ΓΕΩΜΕΤΡΙΑ— **On maxima or minima of a group of plane curves***, *by*
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Let an initial curve C^0 with equation $f(x, y)=0$ and two plane coordinate systems (x, y) and (x', y') be given. These systems are supposed to be analytically and geometrically equivalent[1]. Let the formulae of transformation from one system to the other be

$$\left.\begin{array}{l} x'=a_1x+b_1y \\ y'=a_2x+b_2y \end{array}\right\}, \qquad \begin{array}{l} x=\begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \Big/ \Delta \\ y=\begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \Big/ \Delta \end{array} \tag{1}$$

where $\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $\Delta \neq 0$. The y maximum or minimum, if it exists, of $f(x, y)=0$ is denoted by M_{0y} (fig. 1). To determine the values of the coordi-

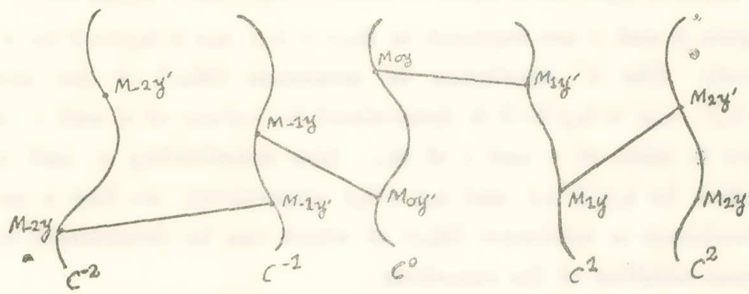

Fig. 1.

nates x and y of M_{0y} suffices to solve the simultaneous equations:

$$f(x, y)=0, \qquad f_x(x, y)=0 \tag{2}$$

If $f_x(x, y) \equiv \frac{\partial}{\partial x} f(x, y)$ is symbolized by $\varphi_1(x, y)$ then the above relations may be written:

$$f(x, y)=0, \qquad \varphi_1(x, y)=0 \tag{2a}$$

Let x and y in $f(x, y)=0$ be substituted by x' and y' respectively. Then $f(x, y)=0$ and $f(x', y')=0$ are essentially the same analytic relations but represent geometrically equivalent plane curves[2].

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* ΧΡΗΣΤΟΥ Β. ΓΚΛΑΒΑ, Μέγιστα ἢ ἐλάχιστα ὁμάδος ἐπιπέδων καμπυλῶν.

[1] C. B. GLAVAS, The principle of geometrical equivalence and some of its consequences to the theory of curves, *Proceedings of the Academy of Athens* 32 (1957), 122 - 24.

[2] Loc. cit.

Clearly the values of x' and y' of the y' maximum or minimum of $f(x', y')=0$, denoted by M_{1y}', are the same respectively with those of x and y of M_{0y}. Of course the points M_{0y} and M_{1y}' are different.

Now let x' and y' in $f(x', y')$ be substituted by their equals in formulae (1). Then we find the curve $f(a_1x+b_1y,\ a_2x+b_2y)=0$. The y maximum ($M_{1y}$) of this curve is determined by the solution of the equations:

$$f(a_1x+b_1y\,,\ a_2x+b_2y)=0\,,\qquad f_x(a_1x+b_1y\,,\ a_2x+b_2y)=0 \tag{3}$$

Let $f_y(x, y)\equiv\dfrac{\partial}{\partial y}f(x, y)$ be symbolized by $\varphi_2(x, y)$. Then:

$$f_x(a_1x+b_1y\,,\ a_2x+b_2y)=$$
$$=\varphi_1(a_1x+b_1y\,,\ a_2x+b_2y)a_1+\varphi_2(a_1x+b_1y\,,\ a_2x+b_2y)a_2=0$$

Therefore the relations (3) become:

$$\left.\begin{aligned}&f(a_1x+b_1y\,,\ a_2x+b_2y)=0\\&\varphi_1(a_1x+b_1y\,,\ a_2x+b_2y)a_1+\varphi_2(a_1x+b_1y\,,\ a_2x+b_2y)a_2=0\end{aligned}\right\} \tag{3a}$$

Again x and y are replaced in $f(a_1x+b_1y,\ a_2x+b_2y)=0$ by x' and y' respectively. The y' maximum or minimum (M_{2y}') of the new curve $f(a_1x'+b_1y',\ a_2x'+b_2y')=0$ is determined by values of x' and y' equal respectively to those of x and y of M_{1y}. Also substituting x' and y' in the last relation by a_1x+b_1y and a_2x+b_2y respectively we find a new curve the y maximum or minimum (M_2y) of which can be determined by the simultaneous solution of the equations

$$\left.\begin{aligned}&f\Big((a_1^2+b_1a_2)x+(a_1b_1+b_1b_2)y\,,\ (a_1a_2+a_2b_2)x+(a_2b_1+b_2^2)y\Big)=0\\&\varphi_1\Big(A_1(x,y), A_2(x,y)\Big)\,(a_1^2+b_1a_2)+\varphi_2\Big((A_1(x,y),\ A_2(x,y)\Big)\,(a_1a_2+a_2b_2)=0\end{aligned}\right| \tag{4}$$

where $A_1(x, y)$ and $A_2(x, y)$ are equal respectively to the first and second expressions inside the parenthesis of the first of the above two equations.

