

stress and it is represented by a cylindrical surface with axis the principal stress space diagonal. It is convenient only for materials which obey the equality $\sigma_{oc}=\sigma_{ot}$. Since none of the engineering materials follows this assumption this criterion is convenient only for monocrystals and very ductile materials.

ii) The other two criteria, the conic and the circular paraboloid, depend on both the octahedral shear and normal stresses. Therefore, they take into consideration not only the influence of the distortional part of the strain energy but also its dilatational part. Moreover, these criteria are better adapted to the real behavior of the engineering substances since they take care, in a rational way, of the existing differences in the failure characteristics of the materials when they are submitted either to simple tension, or to simple compression. Such differences cannot be encountered either by the Mises-Hencky or the Tresca types of yield or failure criteria.

iii) Comparing the Coulomb cone, expressed by relation (12), or its intersection with one principal stress plane expressed by relation (16) with the respective relationships for the circular paraboloid given by Eqs. (21) and (26), it becomes clear that the expressions for the paraboloid are much simpler than those of the Coulomb criterion. It constitutes *the simplest variant* of a general criterion, which describes with high accuracy an extremely wide class of experimental observations.

The elliptic paraboloid failure criterion avoids the physically unlikelihood to dispose the materials in their all equal tensile loading modes an apex in their failure locus. Based on theoretical considerations for the failure envelope in the Coulomb-Mohr theory it has been shown (see Ref. [**5**], chapt. XI. 3 pp. 294-300) that only for certain conditions, fulfilled in the compression zone of loading, there is a real contact between the failure stress circles and the failure envelope. Moreover, experimental evidence with brittle materials [**11**] indicated clearly that the angular apex of the cone in the Coulomb-Mohr criterion is rather physically improbable and the failure behavior of substances in this zone of loading fits better to a smooth curve resembling the neighbour zone of an circular paraboloid.

We shall see in the following that such a criterion based on an circular paraboloid surface fits very well with the existing experimental evidence with a variety of substances and explains satisfactorily their failure behaviour presenting a considerable versatility and adaptability to become a generalized failure criterion.

EXPERIMENTAL EVIDENCE FOR FAILURE CRITERIA OF THE CIRCULAR PARABOLOID TYPE

Extensive experimental evidence on metallic, polymeric and geological materials has indicated a clear dependence of the yield loci of these materials on their respective *strength differential factor* R. The very meticulous early experiments by Coffin [**12**] on gray cast-iron, a very brittle material, with $\sigma_{oc}=100\times10^3$psi and $\sigma_{ot}=33\times10^3$psi and Grassi and Cornet [**13**] with $\sigma_{oc}=96\times10^3$psi and $\sigma_{ot}=28.5\times10^3$psi gave, both of them, a value for R=3.0. Fig. 7 presents the yield locus of these materials, normalized to the yield stress σ_{0t} in simple tension. It is clear from this figure that all experimental results fit excellently the *circular paraboloid failure criterion.*

Fig. 7 presents a conical section of the circular paraboloid with a strength differential factor R=3.0 with the plane $\sigma_3/\sigma_{0t}=0$. In this principal plane $\sigma_1/\sigma_{0t}=\sigma_2/\sigma_{0t}=1.0$ whereas $-\sigma_1/\sigma_{0t}=-\sigma_2/\sigma_{0t}=-3.0$.

Although there are not sufficient data in the compression-compression quadrant, it is clear that the material follows such a form of criterion. If one considers, further, all eventual discarding of results, which were assumed as non-compatible with existing theories in that early time of the execution of the experiments, one may assumes that the results given in the literature should be taken as mandatory.

On the other hand, concerning the yielding mode of various ductile materials, the most famous experiments by Taylor and Quinney [**14**] indicate clearly that aluminium and copper with R approaching unity, obey satisfactorily the Mises yield criterion. However, mild-steel specimens deviate considerably with all the existing experimental points by various studies lying

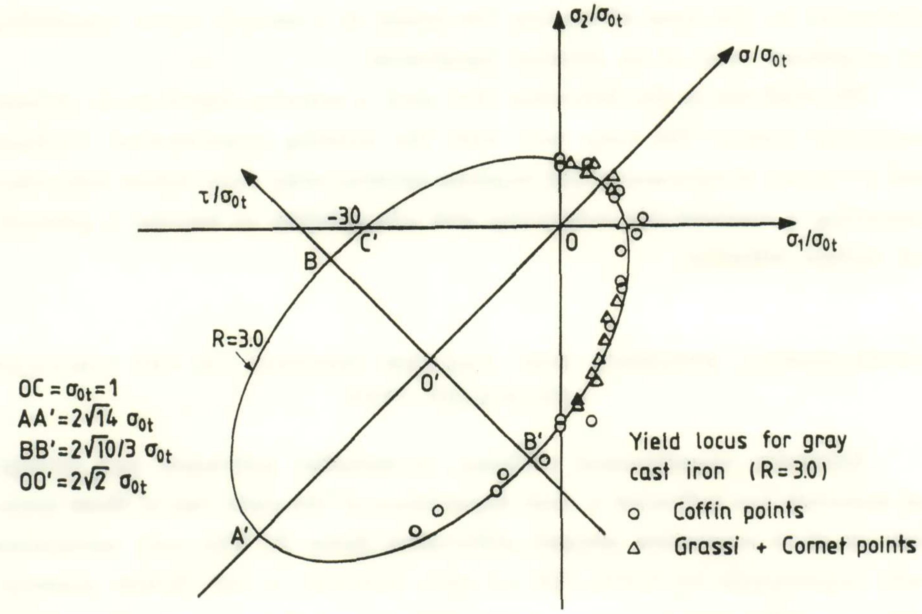

Fig. 7. The yield locus for gray cast-iron with R=3.0 and the experimental points derived from tests of Coffin and Grassi and Cornet.

