ΑΝΩΤΕΡΑ ΜΑΘΗΜΑΤΙΚΑ.– **On the singularities of a system of differential equations, where the time figures explicitly,** *by* ***Dem. G. Magiros****. Ἀνεκοινώθη ὑπὸ τοῦ κ. Βασ. Αἰγινήτου.

1. In my first paper of this volume is referred, without any explanation, that the singular points of the system:

$$\frac{du_1}{dt} = \varepsilon F_1(u_1, u_2, t), \qquad \frac{du_2}{dt} = \varepsilon F_2(u_1, u_2, t), \tag{1}$$

fulfill the equations:

$$A_0(u_1, u_2) = 0, \qquad C_0(u_1, u_2) = 0, \tag{2}$$

where A_0 and C_0 are the first terms of the Fourier series expansions of the functions F_1 and F_2 respectively.

In the following we discuss the above subject. We restrict ourselves to the system (1), although the theory can be applied to more general systems.

A *constant solution* $\{u_1, u_2\}$, of the system (1), determines a point in the u_1, u_2 - plane independent of the time t, and this point is, by definition, a *singular point* of the system (1).

In the following we try to find how to determine approximately the singular points of the system (1).

2. Suppose we are given that the functions F_1 and F_2 fulfill the *expansibility conditions* into Fourier series in t[1], according to which we have:

$$\left.\begin{aligned} \frac{du_1}{dt} &= \varepsilon\{A_0 + A_1 \cos t + B_1 \sin t + \ldots + A_m \cos mt + B_m \sin mt + \ldots\} \\ \frac{du_2}{dt} &= \varepsilon\{C_0 + C_1 \cos t + D_1 \sin t + \ldots + C_m \cos mt + D_m \sin mt + \ldots\} \end{aligned}\right\} \tag{3}$$

where the coefficients A, B, C, D are functions of u_1, and u_2.

By the above we mean that F_1 and F_2 fulfill the conditions of the *Fourier's theorem*[1], then:

a) F_1, F_2 are periodic in t of period, say, 2π,

b) F_1, F_2 are integrable, say Riemann - integrable, in $[t_0, t_0 + 2\pi]$,

c) F_1, F_2 have limited total fluctuations in $[t_0, t_0 + 2\pi]$, and

d) the coefficients in (3) can be found according to the standard manner.

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* ΔΗΜ. ΜΑΓΕΙΡΟΥ, Ἐπὶ τῶν ἀνωμάλων σημείων διαφορικοῦ συστήματος, ὅπου ὁ χρόνος εἰσέρχεται ἐκπεφρασμένως.

Let us take the system:

$$\frac{du_1}{dt} = \varepsilon A_o(u_1, u_2), \qquad \frac{du_2}{dt} = \varepsilon C_o(u_1, u_2), \tag{4}$$

where the time t does not figure explicitly.

Each of the above systems accepts a unique solution $\{u_1, u_2\}$, which assumes given values: $\{u_{1o}, u_{2o}\}$ at $t = t_o$, provided that the functions in their right-hand sides fulfill a Lipschitz condition, when their arguments are restricted to be in the domain:

$$D: \qquad u_1 - u_{1o} | \leqq k_1, \qquad | u_2 - u_{2o} | \leqq k_2, \qquad | t - t_o | < T.$$

Apply Picard's method of succesive approximations for calculation of the solution $\{\bar{u}_1, \bar{u}_2\}$ of the system (4), by taking as zeroth approximation arbitrary conditions $\{\bar{u}_{1o}, \bar{u}_{2o}\}$. The successive approximations, which converge to the solution $\{\bar{u}_1, \bar{u}_2\}$ of (4), are:

$$\left.\begin{array}{l}
\overset{(1)}{u}_1 = \bar{u}_{1o} + \varepsilon \int_{t_o}^{t} A_o(\bar{u}_{1o}, \bar{u}_{2o})\, dt, \qquad \overset{(1)}{\bar{u}}_2 = \bar{u}_{2o} + \varepsilon \int_{t_o}^{t} C_o(\bar{u}_{1o}, \bar{u}_{2o})\, dt, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{(n)}{u}_1 = \bar{u}_{1o} + \varepsilon \int_{t_o}^{t} A_o(\overset{(n-1)}{\bar{u}_1}, \overset{(n-1)}{\bar{u}_2})\, dt, \qquad \overset{(n)}{\bar{u}}_2 = \bar{u}_{2o} + \varepsilon \int_{t_o}^{t} C_o(\overset{(n-1)}{\bar{u}_1}, \overset{(n-1)}{\bar{u}_2})\, dt
\end{array}\right\} \tag{5}$$