This process may be repeated indefinitely. But if we reverse the process and x and y are substituted in $f(x, y)=0$ by their equals in formulae (1), then we get the equation:

$$f\left(\begin{vmatrix}x' & b_1\\ y' & b_2\end{vmatrix}\Big/\Delta\,,\ \begin{vmatrix}a_1 & x'\\ a_2 & y'\end{vmatrix}\Big/\Delta\right)=0$$

It is well understood that the latter equation represents the same curve C^0 with $f(x, y)=0$ but in the coordinate system (x', y'). The y' maximum or mi- (M_{0y}') of C^0, if it exists, is determined by the simultaneous solution of the following equations:

$$\left.\begin{aligned} & f\left(\begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \Big/ \Delta,\ \begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \Big/ \Delta\right) = 0 \\ & \varphi_1\left(\begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \Big/ \Delta,\ \begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \Big/ \Delta\right) \cdot \frac{b_2}{\Delta} + \varphi_2\left(\begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \Big/ \Delta,\ \begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \Big/ \Delta\right) \cdot \frac{-a_2}{\Delta} = 0 \end{aligned}\right\} \quad (5)$$

Let x' and y' be substituted by x and y respectively in the first of the equations (5). Then the new equation $f\left(\begin{vmatrix} x & b_1 \\ y & b_2 \end{vmatrix} \Big/ \Delta,\ \begin{vmatrix} a_1 & x \\ a_2 & y \end{vmatrix} \Big/ \Delta\right) = 0$ represents a curve C^{-1} geometrically equivalent to the curve C^0. As before the y maximum or minimum (M_{-1y}) of C^{-1} corresponds to the same numerical values of x and y with those of x' and y' respectively of M_{0y}'.

Now let x and y in the latter equation be substituted by their equals in the formulae of transformation (1). Then the y' maximum or minimum (M_{-1y}') of C^{-1} can be determined by the solution of the following simultaneous equations

$$\left.\begin{aligned} & f\left(\begin{vmatrix} \begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} & b_1 \\ \begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} & b_2 \end{vmatrix} \Big/ \Delta^2,\ \begin{vmatrix} a_1 & \begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \\ a_2 & \begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \end{vmatrix} \Big/ \Delta^2\right) = 0 \\ & \varphi_1\left(A_1'(x', y'),\ A_2(x', y')\right) \cdot \left(\frac{b_2^2 + b_1 a_2}{\Delta^2}\right) + \\ & + \varphi_2\left(A_1'(x', y'),\ A_2'(x', y')\right) \cdot \left(\frac{-a_1 a_2 - a_2 b_2}{\Delta^2}\right) = 0 \end{aligned}\right\} \quad (6)$$

where $A_1'(x', y')$ and $A_2'(x', y')$ represent the first and second expressions respectively within the parenthesis of the first of the above equations (6).

It is now clear that one can continue in the same way and write the equations which determine the maxima or minima M_{-2y}, M_{-2y}', M_{-3y}, M_{-3y}', ... The first observation is that each of the pairs of points (M_{0y}, M_{1y}'), (M_{1y}, M_{2y}'), (M_{2y}, M_{3y}'), ..., $(M_{ny}, M_{(n+1)y}')$, ... and (M_{0y}', M_{-1y}) (M_{-1y}', M_{-2y}), (M_{-2y}', M_{-3y}), ..., $(M_{-ny}', M_{-(n+1)y})$ ... represent maximum or minimum points which are determined by the same numerical values of the coordinates x, y and x', y' respectively. It now remains to investigate the rest of the maximum or minimum points.

Looking at the systems of equations (3), (4), ..., and (5), (6), ..., which determine the values of the maximum or minimum points M_{1y}, M_{2y}, ... and M_{0y}', M_{-1y}', ... respectively we see that the first of equations (3) is produced from the first of (2) by the substitution in the latter of x, y by $a_1x + b_1y$ and $a_2x + b_2y$ respectively. Similarly the first of equations (4) is produced from the first of (3) by the same type of substitution and so similar remarks hold for the second equations of the systems (5), (6),.... The first of equations (5) can be found from the first of (2) by the substitution of x, y by $\begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \Big/ \Delta$, $\begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \Big/ \Delta$ respectively. Also (6) can be found from (5) by the same type of substitution and so on and so forth.

