ΑΝΑΚΟΙΝΩΣΙΣ ΜΗ ΜΕΛΟΥΣ

ΜΑΘΗΜΑΤΙΚΑ.— **Products and lengths in halfgroupoids (second part)**, *by S. P. Zervos* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κ. Π. Παπαϊωάννου.

REFERENCES, NOTATIONS AND CONTENTS

The present paper is the second part of a study of the same author under the same title, communicated to this Academy on May 8, 1969.

Additional notations. Given a hgr $(A,\cdot)$, $L^{(\nu)}(A)$ [resp. $L^{f}(A)$] = the set of all elements of A having, in $(A,\cdot)$, length $\leqq \nu+1$ (resp. $<+\infty$). *Abridged notations.* $L^{(\nu)}$, L^{f}, $L^{(\nu)}_{A_0}$ and $L^{f}_{A_0}$ for, resp., $L^{(\nu)}(A)$, $L^{f}(A)$, $L^{(\nu)}_{A_0}(A)$ and $L^{f}_{A_0}(A)$.

Contents. Section VI has a preliminary character. The rest of this second part was motivated by the problem of finding *sufficient* conditions for $(A_0,\cdot) \underset{f,g}{\sqsubseteq} (A,\cdot)$. (This problem is attacked in R. H. B., but receives there a partially inexact solution; see our section VIII for a counter-example.) Independently of the solution and further elucidation of this problem, some of the new results and methods seem to us interesting in themselves, at least as applications of the notion of length; in particular, and independently of length, the new (as far as I know) notion of «finite set of divisors», presented in IX. Further applications will be given in a forthcoming paper.

VI. NECESSARY CONDITIONS IN ORDER THAT A HALFGROUPOID BE FREELY GENERATED BY A SUBHALFGROUPOID OF IT

Three such conditions are stated and proved in R. H. B.; we shall consider here a somewhat richer set of such conditions.

Proposition 9. *Let* $(A_0,\cdot) \sqsubseteq (A,\cdot)$ *and* $((A_\nu,\cdot))_{\nu\in N}$ *be mec* (A_0, A). *The following are necessary conditions in order that* $(A_0,\cdot) \underset{f,g}{\sqsubseteq} (A,\cdot)$ *: 1) If, for*

* Σ. Π. ΖΕΡΒΟΥ, **Γινόμενα καὶ μήκη εἰς τὰς δομὰς μιᾶς μερικῶς ὡρισμένης ἐσωτερικῆς πράξεως (μέρος δεύτερον).**

some ν, $a \in A_\nu$ *is strictly prime in* $(A_\nu, \cdot)$, *then it is also strictly prime in* $(A, \cdot)$. *1. b) Special case:* $\nu = 0$. *2) If* $a \in A - A_0$, *then* $a = b \,.\, c$ *in* $(A, \cdot)$ *for one and only one ordered pair* (b, c) *of elements of* A. *3. a) If an equality of the form* $a_\nu' \cdot a_\nu'' = a_\nu'''$, *with* $(a_\nu', a_\nu'', a_\nu''') \in A_\nu^3$, *holds in* $(A, \cdot)$, *then it already holds in* $(A_\nu, \cdot)$. *The special case obtained for* $\nu = 0$, *will be denoted by 3. b). 4) For every* ν, *there is not, in* $(A_\nu, \cdot)$, *any divisor of an element of* A_ν *not belonging to* A_ν. *5) If in a divisor chain in* $(A, \cdot)$ *there is some term belonging to* A_ν, *every (possibly existing) term of the chain coming after it also belongs to* A_ν. *6) For every* ν, *there is no divisor of an element of* $A_{\nu+1}$, *in* $(A, \cdot)$, *not belonging to* A_ν. *7) If* $a \in A_{\nu+1} - A_\nu$, *then* $a = b \,.\, c$, *in* $(A_{\nu+1}, \cdot)$, *for one and only one ordered pair* $(b, c) \in A_\nu$. *8) If, in a divisor chain* $(\gamma_\lambda)_{\lambda \in M}$ *in* $(A, \cdot)$, $\gamma_0 \in A_\nu$, *then, for every* $\lambda \leq \nu$, $\gamma_\lambda \in A_{\nu-\lambda}$. *Hence, if, in addition to* $\gamma_0 \in A_\nu$, $\nu \in M$, *then* $\gamma_\nu \in A_0$ *and the chain is finite over* $(A_0, \cdot)$. *9) Every divisor chain in* $(A, \cdot)$ *is either finite or finite over* $(A_0, \cdot)$. *10) Every finite divisor chain in* $(A, \cdot)$, *the last term of which does not belong to* A_0, *can, by the adjunction of a finite number of terms, be extended to a divisor chain finite over* $(A_0, \cdot)$. *11) For every* ν, $L_{A_0}^{(\nu)} = A_\nu$. *12)* $L_{A_0}^{f} = A$.