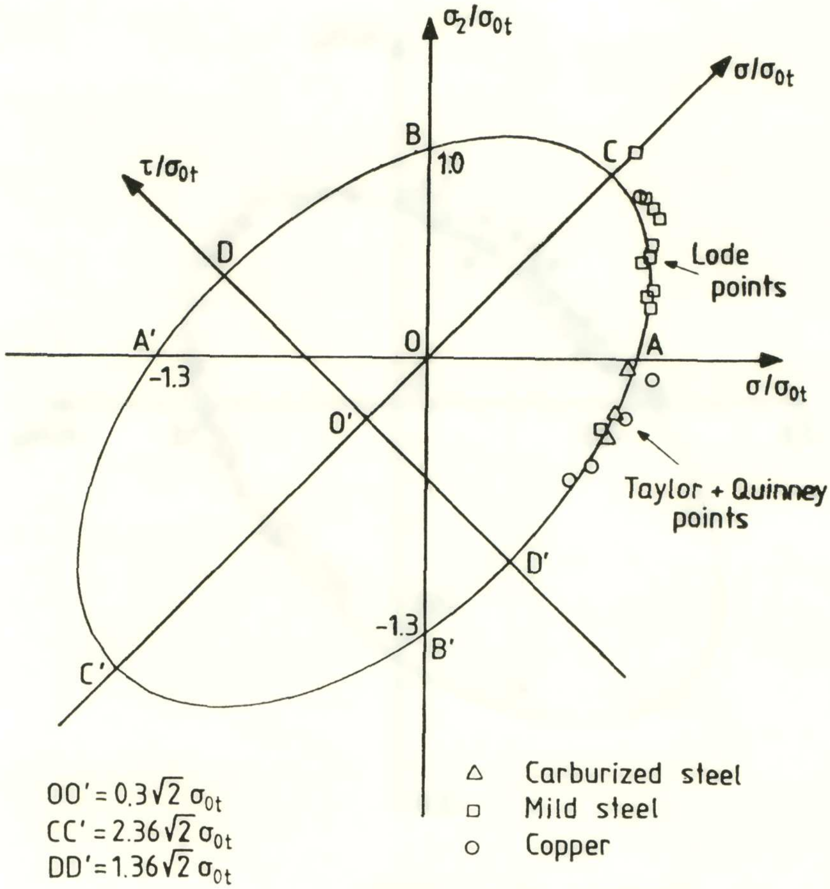

Fig. 8. The experimental results of Taylor and Quinney and Lode for the yield stresses of various kinds of steels and coppers and their coincidence with the circular paraboloid yield criterion with R=1.30.

consistently outside the Mises yield locus. It can, then, readily be proven that these values obey a circular paraboloid criterion with a strength differential factor equal R=1.30. Fig. 8 presents the results of Taylor and Quinney, as well as the equally reliable results of Lode [**15**] for various types of steels and copper, which again show an excellent agreement with the circular paraboloid type of criterion with R=1.30.

Yield criteria based on circular paraboloid type of criterion may be extended to predict the failure behaviour of high-polymers.

Fig. 9 shows the yield locus for a series of polymers plotted in the (σ_1, σ_2)