The set of the first equations of the systems (2), (3), (4),... and (5), (6),... constitutes a cyclic group under the following law of composition symbolized by a small circle «o».

Given two of the above equations, say $f\big(A_1(x, y), A_2(x, y)\big) = 0$ and $f\big(B_1(x, y), B_2(x, y)\big) = 0$, then:

$$\left[f\big((A_1(x, y), A_2(x, y)\big) = 0\right] \circ \left[f\big(B_1(x, y), B_2(x, y)\big) = 0\right] =$$
$$= \left[f\big(A_1(B_1, B_2), A_2(B_1, B_2)\big) = 0\right]$$

In other words x and y in $A_1(x, y)$, $A_2(x, y)$ are replaced by $B_1(x, y)$, $B_2(x, y)$ respectively. The proof that the above set is a cyclic group can be made on the same lines with the one already given[1]. The identity element in this group is the initial curve $f(x, y) = 0$.

Let the second equation of (2a) be written as $\varphi_1(x, y) \cdot 1 + \varphi_2(x,y) \cdot 0 = 0$ Then the second equations of (2), (3), (4),... can be found from their preceding ones if in the latter x and y in φ_1 and φ_2 are substituted by $a_1x + b_1y$ and $a_2x + b_2y$ respectively. The coefficients of φ_1 and φ_2 in each of (2), (3a) (4),... are the product of the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ by the matrix of the coefficients of φ_1 and φ_2 of the preceding equation. For example the coefficients of φ_1 and φ_2 in (4) are the product:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1^2 + b_1a_2 \\ a_1a_2 + a_2b_2 \end{pmatrix},$$

where a_1 and a_2 are the coefficients of φ_1 and φ_2 in (3a).

Similar remarks hold for the second equations of the systems (5) 6),... . Here φ_1 and φ_2 of each equation can be found from φ_1 and φ_2 of its preceding one if the variables x and y in the latter are replaced by

[1] GLAVAS, op. cit., p. 126-28.

$\begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \Big/ \Delta$ and $\begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \Big/ \Delta$ respectively. The coefficients of φ_1 and φ_2 in (5), (6), ... are the «inverse» of those in (3a), (4), ... respectively. Let equations (1) be written as:

$$\left.\begin{aligned} x' &= a_1x + b_1y \\ y' &= a_2x + b_2y \end{aligned}\right| , \quad \left.\begin{aligned} x &= \frac{b_2}{\Delta} x' - \frac{b_1}{\Delta} y' \\ y &= \frac{-a_2}{\Delta} x' + \frac{a_1}{\Delta} y' \end{aligned}\right\} \tag{1a}$$

One can see that to go from the first system of (1a) to the second x' and y' should be interchanged in the former by x and y respectively. Also the coefficients a_1, b_1, a_2, b_2, in the first system of (1a) should be substituted by b_2/Δ, $-b_1/\Delta$, $-a_2/\Delta$ and a_1/Δ respectively.

Now the coefficients of φ_1 and φ_2 in (3a) are a_1 and a_2 respectively. The coefficients of φ_1 and φ_2 in (5) are b_2/Δ and $-a_2/\Delta$. The meaning of the term inverse coefficients is based on the fact that the two transformations of (1a) are such that if one replaces x and y in the first (or x', y' in the second) by x' and y' (by x and y) respectively taken from the second one (from the first one) then the result is the identity transformation $x'=x$ and $y'=y$ (or $x=x'$, $y=y'$)[1].

It is now easy to realize that one can find all second equations of the systems in question if one knows the initial curve $f(x, y)=0$ (and consequently $\varphi_1(x, y)$ and $\varphi_2(x, y)$) and the transformations (1a).

Let the first equation of (2a) be solved with respect to y. Let the solution be $y=\sigma(x)$. If y is substituted by $\sigma(x)$ in the second equation of (2a), i. e. $\varphi_1(x, y)=0$, then $\varphi_1\big((x, \sigma(x)\big)=0$. Suppose that one root of this equation is $x=x_o$. Then by substitution $y_o=\sigma(x_o)$. Therefore $\big(x_o, \sigma(x_o)\big)$ are the values for which (1a) may have maximum or minimum.

It now follows that the first equation (3a) solved for a_2x+b_2y will give:

$$a_2x + b_2y = \sigma(a_1x + b_1y)$$

Substituting this value of a_2x+b_2y in the second equation (3a) we shall get:

$$\varphi_1\big(a_1x + b_1y, \sigma(a_1x + b_1y)\big)\, a_1 + \varphi_2\, \big(a_1x + b_1y, \sigma(a_1x + b_1y)\big)\, a_2 = 0 \tag{A}$$

[1] A. D. Campbell, Advanced analytic geometry, New York, John Wiley and Sons, Inc., 1938, p. 14-15.