Proof. Our references to R.H.B are to his proof of lemma 1.4. 1) and 2) are directly stated and proved there. Immitating an argument used there, we prove 3. a) [in detail: Since $(A_0, \cdot) \underset{f,\,g}{\subseteq} (A, \cdot)$ and $((A_\nu, \cdot))_{\nu \in \mathbb{N}}$ is mec (A_0, A), the product of any two elements of A_ν is defined in $(A_{\nu+1}, \cdot)$; if this product is equal to an element of A_ν, it is already defined in $(A_\nu, \cdot)$. From this and the fact that $(A_\nu, \cdot) \underset{e}{\subseteq} (A_{\nu+1}, \cdot)$ is open comes the assertion.]. The method of finite descent used there can also be used to prove 4) [in detail: Let $a \,.\, b = a_\nu$, with $(a, b) \in A^2$ and $a_\nu \in A_\nu$, hold in $(A, \cdot)$. Then, for some $\varrho \geq 0$, $a \,.\, b = a_\nu$ holds in $(A_{\nu+\varrho}, \cdot)$. Since $((A_\nu, \cdot))_{\nu \in \mathbb{N}}$ is an open extension chain, one can apply finite descent to prove that $a \,.\, b = a_\nu$ holds in $(A_\nu, \cdot)$; then, $(a, b) \in A_\nu^2$.] 4) implies 5). 6) is a consequence of the definition of $A_{\nu+1}$, the special case $\nu = 0$ of 4) and 2). 7) is a consequence of the definition of $(A_{\nu+1}, \cdot)$ and of 2). 6) and an obvious use of finite descent prove 8). 9) and 10) are immediate consequences of 8), 7) and 4) (special case $\nu = 0$). We come, at last, to the applications 11) and 12) of the notion of length over $(A_0, \cdot)$, introduced in the first part of the present paper. By proposition 6 (proved in this part),

$L^{(\nu)}_{A_0} \subseteq A_\nu$; now, 8) implies that $A_\nu \subseteq L^{(\nu)}_{A_0}$; hence, $L^{(\nu)}_{A_0} = A_\nu$, i. e. 11). Finaly, 11) clearly entails 12). |

VII. NECESSARY AND SUFFICIENT CONDITIONS IN ORDER THAT A HALFGROUPOID BE FREELY GENERATED BY A SUBHALFGROUPOID OF IT

Let $(A_0, \cdot) \subseteq (A, \cdot)$. In order that $(A_0, \cdot) \underset{f,g}{\subseteq} (A, \cdot)$, it is necessary and sufficient that a) $(A_0, \cdot) \underset{g}{\subseteq} (A, \cdot)$ and b) $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$.

According to proposition 7, a sufficient condition for a) is $A = L^{f}_{A_0}$, i. e. condition 12) of proposition 9.

Suppose, now, that a) is fulfilled, consider b) and let $((A_\nu, \cdot))_{\nu \in N}$ be mec (A_0, A). According to previous remarks, in order that $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$ it is sufficient that, for every ν, $(A_\nu, \cdot) \underset{f}{\subseteq} (A_{\nu+1}, \cdot)$, hence, that, for every ν, $(A_{\nu+1}, \cdot)$ is an open extension of $(A_\nu, \cdot)$. Hence, sufficient conditions for $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$ are the necessary conditions 7) and 3 a) of proposition 9. Hence, $\{3. a), 7), 12)\}$ (numbers refer to proposition 9) *is a first set of necessary and sufficient conditions for* $(A_0, \cdot) \underset{f,g}{\subseteq} (A, \cdot)$. Hence, $\{3. a), 2), 12)\}$ *is also such a set.*

We proceed, now, to find other interesting sets of necessary and sufficient conditions for $(A_0, \cdot) \underset{f,g}{\subseteq} (A, \cdot)$. *Notation.* By μ) we always denote the corresponding condition in proposition 9. The special case of 4) obtained for $\nu = 0$ will be labelled 4. b). $((A_\nu, \cdot))_{\nu \in N}$ will always denote, in the sequel, mec (A_0, A).