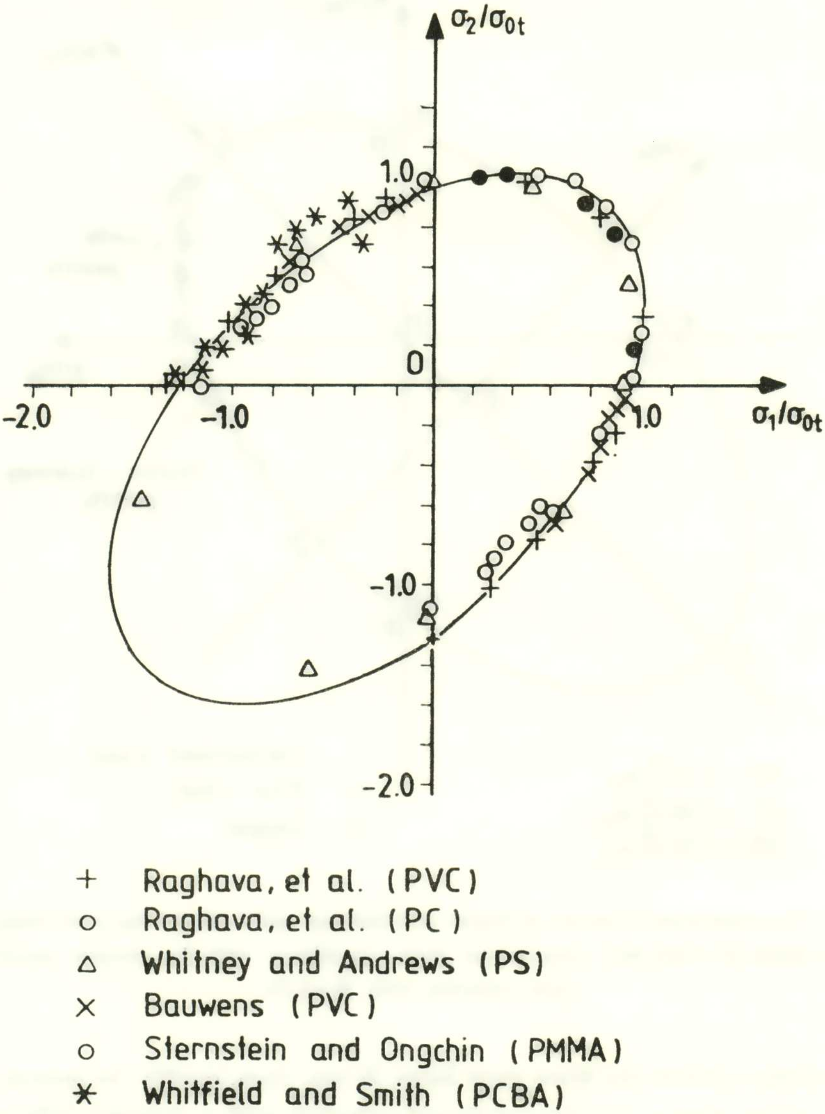

Fig. 9. The yield locus from a series of experiments for various polymers plotted in the (σ_1, σ_2)-plane and the corresponding paraboloid locus for R=1.30.

-principal stress plane and taken from ref. [**16**]. The strength-difference effect for all these materials was found to be R=1.3 approximately. In the same figure the conical section of the circular paraboloid surface with the $\sigma_3=0$ plane was plotted for R=1.30 and represented by the continuous ellipse. It is clear from this figure that again the elliptic paraboloid failure criterion corroborates all experimental results.

On the other hand, experiments executed by Spitzig et al. [**17**] on various types of steels, presenting strength differential ratios $\sigma_{oc}/\sigma_{ot} = 1.055$, compared their results with the conical type of criterion. The values for the tangent of the semi-angle, α, of the apex of the cone were found to be $\tan\alpha = 0.026$ and 0.028 respectively, whereas the distance, $\sqrt{3}\,\sigma_A$, of the apex of the cone from the origin of the coordinates, multiplied by the factor $\frac{\sqrt{2}}{3}R$, were given as $\sqrt{2}\,R\sigma_A = 1.480$ and 1.066 MPa. However, from the respective values of σ_{oc} and σ_{ot} these quantities are 1.47 MPa and 1.070 MPa respectively. Therefore, the theory by Spitzig et al. based on the conical failure criterion yield satisfactory results. However, these results, with ratios σ_{oc}/σ_{ot} of the order of $R=1.10$, correspond to yielding loci, which differ only slightly between theories and, therefore, they are not decisive for the selection of the correct criteria.

In order to show the weakness of conical failure criterion, we have plotted in Fig. 10 the yield loci derived from both criteria and for identical values of the strength-differential factor $R = \sigma_{oc}/\sigma_{ot}$. It can be readily seen from the corresponding loci that, whereas, for values of R close to unity, there is a small difference between the loci derived from both criteria, for larger values of $R (R > 1.10)$ the differences increase and they become significant, so that, for brittle materials with R approaching values of 3.00 the ellipses of the conical failure criterion degenerate into *a parabola* passing through the points (1,0), (0,1) and (0,-3), (-3,0), whereas for $R > 3.00$ these curves become *hyperbolas*.

The conical failure criterion has been introduced and extensively applied by Nadai [**19**], who reports also experiments and applications of the Coulomb-Mohr criterion, which is a precursor and an outcome of the conical failure criterion. Moreover Bauwens [**18**] and Sternstein and Ongchin [**19**] applied it to polymers under the form:

$$\tau_n + Ap = C \tag{29}$$

where τ_n is the octahedral shear stress, which is directly related to the second stress invariant J_2, p is the mean normal stress, and A and C real constants, with the constant C, having dimensions of stress. As it is pointed out by Ra-

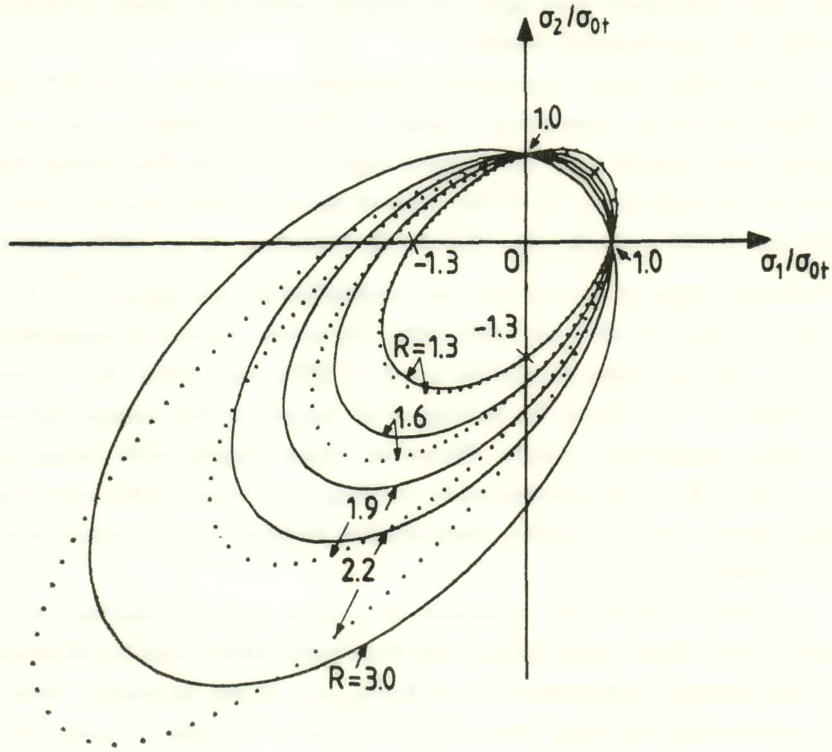

Fig. 10. The families of yield loci for the two types of strength-differential modified criteria for constant yield stress in simple tension and R varying between R=1.3 and R=3.0. Full lines correspond to the paraboloid criterion, whereas dotted lines correspond to the conic criterion.