Suppose that the last equation has been solved for $a_1x + b_1y$. Then its solution will be a function of the coefficients a_1 and a_2. Hence we may write:

$$a_1x + b_1y = \varphi(a_1, a_2)$$

To find therefore the values x and y of the M_{1y} maximun or minimum suffices to solve the following system of equations:

$$a_1x + b_1y = \varphi(a_1, a_2), \quad a_2x + b_2y = \sigma\big(\varphi(a_1, a_2)\big) \tag{2a)$'$}$$

It is not difficult now to see that the next maximum or minimum will be determined by the system:

$$\left.\begin{aligned} (a_1^2 + b_1a_2)\,x + (a_1b_1 + b_1b_2)\,y &= \varphi(a_1^2 + b_1a_2,\ a_1a_2 + a_2b_2) \\ (a_1a_2 + a_2b_2)\,x + (a_2b_1 + b_2^2)\,y &= \sigma\big(\varphi(a_1^2 + b_1a_2,\ a_1a_2 + a_2b_2)\big) \end{aligned}\right\} \quad (3a)'$$

This process may be continued on indefinitely.

In a similar way one can solve the systems of equations (5), (6),.... The solutions of (5) and (6) may be written:

$$\left.\begin{aligned} &\begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \Big/ \Delta = \varphi\left(\frac{b_2}{\Delta}, \frac{-a_2}{\Delta}\right) \\ &\begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \Big/ \Delta = \sigma\left[\varphi\left(\frac{b_2}{\Delta}, \frac{-a_2}{\Delta}\right)\right] \end{aligned}\right\}, \quad \left.\begin{aligned} &\begin{Vmatrix} \begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} & b_2 \\ \begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} & b_2 \end{Vmatrix} \Big/ \Delta^2 = \varphi\left(\frac{b_2^2 + b_1a_2}{\Delta^2}, \frac{-(a_1a_2 + a_2b_2)}{\Delta^2}\right) \\ &\begin{Vmatrix} a_1 & \begin{vmatrix} x' & b_1 \\ y' & b_2 \end{vmatrix} \\ a_2 & \begin{vmatrix} a_1 & x' \\ a_2 & y' \end{vmatrix} \end{Vmatrix} \Big/ \Delta^2 = \sigma\left[\varphi\left(\frac{b^2_2 + b_1a_2}{\Delta^2}, \frac{-(a_1a_2 + a_2b_2)}{\Delta^2}\right)\right] \end{aligned}\right\}$$

Or:

$$\left.\begin{aligned} b_2x' - b_1y' &= \Delta\varphi(b_2/\Delta, -a_2/\Delta) \\ -a_2x' + a_1y' &= \Delta\sigma\big(\varphi(b_2/\Delta, -a_2/\Delta)\big) \end{aligned}\right\} \quad (5)'$$

$$\left.\begin{aligned} (b_2^2 + a_2b_1)x' - (b_1b_2 + a_1a_2)\,y' &= \Delta^2\varphi\big((b_2^2 + b_1a_2)/\Delta^2,\ -(a_1a_2 + a_2b_2)/\Delta^2\big) \\ -(a_1a_2 + a_2b_2)x' + (a_1^2 + a_2b_1)y' &= \Delta^2\sigma\big(\varphi\big((b_2^2 + b_1a_2)/\Delta^2,\ -(a_1a_2 + a_2b_2)/\Delta^2\big)\big) \end{aligned}\right\} \quad (6)'$$

The solution of the system $(2a)'$ gives:

$$x_1 = \begin{vmatrix} \varphi(a_1, a_2) & b_1 \\ \sigma\big(\varphi(a_1, a_2)\big) & b_2 \end{vmatrix} \Big/ \Delta, \quad y_1 = \begin{vmatrix} a_1 & \varphi(a_1, a_2) \\ a_2 & \sigma\big(\varphi(a_1, a_2)\big) \end{vmatrix} \Big/ \Delta \qquad (2a)''$$

The system (3a)′ also gives:

$$\left.\begin{aligned} x_2 &= \begin{vmatrix} \varphi(a_1^2 + b_1a_2,\ a_1a_2 + a_2b_2) & (a_1b_1 + b_1b_2) \\ \sigma(\varphi(a_1^2 + b_1a_2,\ a_1a_2 + a_2b_2)) & (a_2b_1 + b_2^2) \end{vmatrix} \Big/ \Delta^2 \\ y_2 &= \begin{vmatrix} (a_1^2 + b_1a_2) & \varphi(a_1^2 + b_1a_2,\ a_1a_2 + a_2b_2) \\ (a_1a_2 + a_2b_2) & \sigma(\varphi(a_1^2 + b_1a_2,\ a_1a_2 + a_2b_2)) \end{vmatrix} \Big/ \Delta^2 \end{aligned}\right\} \quad (3a)''$$

It now becomes clear that to find x_n, y_n $(n=0, 1, 2, \ldots)$ suffices to calculate Δ^n. Then the two elements of the first column of Δ^n are the expressions inside the parenthesis of the functions φ and σ of the determinant of the numerator of x_n and y_n. The other two elements of the determinant of x_n are the two elements of the second column of Δ^n. Also the elements of the first column of Δ^n are the two elements of the first column of the determinant of the numerator of y_n.