Proposition 10. *Let* $(A_0, \cdot) \subseteq (A, \cdot)$. $\{2), 4. b)\}$ *is, then, a set of sufficient conditions in order to have* $A_\nu = L^{(\nu)}_{A_0}$.

Proof. We suppose that 2) and 4. b) hold. It is, then, sufficient to show that $A_\nu \subseteq L^{(\nu)}_{A_0}$. We observe, first, that 4. b) ensures that $A_0 = L^{(0)}_{A_0}$. The proof will be completed by induction. Suppose that the assertion holds for all $\mu \leqslant \nu$ and consider $a_{\nu+1} \in A_{\nu+1} - A_\nu$. According to 2), one and only one equality of the form $a_{\nu+1} = a'_\nu \cdot a''_\nu$ holds in $(A, \cdot)$. Then, by the definition of $A_{\nu+1}$, $(a'_\nu, a''_\nu) \in A^2_\nu$, hence, by the hypothesis of the induction $l_{A_0}(a'_\nu) \leqslant \nu + 1$ and $l_{A_0}(a''_\nu) \leqslant \nu + 1$; hence, $l_{A_0}(a_{\nu+1}) \leqslant (\nu + 1) + 1 = \nu + 2$. Hence, $A_{\nu+1} \subseteq L^{\nu+1}_{A_0}$. ❙

C o r o l l a r y. *Let* $(A_0, \cdot) \subseteq (A, \cdot)$. $\{2), 4.b)\}$ *is, then, a set of necessary and sufficient conditions in order that* $A_{\nu+1} - A_\nu$ *be the set of the elements of length* $\nu + 2$ *over* A_0.

Conditions 2) and 4. b) are essential for the validity of the assertion of this corollary, hence, also, of proposition 10. We shall see it in two examples. *Example 1. [4. b) holds, 2) does not hold and the assertions do not hold.]* $A_0 = \{a_0\}$ and $a_0 . a_0$ is not defined in $(A_0, \cdot)$. $A = \{a_0, a_1\}$ and $a_0 . a_0 = a_0 . a_1 = a_1$ in $(A, \cdot)$. Then, $A_1 = \{a_0, a_1\}$, with $a_0 . a_0 = a_1$ in $(A_1, \cdot)$; $A_2 = A_1$ and $a_0 . a_0 = a_0 . a_1 = a_1$ in $(A_2, \cdot)$; hence, $(A_2, \cdot) = (A, \cdot)$. Here, there is no divisor of an element of A_0 not belonging to A_0, so 4. b) holds; but, $a_1 \in A_1 - A_0$ and $a_1 = a_0 . a_0 = a_0 . a_1$, hence 2) does not hold. The infinite sequence

$$\left[\begin{array}{l} a_1, a_0 \\ a_1, a_0, a_1, a_0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right.$$

of divisor chains finite over $(A_0, \cdot)$ shows that a_1 has length $+\infty$ over $(A_0, \cdot)$. ❙

Example 2. [2) holds, 4. b) does not hold and the assertions do not hold.] Same $(A_0, \cdot)$ and A as in the previous example, with $a_0 . a_0 = a_1$ and $a_0 . a_1 = a_0$ in $(A, \cdot)$. Then, 2) holds, but 4. b) does not hold; and the same infinite sequence of divisor chains finite over $(A_0, \cdot)$ shows that $l_{A_0}(a_1) = +\infty$. ❙

Proposition 11. *Let* $(A_0, \cdot) \subseteq (A, \cdot)$. *If 2) and 4. b) hold, then, for every* $\nu \geqq 1$, *every equality of the form* $a'_\nu . a''_\nu = a'''_\nu$, *with* $(a'_\nu, a''_\nu, a'''_\nu) \in A^3_\nu$, *holding in* $(A_{\nu+1}, \cdot)$ *already holds in* $(A_\nu, \cdot)$.