ghava et al. [**13**], Eq. (29) is another expression of the relationship:

$$\sqrt{3}\, J_2^{1/2} + aI_1 = c \tag{30}$$

in which it was taken into account that the odd dependence of any failure and yield criterion on the J_3 -stress invariant is insignificant and it may be neglected.

In this relation a is the so-called *mean-stress coefficient* and c expresses the basic strength of the material [**20**]. This criterion for constant values of a and c coincides with the Drucker-Prager criterion for soils [**21**].

The constants A and C in the criterion of Eq. (29) are expressed by:

$$A = \frac{\sigma_{oc} - \sigma_{ot}}{\sigma_{oc} + \sigma_{ot}} \quad \text{and} \quad C = \frac{\sigma_{oc}\sigma_{ot}}{\sigma_{oc} + \sigma_{ot}} \tag{31}$$

which when compared with relations (9) and (10) of the conical failure criterion yield:

$$A = \frac{\sqrt{2}}{2}\tan\alpha \quad \text{and} \quad C = \frac{3}{2\sqrt{2}}\sigma_A \tan\alpha \tag{32}$$

In the same context the circular paraboloid criterion was suggested for the first time by Schleicher [**3**], as early as 1925, and elaborated by Stassi d' Alia [**22**] and Tschoegl [**8**]. Raghava, Caddell and Yeh [**23**] have applied it to the yield behaviour of some polymers and compared it with other forms of the same idea. Theocaris [**16**] has discussed its application and Theocaris et al. [**24**] have used both criteria, the conical and the circular paraboloid, to show the influence of mechanical properties of a bimaterial plate when a crack existing in the one phase approaches the interface.

The *Schleicher-Stassi* criterion for plane stress conditions becomes:

$$(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) + 3\ (\sigma_{oc} - \sigma_{ot})p = \sigma_{oc}\sigma_{ot} \tag{33}$$

It consists of three terms, the first of which expresses the distortional component of energy and corresponds to the classical Mises yield condition, the second term expressed an elastic energy, depending on hydrostatic stress, p, and the difference in yield stresses for compression and tension, whereas the right-hand side term is the geometric mean of these two yield stresses.

Addition of the three terms in Eq. (33) is legitimate, since these terms express energy quantities.They tend to a limit, when $\sigma_{oc}=\sigma_{ot}$, which reduces to the classical Mises yield condition for ductile materials. Moreover, for $\sigma_{oc} >> \sigma_{ot}$ when it may be assumed that $\sigma_{ot}/\sigma_{oc} \to 0$, Eq. (33) represents an ellipse, which is equal in size to the typical Mises ellipse with $\sigma_{oc}=\sigma_{ot}$ and it passes through the origin of coordinates in a $(\sigma_1/\sigma_{oc}, \sigma_2/\sigma_{oc})$-diagram, as well as through the points (-1, 0) and (0, -1). All other ellipses, if they are referred to the same yield stress in simple compression, σ_{oc}, are smaller in size than these two limit curves, they pass, all of them, through the points (-1, 0) and (0, -1). Fig. 11 presents the family of the yield loci according to the Schleicher-Stassi criterion and normalized to the same yield stress in simple compression. It is clear from this figure that the ellipses for $R=1.0$ and $R=\infty$