Solving in a similar way equations (5)′ we find:

$$x_o' = \begin{vmatrix} \Delta\varphi(b_2/\Delta,\ -a_2/\Delta) & -b_1 \\ \Delta\sigma(\varphi(b_2/\Delta,\ -a_2/\Delta)) & a_1 \end{vmatrix} \Big/ \begin{vmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{vmatrix},$$

$$y_o' = \begin{vmatrix} b_2 & \Delta\varphi(b_2/\Delta,\ -a_2/\Delta) \\ -a_2 & \Delta\sigma(\varphi(b_2/\Delta,\ -a_2/\Delta)) \end{vmatrix} \Big/ \begin{vmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{vmatrix}$$

But the common denominators of x_o', y_o' are equal to Δ. We may therefore write x_o' and y_o' as follows:

$$\begin{aligned} x_o' &= \begin{vmatrix} \varphi(b_2/\Delta,\ -a_2/\Delta) & -b_1/\Delta \\ \sigma(\varphi(b_2/\Delta,\ -a_2/\Delta)) & a_1/\Delta \end{vmatrix} \Big/ \Delta^{-1}, \\ y_o' &= \begin{vmatrix} b_2/\Delta & \varphi(b_2/\Delta,\ -a_2/\Delta) \\ -a_2/\Delta & \sigma(\varphi(b_2/\Delta,\ -a_2/\Delta)) \end{vmatrix} \Big/ \Delta^{-1} \end{aligned} \qquad (5)''$$

Here $\Delta^{-1} = \begin{vmatrix} b_2/\Delta & -b_1/\Delta \\ -a_2/\Delta & a_1/\Delta \end{vmatrix}$, i.e. the elements of this determinant are the inverse of those of Δ. Also we find:

$$\left.\begin{aligned} x'_{-1} &= \begin{vmatrix} \Delta^2\varphi((b^2_2 + b_1a)_2/\Delta^2,\ -(a_1a_2 + a_2b_2)/\Delta^2) & -(b_1b_2 + a_1a_2) \\ \Delta^2\sigma(\varphi(b_2^2 + b_1a_2/\Delta^2,\ -(a_1a_2 + a_2b_2)/\Delta^2)) & (a_1^2 + a_2b_1 \end{vmatrix} \Big/ \Delta^2 \\ y'_{-1} &= \begin{vmatrix} b_2^2 + a_2b_1 & \Delta^2\varphi((b_2^2 + b_1a_2)/\Delta^2,\ -(a_1a_2 + a_2b_2)/\Delta^2) \\ -(a_1a_2 + a_2b_2) & \Delta^2\sigma(\varphi((b_2^2 + b_1a_2)/\Delta^2,\ -(a_1a_2 + a_2b_2)/\Delta^2)) \end{vmatrix} \Big/ \Delta^2 \end{aligned}\right\}$$

Or:

$$\left.\begin{aligned} x'_{-1} &= \begin{vmatrix} \varphi\left(b^2_2 + b_1a_2/\Delta^2, \ -(a_1a_2 + a_2b_2)/\Delta^2\right) & -(b_1b_2 + a_1a_2)/\Delta^2 \\ \sigma\left(\varphi\left(b_2^2 + b_1a_2/\Delta^2, \ -(a_1a_2 + a_2b_2)/\Delta^2\right)\right) & a_1^2 + a_2b_1/\Delta^2 \end{vmatrix} \Big/ \Delta^{-2} \\ y'_{-1} &= \begin{vmatrix} b_2^2 + a_2b_1/\Delta^2 & \varphi\left(b_2^2 + b_1a_2/\Delta^2, \ -(a_1a_2 + a_2b_2)/\Delta^2\right) \\ -(a_1a_2 + a_2b_2)/\Delta^2 & \sigma\left(\varphi\left(b_2^2 + b_1a_2/\Delta^2, \ -(a_1a_2 + a_2b_2)/\Delta^2\right)\right) \end{vmatrix} \Big/ \Delta^{-2} \end{aligned}\right\} \quad (6)''$$

The remarks made before for the values of x_n, y_n are valid for the values of $x'_{(n-1)}$, $y'_{(n-1)}$ $(n = 1, 2, 3, 4, \ldots)$. It is also easy to realize that the expressions for $x'_{(n-1)}$, $y'_{(n-1)}$ are the «inverse» of those for x_n, y_n. Really if one substitutes a_1, b_1, a_2, b_2 in the expressions for x_n, y_n by b_2/Δ, $-b_1/\Delta$, $-a_2/\Delta$, a_1/Δ respectively one finds $x'_{-(n-1)}$, $y'_{-(n-1)}$ and conversely.