Proof. If $a'''_\nu \notin A_0$, the assertion is a consequence of 2) and the definition of mec (A_0, A). If $a'''_\nu \in A_0$, 4. b) implies that $(a'_\nu, a''_\nu) \in A^2_0$; this and the definition of A_1 imply that $a'_\nu . a''_\nu = a'''_\nu$ (i. e. $a'_0 . a''_0 = a'''_0$) holds in $(A_1, \cdot)$, hence, in all $(A_\nu, \cdot)$, for $\nu \geqq 1$. ❙

C o r o l l a r y. *Let* $(A_0, \cdot) \subseteq (A, \cdot)$. *If 2) and 4. b) hold, then, for every* $\nu \geqq 1$, $(A_{\nu+1}, \cdot)$ *is an open extension of* $(A_\nu, \cdot)$. However, $(A_1, \cdot)$ *is not necessarily an open extension of* $(A_0, \cdot)$. [*Example 3.* $A_0 = A = \{\alpha, \beta, \gamma\}$. $(A_0, \cdot)$ is defined by $\alpha . \beta = \gamma$ and $(A, \cdot)$ by $\alpha . \beta = \gamma$ and $\gamma . \alpha = \gamma$. Then, $(A_1, \cdot) = (A, \cdot)$, 2) and 4. b) hold trivially since $A_1 - A_0 = \varnothing$ and $\gamma . \alpha = \gamma$

holds in $(A_1, \cdot)$ but not in $(A_0, \cdot)$, so that $(A_1, \cdot)$ is not an open extension of $(A_0, \cdot)$. |] Hence, if we want that $(A_1, \cdot)$ is an open extension of $(A_0, \cdot)$, we have to suppose it, or to make an even stronger assumption, as 3. b). This last choice gives

Proposition 12. *Let* $(A_0, \cdot) \underline{\subset} (A, \cdot)$. $\{$*2), 3. b), 4. b), 12)*$\}$ *is a set of necessary and sufficient conditions for* $(A_0, \cdot) \underset{f, g}{\underline{\subset}} (A, \cdot)$.

Proof. 12) implies $(A_0, \cdot) \underset{g}{\underline{\subset}} (A, \cdot)$. $\{$2), 3. b), 4. b$\}$ implies that $((A_\nu, \cdot))_{\nu \in N}$, which (by definition) is mec (A_0, A), is an open extension chain, hence that $(A_0, \cdot) \underset{f}{\underline{\subset}} (A, \cdot)$. Hence, $(A_0, \cdot) \underset{f, g}{\underline{\subset}} (A, \cdot)$. The necessity of the conditions 2), 3. b), 4. b) and 12) follows from proposition 9. |

This set of necessary and sufficient conditions is convenient, *from the point of view that their statement uses solely* $(A_0, \cdot)$ *and* $(A, \cdot)$.

VIII. SEARCHING FOR CONDITIONS OF «FINITE» CHARACTER

Note. The word «character» is used here loosely and bears no necessary relation to the various accepted mathematical uses of fhe term.

In this stage, given $(A_0, \cdot) \underline{\subset} (A, \cdot)$, it is natural to ask for a set of necessary and sufficient conditions for $(A_0, \cdot) \underset{f, g}{\underline{\subset}} (A, \cdot)$, *not using the notion of length etc. and, in this sense, of «finite» character.*

Much before the introduction of the notion of length, the following answer to this question was inserted and «proved» in R. H. B. : «$\{$1. b), 2), 9)$\}$ is such a set.» Unfortunately, this is not completely true, as we readily show by the same example 3. In it, $(A_1, \cdot) = (A, \cdot)$ and this is not an open extension of $(A_0, \cdot)$, hence, $(A_0, \cdot) \underset{f}{\underline{\subset}} (A, \cdot)$ is *not* true. However, since $A - A_0 = \varnothing$, 1), 2) and 9) are trivially satisfied. Hence, $\{$1), 2), 9)$\}$ is *not* sufficient for $(A_0, \cdot) \underset{f, g}{\underline{\subset}} (A, \cdot)$. |

$\{$1. b), 2), 9)$\}$ is so weak, that it is not sufficient even for $A = L^f_{A_0}$, i. e. 12). This we now show by a somewhat more complicated example.