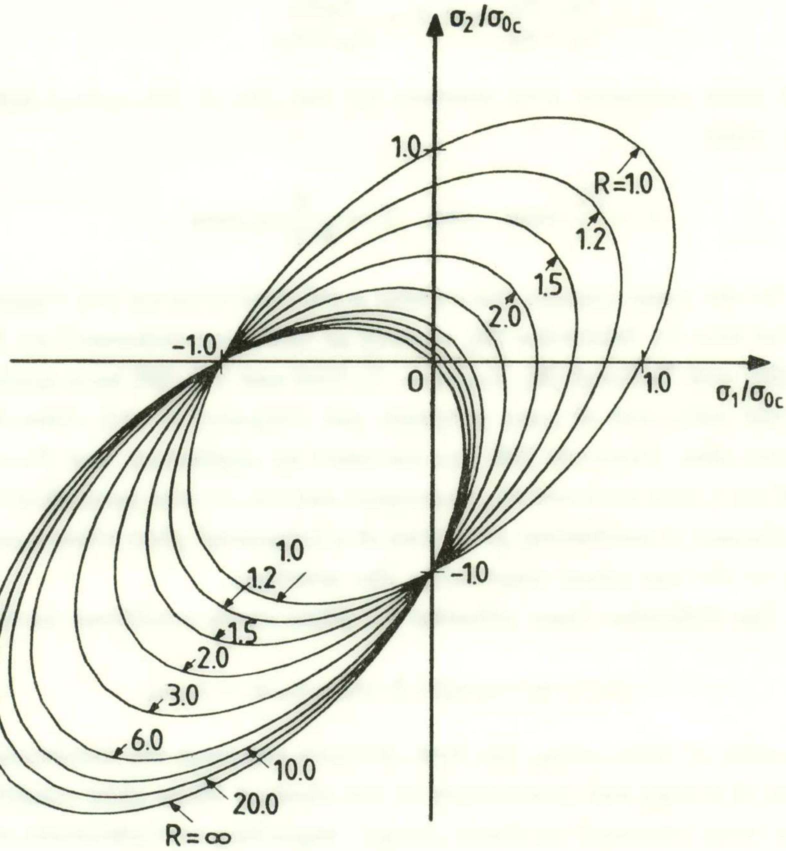

Fig. 11. The family of the circular paraboloid yield criteria with R varying between unity and infinity for the same yield stress in uniaxial compression.

are equal. Contrariwise, Fig. 12, which presents a similar family of yield loci but for the same yield stress in simple tension contains ellipses, whose sizes increase progressively as R is increasing, but all ellipses pass through the points (1, 0) and (0, 1) in the σ_1/σ_{0t}-σ_2/σ_{0t} principal stress space.

Comparing the yield loci resulting from the two models and the experimental data available for various materials, it may be concluded that, whereas the Schleicher-Stassi criterion corroborates satisfactorily the experimental evidence with various materials, the Nadai-Bauwens-Sternstein criterion deviates significantly, especially in the critical compression-compression quadrant.

Furthermore, a noticeable difference between the two types of criteria

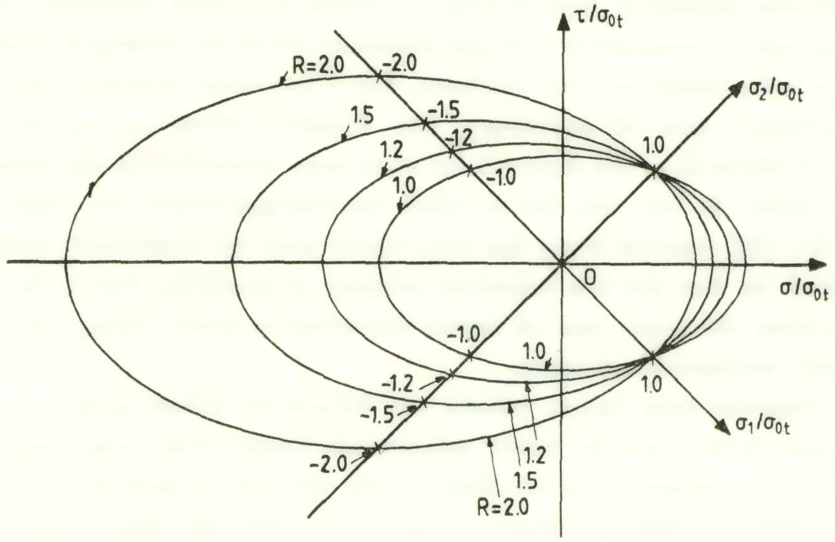

Fig. 12. The family of the circular paraboloid yield criteria with R varying between unity and R=2.0, for the same yield-stress in uniaxial tension in the (σ, τ)-plane.

exists which influences considerably the reliability of their results. Indeed, the Nadai-Bauwens Sternstein failure criterion, derived from the conical criterion, as it is expressed by relation (29), considers an algebraic addition of stresses, which are not collinear. The octahedral shear stress, τ_n, lies always on the deviatoric plane, whereas the hydrostatic component, p, is always normal to this plane. Therefore, any algebraic addition of these stresses is meaningless. Their addition is explained if we consider from relation (8) that only the component of octahedral shear stress is multiplied by cotα and therefore collinear to σ_{nn}-stress added to it.

On the contrary, addition of the terms in Eq. (26) is legitimate since these terms express energy quantities. This is another reason to consider this criterion and therefore the circular paraboloid criterion as a reliable criterion describing satisfactorily the failure mode of engineering substances.

THE FAILURE CRITERION IN ANISOTROPIC BODIES

A final remark in closing this review paper is worthwhile. This remark concerns the extension and generalization of the circular paraboloid failure criterion to initially anisotropic materials. For this type of materials the simple circular paraboloid type of failure criterion is no more sufficient. In the simple case of transversely isotropic materials, where the substance is defined by five independent material constants, that is two elastic moduli E_L, E_T along the principal axes of anisotropy, two Poisson's ratios ν_{LT}, ν_{TT} and one shear modulus G_{LT}, the failure locus is no more symmetric in the principal stress space. In this case the so-called *tensorial-type criteria* are valid. The Tsai-Wu [**25**] *tensorial theory* has been widely used for transversely isotropic materials as they are this important category of materials, that is the fiber composites, the paper, and all metals submitted to severe rolling and other oriented mechanical processes.

Tensorial type failure criteria are defined by failure surfaces in the principal stress space by tensor polynomials. Under plane stress conditions when the coordinate axes are aligned with the axes of material symmetry these criteria contain only linear and quadratic terms and they are expressed by:

$$F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_1\sigma_1 + F_2\sigma_2 + F_{66}\tau_{12}^2 = 1 \qquad (34)$$

where the subscripts $_1$ and $_2$ on stresses σ and τ denote the strong and the weak directions respectively. Because of the symmetry conditions for plane-stress or for transversely isotropic materials the factors F_{16}, F_{26} and F_{66} are about or are equal to another factor because these symmetry conditions on the τ_{12}-stress require these factors to vanish.