We have thus far determined the coordinates of the points M_{0y}, M_{1y} M_{2y}, M_{3y}, ... and M_{0y}', M_{-1y}', M_{-2y}', ... Consider separately the set of the «abscissas» of these points. We shall prove that this set constitutes a group under a certain law of composition.

Since the value of each abscissa depends upon a power of the determinant Δ it is natural to define as law of composition «o» the one by which the combination of any two elements (abscissas) produces a new element whose denominator will be the ordinary product of the powers of Δ of the dominators of the given elements. Then the numerator is formed as it was seen before by the elements of the determinant of the denominators.

It now becomes clear that the set in question is closed under the above law of composition, is associative with a neutral element and every element has an inverse in the set. In addition it is commutative constituting a cyclic group whose elements are generated by the initial element x_o.

Here we shall show only that the combination of two inverse elements produces the neutral element. From (3a)'' and (6)'' we have:

$$x_2 o x'_{-1} = \begin{vmatrix} \varphi(1,0) & 0 \\ \sigma(\varphi(1,0)) & 1 \end{vmatrix} \Big/ \Delta^o = \varphi(1,0) \Big/ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \varphi(1,0)$$

The determinant $\Delta^o = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ can be produced from $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ if

$a_1 = 1$, $b_1 = 0$, $a_2 = 0$ and $b_2 = 1$. But $\varphi(a_1, a_2)$, now $\varphi(1, 0)$, is the solution of the equation (A) which now after the necessary substitutions becomes $\varphi_1(x, \sigma(x)) = 0$. The solution of the latter equation has been found to be x_o.

Therefore $\varphi(1,0)=x_0=x_2 o x'_{-1}$. The remarks made for the set of abscissas hold equally well for the «ordinates».

A corollary of the above discussion is that the combination under the previously defined law of composition of the values of the abscissas (or the ordinates) of the pairs of points $(M_{oy}, M_{oy'})$, $(M_{1y}, M_{-1y'})$, $(M_{2y}, M_{-2y'})$, ... is constant and equal to the value of the abscissa (or the ordinate) of $M_{oy'}$. For since the abscissa of M_{oy} is the identity element in the set of the abscissas (or the ordinates) it follows that the combination of the latter abscissa with the one of $M_{oy'}$ will produce again the abscissa (or ordinate) of $M_{oy'}$. Also the denominators of M_{1y} and $M_{-1y'}$ are Δ and Δ^{-2}. Therefore the combination of the corresponding abscissas will give an element with denominator Δ^{-1}, i. e. the abscissa of $M_{oy'}$, and so on and so forth.

The examination of maxima or minima of a group of curves has been made under the general linear form of analytic transformations (1) of the general coordinate systems (x, y) and (x', y'). The only known coordinate systems which are both analytically and geometrically equivalent are the cartesian orthogonal and oblique ones on the one hand and the polar and cathetic on the other. The formulae of transformation for the first are

$$x=x'\cos\varphi + y'\cos(\omega+\varphi), \qquad y=x'\sin\varphi + y'\sin(\omega+\varphi),$$

where ω and φ represent the angles between the two oblique axes and the oblique x' axis with the rectangular x axis respectively. The formula of transformation from the polar system to the cathetic one is $r=g\cos\vartheta$. Therefore it is clear that the above formulae of transformation are a special case of (1).

Example. — Let the equation of an initial curve be $x^2-2x+y=0$. Let the formulae of transformation be $x'=x$, $y'=x+y$ and $x=x'$, $y=y'-x'$. Taking the partial derivative of the given equation with respect to x and putting the result equal to zero we get $x=1$.

Substituting this value of x in $x^2-2x+y=0$ we get $y=1$. Therefore $x=1$ and $y=1$ are the values of maximum or minimum, if it exists, of the given curve. The geometrically equivalent of the given curve is $x'^2-2x'+y'=0$. Substituting x' and y' by x and $x+y$ respectively we finally take $x=1/2$ and $y=1/4$ as the values of the coordinates of the maximum or minimum of the latter curve. Continuing the process we find for the next maxima or minima $x=0$ and $y=0$, $x=-1/2$ and $y=1/4$, etc.