Example 4. $(\nu, \varrho) \in N^2$, with $\nu \geqslant \varrho$. $(\nu_1, \varrho_1) \neq (\nu_2, \varrho_2)$ implies $a_{\nu_1 \varrho_1} \neq a_{\nu_2 \varrho_2}$ and $a'_{\nu_1 \varrho_1} \neq a'_{\nu_2 \varrho_2}$. A consists of all $a_{\nu\varrho}$ and $a'_{\nu\varrho}$, while A_0 consists of all

$a_{\nu\nu}$. $(A,\cdot)$ is defined by the following infinite set of equalities: $a_{oo} = a_{oo}^2$. For all $(\nu, \varrho) \in \mathbb{N}^2$, with $\nu \geqslant \varrho$, $a'_{\nu\varrho} = a_{\nu\nu} . a_{\varrho\varrho}$.

$a_{oo} = a_{11} . a'_{11}$,

$a_{oo} = a_{21} . a'_{21}$, $a_{21} = a_{22} . a'_{22}$,

$a_{oo} = a_{31} . a'_{31}$, $a_{31} = a_{32} . a'_{32}$, $a_{32} = a_{33} . a'_{33}$,

. . . .

$a_{oo} = a_{\nu 1} . a'_{\nu 1}$, $a_{\nu 1} = a_{\nu 2} . a'_{\nu 2}$, $a_{\nu 2} = a_{\nu 3} . a'_{\nu 3}$, . . . $a_{\nu\nu-1} = a_{\nu\nu} . a'_{\nu\nu}$

. . . .

We now define $(A_o,\cdot)$ by stipulating that, from all the above equalities, only $a_{oo} = a_{oo}^2$ holds in $(A_o,\cdot)$. With $(A_o,\cdot) \subseteq (A,\cdot)$ so defined, it is obvious that 2) and 9) hold. $a_{oo} \in A_o$ is not strictly prime in $(A_o,\cdot)$, while, for $\nu \neq 0$, all $a_{\nu\nu} \in A_o$ are strictly prime in both $(A_o,\cdot)$ and $(A,\cdot)$, so that 1), also, holds. Hence $\{1. b), 2), 9)\}$ holds. But, obviously, $l_{A_o}(a_{oo}) = +\infty$, hence, $L^f_{A_o} \subset A$. |

So arises the problem: *Given* $(A_o,\cdot) \subseteq (A,\cdot)$, *to find a set of conditions, as weak as possible but containing 9), sufficient for* $A = L^f_{A_o}$, *i. e. for 12).*

Trying to find an answer to this problem, we were, naturaly led to an apparently new notion, which, in turn, proved also useful in the alucidation of the initial problem of finding sufficient conditions for $(A_o,\cdot) \underset{f,g}{\subseteq} (A,\cdot)$. This notion, which is presented below, seems, also, to us interesting in itself.

IX. THE PROPERTY OF HAVING A FINITE SET OF DIVISORS

Definition 3. Let $(A,\cdot)$ be a hgr. Then, an element $a \in A$ will be said to have the *«finite divisor»* (abbreviation: *«f. d.»*) property in $(A,\cdot)$, iff the set of all its divisors in $(A,\cdot)$ is finite (possibly, empty). A subset B of A will be said to have the *«finite divisor»* (same abbreviation) property in $(A,\cdot)$, iff all elements of B have this property in $(A,\cdot)$. *Special case:* $B = A$; we shall, then, say simply that A, or $(A,\cdot)$, has the finite divisor property.

Examples. a) The multiplicative hgr $(\mathbb{N}^*,\cdot)$ has the f. d. property. b) Also, the additive hgr $(\mathbb{N}, +)$ has it. c) Also, any finite hgr. d) If an infinite multiplicative hgr has an one-sided «zero», this zero has not the f. d. property; for instance, 0 in $(\mathbb{N},\cdot)$. e) If, in a hgr $(A,\cdot)$, there is an one - sided identity and an infinite number of elements having one-sided inverses, no element of A has the f. d. property. This is the case, for instance, for all infinite fields.

Proposition 13. *If* $(A,\cdot)$ *has the f. d. property, then, every (possibly existing) element of* A *of length* $+\infty$ *has, in* $(A,\cdot)$, *at least one divisor of length* $+\infty$.