Further restrictions on these factors require that:

$$\begin{gathered} F_{11}F_{66} > 0 \\ F_{22}F_{66} > 0 \\ \text{and} \\ F_{12}^2 - F_{11}F_{22} < 0 \end{gathered} \qquad (35)$$

Since again the failure surface should be open in the purely compressive octant the choice of the strength factors, F_{ij}, should give always forms of the failure surface obeying this requirement. This restriction derives from the fact that hydrostatic compression for moderate pressures cannot lead to failure. Then, the argumentation made previously for the isotropic materials is still valid and the circular paraboloid becomes an elliptic one.

For many anisotropic materials, like paper and unidirectional composites, which are assumed as transversely isotropic the factor F_{12}, characterizing the strength interaction between normal stresses σ_1 and σ_2, takes insignificant values.

This factor determines the inclination of the failure ellipses, which are formed by intersections of the failure surface described by relation (34) and planes of constant shear, τ_{12}.

On the other hand, all other factors of equation (34) are related to the intercepts of the failure elliptic paraboloid with the principal-stress axes.

From these considerations it may be derived that the value of the F_{12} factor determines the deviation of this elliptic paraboloid from the circular paraboloid corresponding to an equivalent isotropic material, that is the material with the same mechanical characteristic properties but with zero anisotropy.

For orthotropic materials the space diagonal is parallel to the axis of symmetry of the failure surface. The value of the factor F_{12} defines the angular displacement of the corresponding failure surface. There are many methods to evaluate the F_{12}-factor. These are not described here. We are limited to show only in Fig. 13 the influence of the F_{12}-factor on the shape and orientation of ellipses corresponding to conic sections of the elliptic paraboloid with the plane $\tau_{12}=0$ of a paperboard taken from ref. [**26**].

It is clear from this figure that, although the intercepts of the intersec-

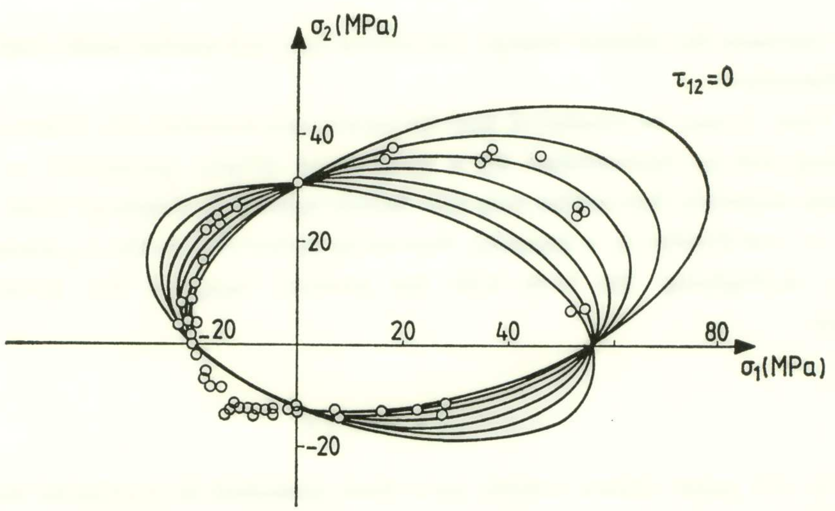

Fig. 13. Conic sections of the failure locus for a type of paperboard (anisotropic material) for $\tau_{12}=0$ and variying values of the strength factor F_{12} (after ref. [**26**]).

tions of the family of the ellipses in the plane $\sigma_3=0$ with $\tau_{12}=0$, with the $\sigma_2=0$ and $\sigma_1=0$ axes are constant, and the values of the principal stresses $\pm\sigma_1$ and $\pm\sigma_2$ in tension and compression are unchanged, the shape and orientation of the ellipses are drastically depending on the values of the F_{12}-factor.

Fig. 14 presents the conic sections of the same material, a paperboard with $F_{12}=0$, for different values of the principal shear stress τ_{12}. As the shear

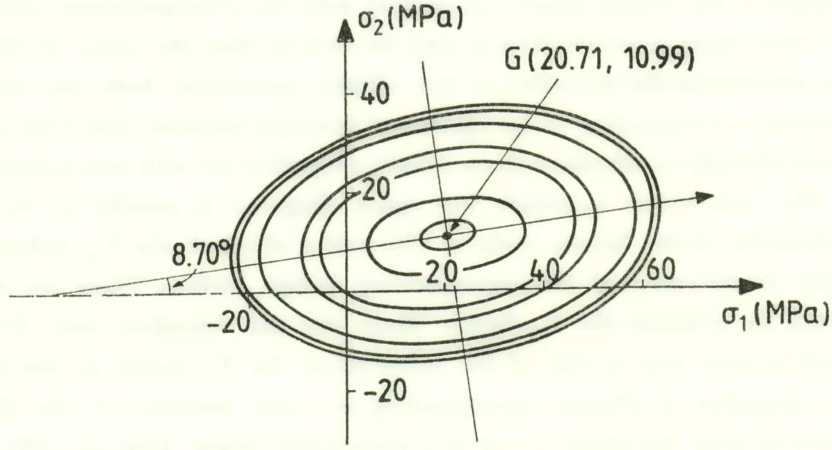

Fig. 14. Conic sections of the failure locus for a type of paperboard with strength factor$F_{12}=0$, for different values of the applied τ_{12}-shear stress (after ref. [26]).

stress increases the ellipses change dimensions in a self-similar mode without any distortion.