Now reversing the process we transform analytically $x^2-2x+y=0$ to the (x', y') system taking the equation $x'^2 - 2x' + y' = 0$. The values of the maximum or minimum of this curve is given by $x' = 3/2$, $y' = 9/4$. The geometrically equivalent of $x'^2 - 3x' + y' = 0$ is $x^2 - 3x + y = 0$. Transforming again analytically the latter curve to the (x', y') system we get $x'^2 - 4x' + y' = 0$. The values of the maximum or minimum of the latter curve are $x' = 2$ and $y' = 4$. Continuing this process we get as the values of the next maxima or minima $x' = 5/2$ and $y' = 25/4$, etc. The abscissas therefore of the above maxima or minima form the following sequences with the abscissa 1 of the maximum or minimum of the initial curve:

$$1 \begin{cases} 1/2,\ 0,\ -1/2,\ -1,\ -3/2, \ldots \\ 3/2,\ 2,\ 5/2,\ 3,\ 7/2, \ldots \end{cases}$$

Applying the Calculus of Finite Differences to the above sequences we find that the general terms are respectively $1 - \frac{x-1}{2}$ and $1 + \frac{x-1}{2}$.

Therefore we may write those sequences as:

$$1 \begin{cases} 1 - 1/2,\ 1 - 2 \cdot 1/2,\ 1 - 3 \cdot 1/2,\ 1 - 4 \cdot 1/2, \ldots,\ 1 - n \cdot 1/2, \ldots \\ 1 + 1/2,\ 1 + 2 \cdot 1/2,\ 1 + 3 \cdot 1/2,\ 1 + 4 \cdot 1/2, \ldots,\ 1 + n \cdot 1/2, \ldots \end{cases}$$

These sequences constitute a group under the law of composition $(1 + m/2) \circ (1 + n/2) = 1 + (n + m)/2$, where n, m are integers. It is easy to see that if $n = -m$, we get the identity element 1. Also all the other properties of a cyclic group are easily derived.

The values on the other hand of the ordinates of maxima or minima form the two sequences:

$$1 \begin{cases} 1/4,\ 0,\ 1/4,\ 1, \ldots \\ 9/4,\ 16/4,\ 25/4,\ 36/4, \ldots \end{cases}$$

Applying again the Calculus of Finite Differences to the second sequence we get as its general term $(x^2 + 4x + 4)/4$. Therefore to find the general term of the first one suffices to change x to its additive inverse, i.e. $-x$. Thus the general term of the second sequence is $\left((-x)^2 - 4x + 4\right)/4$.

We can thus write both sequences as:

$$1 \begin{cases} 1 - 1 + (-1/2)^2,\ 1 - 2 + (-2/2)^2,\ 1 - 3 + (-3/2)^2, \ldots,\ 1 - n + (-n/2)^2, \ldots \\ 1 + 1 + (1/2)^2,\ 1 + 2 + (2/2)^2,\ 1 + 3 + (3/2)^2, \ldots,\ 1 + n + (n/2)^2, \ldots \end{cases}$$

The law of composition of the above set of the values of the ordinates is $[1+m+(m/2)^2] \circ [1+n+(n/2)^2] = \left[1+(m+n)+\left(\frac{m+n}{2}\right)^2\right]$ where m and n are integers. If $m=-n$ then we get the identity element. That the latter set of ordinates constitutes a group is easily verified.

Remark.—It has been already proven that the two sets of the numerical values of the coordinates of the maximmm or minimum points of a set-group of plane curves constitute groups under a certain operation. Now consider a family of plane curves $f(x, y, a)=0$ where a is a parameter. The geometrically equivalent of the latter family of curves is $f(x', y', a)=0$. Transforming $f(x', y', a)=0$ to the (x, y) system through formulae (1) we get $f(a_1x+b_1y,\ a_2x+b_2y, a)=0$ the geometrically equivalent family of which is $f(a_1x'+b_1y',\ a_2x'+b_2y', a)=0$. Of course we can continue this process.

Reversing the above process we start from $f(x, y, a)=0$ and by substitution from (1) get $f\left(\begin{vmatrix} x & a_1 \\ y & a_2 \end{vmatrix} \Big/ \Delta,\ \begin{vmatrix} b_1 & x \\ b_2 & y \end{vmatrix} \Big/ \Delta\right)=0$. The geometrically equivalent of the latter curve is $f\left(\begin{vmatrix} x' & a_1 \\ y' & a_2 \end{vmatrix} \Big/ \Delta,\ \begin{vmatrix} b_1 & x \\ b_2 & y \end{vmatrix} \Big/ \Delta\right)=0$. Continuing in this way we produce a group of families of curves under a certain law of composition (p. 510).

The envelope of the family $f(x, y, a)=0$ can be determined if a is eliminated between the latter equation and $f_a(x, y, a)=0$. Let $\varphi(x, y)=0$ be the equation of that envelope. Now to find the envelope of $f(a_1x+b_1y,\ a_2x+b_2y, a)=0$ suffices to eliminate a between the latter equation and $f_a(a_1x+b_1y,\ a_2x+b_2y, a)=0$. Evidently the result of such an elimination is the equation $\varphi(a_1x+b_1y,\ a_2x+b_2y)=0$. It now becomes clear that the envelopes of the given group of families of curves constitute another group under the same operation (law of composition).

ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν ἀνακοίνωσιν τῆς 28 Φεβρουαρίου 1957 μὲ θέμα «Ἡ ἀρχὴ τῆς γεωμετρικῆς ἰσοδυναμίας καί τινες τῶν συνεπειῶν της εἰς τὴν θεωρίαν τῶν καμπυλῶν» ἐδείχθη, ὅτι εἶναι δυνατὸς ὁ σχηματισμὸς συνόλων ἐπιπέδων καμπυλῶν τὰ ὁποῖα νὰ συνιστοῦν ὁμάδας ὑπὸ ὡρισμένον νόμον συνθέσεως.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν διατυποῦται ἓν θεώρημα δυνάμει τοῦ ὁποίου τὰ σύνολα τῶν ἀριθμητικῶν τιμῶν τῶν «τετμημένων» ἢ «τεταγμένων» τῶν μεγίστων ἢ ἐλαχίστων σημείων ἑνὸς συνόλου - ὁμάδος ἐπιπέδων καμπυλῶν ἀποτελοῦν πάλιν ὁμάδας ὑπὸ ὡρισμένον νόμον συνθέσεως. Πρὸς τοῦτο λαμβάνεται ἐν ἀρχῇ μία ἀρχικὴ καμπύλη καὶ δύο γενικὰ ἐπίπεδα συστήματα συντεταγμένων. Ἐκ τῆς τελευταίας καμπύλης παράγεται δι' ἐπανειλημμένης ἐφαρμογῆς ἀναλυτικῶν καὶ γεωμετρικῶν μετασχηματισμῶν ἓν σύνολον καμπυλῶν, τὸ ὁποῖον συνιστᾷ ὁμάδα. Ἐν συνεχείᾳ προσδιορίζονται αἱ συντεταγμέναι τῶν μεγίστων ἢ ἐλαχίστων σημείων τῶν καμπυλῶν τῆς ὡς ἄνω ὁμάδος καὶ διαπιστοῦται ὅτι αἱ ἀριθμητικαὶ τιμαὶ τῶν «τετμημένων» καὶ «τεταγμένων» τῶν μεγίστων ἢ ἐλαχίστων τούτων σημείων ἀποτελοῦν κυκλικὰς ὁμάδας μὲ ἀρχικὰ στοιχεῖα (στοιχεῖα ταὐτότητος) ἀντιστοίχως τὴν ἀριθμητικὴν τιμὴν τῆς «τετμημένης» καὶ «τεταγμένης» τοῦ μεγίστου ἢ ἐλαχίστου σημείου τῆς ἀρχικῆς καμπύλης.

Ἐν κατακλεῖδι δίδεται ἓν συγκεκριμένον παράδειγμα, ὅπου τῇ βοηθείᾳ τοῦ Λογισμοῦ πεπερασμένων διαφορῶν διαφαίνεται ἡ ὕπαρξις τῶν ἐν λόγῳ ὁμάδων τῶν συντεταγμένων τῶν μεγίστων ἢ ἐλαχίστων σημείων.

Τέλος ἀποδεικνύεται, ὅτι καὶ τὸ σύνολον τῶν περιβαλλουσῶν μιᾶς δοθείσης ὁμάδος, συνισταμένης ἀπὸ οἰκογενείας καμπυλῶν, συνιστᾷ καὶ τοῦτο ὁμάδα ὑπὸ τὸν αὐτὸν νόμον συνθέσεως.

ΧΗΜΕΙΑ.— **La configuration électronique des terres rares,** *par* ***Paul Sakellaridis****. Ἀνεκοινώθη ὑπὸ τοῦ κ. Ἐμμ. Ἐμμανουήλ.

Les terres rares appartiennent aux éléments dits de transition ; ils contiennent en même temps des couches f et d incomplètes, ces couches sont les couches 4f et 5d.

L'étude des spectres d'absorption et d'émission optiques et de rayons X des éléments permet en principe de distinguer leurs différentes orbitales électroniques et de calculer le nombre d'électrons qui se trouvent sur chacune d'elles.

Dans le cas des terres rares, à cause de la complexité de leurs spectres optiques et de leurs spectres de rayons X, quelques-uns seulement de ces spectres ont été partiellement étudiés, tandis que la plupart restent à étudier ou à classifier. Par conséquent, les configurations électroniques proposées par différents auteurs pour l'ensemble de ces éléments, l'ont été en grande partie par extrapolation. Nous signalons ici (tableau N° 1) la configuration électronique proposée par W. F. Meggers[1] d'après les données expérimentales actuelles, qui proviennent surtout des spectres optiques pour des éléments voisins et pour quelques-unes des terres rares elles-mêmes.

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