Proof. Suppose $\gamma_o \in A$ and $l(\gamma_o) = +\infty$. Then, γ_o has at least one divisor in $(A,\cdot)$. Since A has the f. d. property, the set Δ of all the divisors of γ_o in $(A,\cdot)$ is finite, say $\Delta = \{\gamma_{o1}, \ldots, \gamma_{o\nu}\}$. If all the elements of Δ had finite length, one (or more) of them would have length maximum, say $\mu + 1$. No divisor chain in $(A,\cdot)$ with first term belonging to Δ would then have length $> \mu + 1$. There would not exist, then, in $(A,\cdot)$, a divisor chain with first term γ_o and length $> \mu + 2$. The length of γ_o would, then, be $\mu + 2$, contrary to the hypothesis that it is $+\infty$. Hence, at least one of the element of Δ has length $+\infty$. ❙

C o r o l l a r y. *If* $(A,\cdot)$ *has the f. d. property, then, every (possibly existing) element of A of length* $+\infty$ *is the first term of an infinite divisor chain, in* $(A,\cdot)$, *all terms of which have length* $+\infty$.

An immediate consequence of this corollary is

Proposition 14. *In a hgr* $(A,\cdot)$ *having the f. d. property, the hypothesis that all divisor chains are finite implies that* $A = L^f$ (i. e. that all elements of A have finite length).

Proof. Because otherwise there would exist, by the preceding corollary, at least one infinite divisor chain in $(A,\cdot)$, contrary to the hypothesis. ❙

Proposition 14 gives a sufficient (but, obviously, not necessary) condition for $A = L^f$.

Proposition 15. *Let* $(A_o,\cdot) \subseteq (A,\cdot)$. *If* $A - A_o$ *has the f. d. property in* $(A,\cdot)$, *every (possibly existing) element of* $A - A_o$ *which has length* $+\infty$ *over* $(A_o,\cdot)$ has at least one divisor of length $+\infty$ over $(A_o,\cdot)$.

Proof. Let $\gamma_o \in A - A_o$ and $l_{A_o}(\gamma_o) = +\infty$. Then, the set Δ of the divisors of γ_o in $(A,\cdot)$ is non empty and, because of the f. d. property, finite. If $\gamma_1 \in \Delta$ and has not length $+\infty$ over $(A_o,\cdot)$, either $l_{A_o}(\gamma_1)$ is not defined in $(A,\cdot)$, or it is finite. However, since $l_{A_o}(\gamma_o) = +\infty$, it is not possible that, for all $\gamma_1 \in \Delta$, $l_{A_0}(\gamma_1)$ is not defined. The rest of the proof follows closely the proof of proposition 13. ❙

A corollary of this proposition is

Proposition 16. *Let* $(A_0, \cdot) \underline{\subset} (A, \cdot)$. *If* $A - A_0$ *has the f.d. property in* $(A, \cdot)$ *and if 4.b) holds, every (possibly existing) element* $a \in A$ *with* $l_{A_0}(a) = +\infty$ *is the first term of an infinite divisor chain, in* $(A, \cdot)$, *all the terms of which have length* $+\infty$ *over* $(A_0, \cdot)$.

Proof. We, first, observe that 4.b) implies that all elements of A_0 have length 1 over $(A_0, \cdot)$. Let, now, $\gamma_0 \in A$ satisfy $l_{A_0}(\gamma_0) = +\infty$; then, necessarily, $\gamma_0 \in A - A_0$. An obvious induction completes the proof. ❙

This proposition entails

Proposition 17. *Let* $(A_0, \cdot) \underline{\subset} (A, \cdot)$ *and suppose that* $A - A_0$ *has the f.d. property in* $(A, \cdot)$ *and 4.b), holds. Then, if 9) holds, no element of A has length* $+\infty$ *over* $(A_0, \cdot)$.

Proof. If $\gamma_0 \in A$ with $l_{A_0}(\gamma_0) = +\infty$, there would exist, according to proposition 16, an infinite divisor chain in $(A, \cdot)$ with all its terms γ_ν satisfying $l_{A_0}(\gamma_\nu) = +\infty$ and, hence, $\gamma_\nu \in A - A_0$. This would contradict 9). ❙

The following simple remark will be used below: *Let* $(A_0, \cdot) \underline{\subset} (A, \cdot)$. *If no element of* $A - A_0$ *is strictly prime in* $(A, \cdot)$, *every finite divisor chain in* $(A, \cdot)$ *can be extended, in order to become either a chain finite over* $(A_0, \cdot)$ *or an infinite chain.*