Then, it may be concluded that for anisotropic materials the failure surface may still be repressented by a generalized elliptic paraboloid as for isotropic materials, but in this case this failure surface is displaced from the origin of coordinates in a quantity depending exclusively on the F_{12}-strength factor, multiplying the term with the product $(\sigma_1\sigma_2)$ of the principal stresses.

CONCLUSIONS

In this paper failure criteria have been examined in multiaxial states of stress associated with geometrical representations of surfaces in the principal stress space.

It was shown that the circular paraboloid defined by the σ_{oc}- and σ_{ot}- failure stresses of the material in simple compression and tension respectively, describes satisfactorily the failure mode of all isotropic materials from the brittlest to the most ductile. The Mises cylindrical failure space and the Coulomb conical one are cases which yield approximate results when compared with experiments and only in limited zones of loading.

The circular paraboloid failure surface has its axis symmetry coinciding with the space diagonal for which it is valid that $\sigma_1=\sigma_2=\sigma_3$ and the direction cosines are $\xi_1=\xi_2=\xi_3=1/\sqrt{3}$.

Finally, it was shown that for anisotropic materials a similar failure surface is still valid but the paraboloid becomes elliptic and is displaced from the space diagonal.

ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν εἰσαγωγὴν τῆς ἀνακοινώσεως ἐξετάζονται τὰ παραδεδεγμένα κριτήρια διαρροῆς καὶ ἀστοχίας τῶν ὑλικῶν. Τὰ βασικώτερα ἐξ αὐτῶν ὁρίζουν ὅτι διαρροὴ ἢ ἀστοχία τῶν ὑλικῶν λαμβάνει χώραν ὅταν εἴτε ἡ μέγιστη διατμητικὴ τάσις (κριτήριον Tresca) εἴτε ἡ στροφικὴ ἐνέργεια (κριτήριον Mises-Huber-Hencky) λάβουν κρίσιμον τιμήν, ὁριζομένην εἴτε ἀπὸ τὴν ἀντοχὴν τοῦ ὑλικοῦ εἰς καθαρὰν διάτμησιν, εἴτε ἀπὸ τὸ τετράγωνον τῆς ἰσοδυνάμου τάσεως.

Ἐν τούτοις εὐθὺς ἐξ ἀρχῆς ἔχει παρατηρηθῆ ὅτι ὅλα τὰ ὑλικὰ τῶν κατασκευῶν ἔχουν διάφορον ἀντοχὴν ὅταν καταπονοῦνται εἰς ἁπλὸν ἐφελκυσμὸν ἢ εἰς ἁπλὴν θλίψιν. Ὁ λόγος τῆς τάσεως διαρροῆς (ἢ ἀστοχίας) τυχόντος ὑλικοῦ εἰς θλίψιν ἀναγόμενος εἰς τὴν τάσιν διαρροῆς εἰς ἐφελκυσμὸν καθορίζει τὸν διαφορετικὸν συντελεστὴν ἀντοχῆς του, παριστώμενον διὰ R, ὁ ὁποῖος καὶ ὁρίζει τὸ μέγεθος τῆς ἀνισοτρόπου κρατύνσεως τοῦ ὑλικοῦ.

Ἐκ τῆς μεγάλης σωρείας πειραμάτων ποὺ ἐγένοντο μέχρι σήμερον διὰ τὸν καθορισμὸν τῶν τόπων ἀστοχίας τῶν ὑλικῶν ἀπεδείχθη ὅτι διὰ πολυαξονικὴν φόρτισιν οὐδὲν ἐξ αὐτῶν ὑπακούει εἴτε εἰς τὸ ἑξάγωνον τοῦ Tresca εἴτε εἰς τὴν ἔλλειψιν τοῦ Mises, ὅταν τὸ ὑλικὸν καταπονεῖται εἰς διδιάστατον ἐντατικὴν κατάστασιν. Διὰ τριδιάστατον ἐντατικὴν κατάστασιν ὁ μὲν τύπος τοῦ Tresca λαμβάνει τὴν μορφὴν ἑξαγωνικοῦ πρίσματος μὲ τὴν κύριαν διαγώνιον τοῦ χώρου τῶν κυρίων τάσεων ὡς ἄξονα, τὸ δὲ κριτήριον τοῦ Mises λαμβάνει τὴν μορφὴν κυλίνδρου μὲ τὴν αὐτὴν διαγώνιον ὡς ἄξονα.

Ἀποκλίσεις ἐκ τῶν τόπων αὐτῶν πάντοτε παρουσιάζονται καὶ ἰδίως εἰς τὸν τεταρτημόριον τῆς θλίψεως-θλίψεως, ὅπου ἡ ἐπίδρασις τῆς ἀνισοτρόπου κρατύνσεως εἶναι ἐντονωτέρα. Ἤδη ἀπὸ τὴν ἀρχὴν τοῦ αἰῶνος παρετηρήθη ὑπὸ τῶν μελετητῶν ὅτι ἐὰν εἰς τὸν τόπον διαρροῆς ληφθῇ ὑπ' ὄψιν καὶ ἡ ἐπίδρασις τῆς ἑτέρας συνιστώσης τῆς ἐνεργείας, ἤτοι ἡ διογκωτικὴ ἐνέργεια, ὁ προκύπτων τόπος διαρροῆς προσεγγίζει καλύτερον πρὸς τὰ πειραματικὰ δεδομένα.