We are in a position, now, to formulate and prove:

Proposition 18. *Let* $(A_0, \cdot) \underline{\subset} (A, \cdot)$. *The following conditions form a set of sufficient conditions for* $A = L^f_{A_0}$: *C.1)* $A - A_0$ *has the property f.d. in* $(A, \cdot)$. *C.2) No element of* $A - A_0$ *is strictly prime in* $(A, \cdot)$. *C.3) 4.b), and C.4) 9).*

Proof. Suppositions C.1), C.3) and C.4) imply the validity of the assertion of proposition 17. This and supposition C.2), which implies the validity of the assertion of the last remark, complete the proof. ❙

C o r o l l a r y. *Let* $(A_0, \cdot) \underline{\subset} (A, \cdot)$. $\{$*C.1), C.2), C.3), C.4)*$\}$ *is a set of sufficient conditions for* $(A_0, \cdot) \underset{g}{\underline{\subset}} (A, \cdot)$.

Proposition 18 throws some light on the initial problem of finding sufficient conditions for $(A_0, \cdot) \underset{f,g}{\underline{\subset}} (A, \cdot)$. In fact, the set $\{$2), 3.b), 4.b), 12)$\}$

of sufficient conditions for $(A_o, \cdot) \underset{f,g}{\subseteq} (A, \cdot)$ can, after proposition 19, be replaced by the set $\{2), 3.b), 4.b), C.1), C.2), 9)\}$. But, 2) clearly implies C. 1) and C. 2), hence, the last set can be replaced by the set $\{2), 3.b), 4.b), 9)\}$. Finally, conditions 3. b) and 4. b) can be condensed in the condition: *If* $a_o \in A_o$, $b \cdot c = a_o$ *in* $(A, \cdot)$ *implies* $b \cdot c = a_o$ *in* $(A_o, \cdot)$. We chall denote it by C. 5).

According to proposition 9, 2), 3. b), 4. b) and 9) are, also, necessary conditions for $(A_o, \cdot) \underset{f,g}{\subseteq} (A, \cdot)$. Hence, $\{2), C.5), 9)\}$ *is a set of necessary and sufficient conditions for* $(A_o, \cdot) \underset{f,g}{\subseteq} (A, \cdot)$.

Note. It is easy, now, to explain the lapse in lemma 1. 4, in R.H.B.; «reversing» logically C. 5) seems to give 1. b.) [(i) in his notation], but this is not correct, on closer examination.

ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἀνακοίνωσις ἀποτελεῖ συνέχειαν ἀνακοινώσεως γενομένης κατὰ τὴν συνεδρίαν τῆς 8ης Μαΐου 1969 εἰς τὴν Ἀκαδημίαν Ἀθηνῶν. Ἐπιλύεται καὶ διευκρινίζεται ἐδῶ τὸ πρόβλημα τῆς ἀναζητήσεως ἱκανῶν συνθηκῶν διὰ τὴν ἐλευθέραν παραγωγὴν μιᾶς δομῆς, μιᾶς μερικῶς ὡρισμένης ἐσωτερικῆς πράξεως ὑπὸ μιᾶς ὑποδομῆς αὐτῆς. Εἰσάγεται, ἐπίσης, ἡ νέα ἔννοια τοῦ «πεπερασμένου συνόλου διαιρετῶν».

★

Ὁ Ἀκαδημαϊκὸς κ. **Κ. Π. Παπαϊωάννου** ἀνακοινῶν τὴν ὡς ἄνω ἐργασίαν εἶπε τὰ ἑξῆς:

Ἡ παροῦσα ἀνακοίνωσις ἀποτελεῖ συνέχειαν τῆς ἀνακοινώσεως τοῦ καθηγητοῦ κ. Ζερβοῦ, τὴν ὁποίαν παρουσιάσαμεν κατὰ τὴν συνεδρίαν τῆς 8ης Μαΐου 1969. Ἐπιλύεται εἰς αὐτὴν πρόβλημα, τὸ ὁποῖον ἀπησχόλησε διακεκριμένους ξένους μαθηματικούς. Ἐπίσης, εἰσάγεται μία ἐνδιαφέρουσα νέα ἔννοια. Ἡ ἐφαρμογή της διαφωτίζει ἰδιαιτέρως τὰς συνθήκας τοῦ ἐν λόγῳ προβλήματος.