Εἰς τὴν ἐργασίαν αὐτὴν μελετῶνται μόνον ἀρχικῶς ἰσότροπα ὑλικά, τὰ ὁποῖα παρουσιάζουν ἀνισότροπον κράτυνσιν κατὰ τὴν φόρτισίν των. Λαμβανομένης ὑπ' ὄψιν τῆς συμμετρίας τοῦ τόπου ὡς πρὸς τοὺς τρεῖς κυρίους ἄξονες τῶν τάσεων, οἱ κατάλληλοι τόποι διὰ τοιαῦτα ὑλικὰ πρέπει νὰ ἐκφράζωνται δι' ἐξισώσεων τετραγωνικῆς μορφῆς. Ἐξ ὅλων τῶν μονοχώνων στερεῶν μὲ ἐξίσωσιν τετραγωνικῆς μορφῆς τὰ μόνα ποὺ ὑπακούουν εἰς τὴν συνθήκην συμμετρίας εἶναι ὁ κῶνος καὶ τὸ παραβολοειδές.

Καθορίζονται ἐν συνεχείᾳ αἱ ἐξισώσεις αἱ ὁποῖαι ἐπηρεάζουν τὰ στερεὰ αὐτὰ καὶ προσδιορίζονται τὰ χαρακτηριστικὰ των μεγέθη, ἤτοι ἡ γωνία τῆς κορυφῆς τοῦ κώνου καὶ ἡ ἀπόστασίς της ἀπὸ τῆς ἀρχῆς τῶν ἀξόνων διὰ τὸ κωνικὸν κριτήριον ἀστοχίας, καὶ αἱ συντεταγμέναι τῆς ἑστίας καὶ τῆς κορυφῆς τοῦ κυκλικοῦ παραβολοειδοῦς διὰ τὸ δεύτερον κριτήριον.

Ἐν συνεχείᾳ ὁρίζονται αἱ τομαὶ τοῦ κώνου καὶ τοῦ παραβολοειδοῦς ὑπὸ κυρίων ἐπιπέδων τῶν τάσεων καὶ εὑρίσκεται ὅτι διὰ τὸ κωνικὸν κριτήριον αἱ τομαὶ αὐταὶ εἶναι ἐλλείψεις διὰ τιμὰς τοῦ R κυμαινομένας εἰς τὸ διάστημα $0 \leqq R < 3.0$. Διὰ τὴν τιμὴν $R = 3.0$ αἱ ἐλλείψεις ἐκφυλίζονται εἰς παραβολὴν καὶ διὰ $R > 3.0$ μετατρέπονται εἰς ὑπερβολάς. Ἡ ἀκραία ὑπερβολὴ διὰ $R = \infty$ ἔχει ἀσυμπτώτους παραλλήλους πρὸς τοὺς κυρίους ἄξονας $\sigma_1 = \sigma_2 = 0$.

Διὰ τὸ κυκλικὸν παραβολοειδὲς κριτήριον ὅλαι αἱ τομαί του μὲ οἱονδήποτε τῶν ἐπιπέδων $\sigma_3 = 0$, $\sigma_2 = 0$, $\sigma_1 = 0$ εἶναι ἐλλείψεις μὲ τὴν ἀκραίαν ἔλλειψιν διὰ $R = \infty$ ἐκφυλιζομένην εἰς εὐθεῖαν τέμνουσαν τοὺς ἄξονας $\sigma_2 = 0$ $\sigma_1 = 0$ εἰς ἀποστάσεις ἴσας πρὸς $R = 1.0$. Ἀντιστοίχως τὸ ἕτερον ὅριον τῶν ἐλλείψεων διὰ $R = 0$ ἀποτελεῖ τὴν ἀντίστοιχον ἔλλειψιν τοῦ Mises.

Τὰ δύο αὐτὰ κριτήρια συγκρίνονται πρὸς τὰ πειραματικὰ ἀποτελέσματα ποὺ ὑπάρχουν ἄφθονα εἰς τὴν βιβλιογραφίαν καὶ ἀποδεικνύεται ὅτι τὸ κυκλικὸν παραβολοειδὲς κριτήριον συμπίπτει μὲ ἱκανοποιητικὴν προσέγγισιν μὲ τὴν ὁλότητα τῶν πειραματικῶν ἀποτελεσμάτων διὰ μεταλλικὰ καὶ πλαστικὰ δοκίμια.

Ἐκ τῆς συμπτώσεως αὐτῆς καὶ τῶν προτερημάτων τοῦ κριτηρίου αὐτοῦ θεωρεῖται τοῦτο ὡς τὸ καταλληλότερον ἐξ ὅλων τῶν ὑπαρχόντων κριτηρίων.

Τέλος τὸ κριτήριον αὐτὸ ἐπεκτείνεται καὶ δι' ἀνισότροπα ὑλικὰ ὅπου ὅλαι αἱ μέχρι σήμερον πειραματικαὶ ἐνδείξεις συμφωνοῦν μὲ τὰ ἀποτελέσματα τὰ προκύπτοντα ἐκ τοῦ κριτηρίου αὐτοῦ.

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