Apply also the above method for calculation of the solution $\{u_1, u_2\}$ of (3) by taking as zeroth approximarion arbitrary initial conditions $\{u_{1o}, u_{2o}\}$ at $t = t_o$. The successive approximations are:

$$\left.\begin{array}{l}
\overset{(1)}{u}_1 = u_{1o} + \varepsilon \int_{t_o}^{t} A_o(u_{1o}, u_{2o})\, dt + \varepsilon \int_{t_o}^{t} A_1(u_{1o}, u_{2o}) \cos t\, dt + \varepsilon \int_{t_o}^{t} B_1(u_{1o}, u_{2o}) \sin t\, dt + \ldots \\
\overset{(1)}{u}_2 = u_{2o} + \varepsilon \int_{t_o}^{t} C_o(u_{1o}, u_{2o})\, dt + \varepsilon \int_{t_o}^{t} C_1(u_{1o}, u_{2o}) \cos t\, dt + \varepsilon \int_{t_o}^{t} D_1(u_{1o}, u_{2o}) \sin t\, dt + \ldots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{(n)}{u}_1 = u_{1o} + \varepsilon \int_{t_o}^{t} A_o(\overset{(n-1)}{u_1}, \overset{(n-1)}{u_2})\, dt + \varepsilon \int_{t_o}^{t} A_1(\overset{(n-1)}{u_1}, \overset{(n-1)}{u_2}) \cos t\, dt + \varepsilon \int_{t_o}^{t} B_1(\overset{(n-1)}{u_1}, \overset{(n-1)}{u_2}) \sin t\, dt + \ldots \\
\overset{(n)}{u}_2 = u_{2o} + \varepsilon \int_{t_o}^{t} C_o(\overset{(n-1)}{u_1}, \overset{(n-1)}{u_2})\, dt + \varepsilon \int_{t_o}^{t} C_1(\overset{(n-1)}{u_1}, \overset{(n-1)}{u_2}) \cos t\, dt + \varepsilon \int_{t_o}^{t} D_1(\overset{(n-1)}{u_1}, \overset{(n-1)}{u_2}) \sin t\, dt + \ldots
\end{array}\right\} \tag{6}$$

which couverge to the unique solution $\{u_1, u_2\}$ of the system (3).

Let us take the same initial conditions:

$$u_{1o} = \bar{u}_{1o}, \quad u_{2o} = \bar{u}_{2o}, \tag{7}$$

in the approximations (5) and (6), and subtract properly; the result for the n^{th} approximation is:

$$\left.\begin{aligned} \overset{(n)}{u}_1-\overset{(n)}{\bar{u}}_1 &= \varepsilon\int_{t_o}^{t}\{A_o(\overset{(n-1)}{u}_1, \overset{(n-1)}{u}_2)-A_o(\overset{(n-1)}{\bar{u}}_1, \overset{(n-1)}{\bar{u}}_2)\}dt + \varepsilon\int_{t_o}^{t} A_1(\overset{(n-1)}{u}_1, \overset{(n-1)}{u}_2)\cos t\,dt + \\ &\quad + \varepsilon\int_{t_o}^{t} B_1(u_1, u_2)\sin t\,dt + \ldots \\ \overset{(n)}{u}_2-\overset{(n)}{\bar{u}}_2 &= \varepsilon\int_{t_o}^{t}\{C_o(\overset{(n-1)}{u}_1, \overset{(n-1)}{u}_2)-C_o(\overset{(n-1)}{\bar{u}}_1, \overset{(n-1)}{\bar{u}}_2)\}dt + \varepsilon\int_{t_o}^{t} C_1(\overset{(n-1)}{u}_1, \overset{(n-1)}{u}_2)\cos t\,dt + \\ &\quad + \varepsilon\int_{t_o}^{t} D_1(\overset{(n-1)}{u}_1, \overset{(n-1)}{u}_2)\sin t\,dt + \ldots \end{aligned}\right\} \quad (8)$$

The integrals in (8) are bounded and the right-hand sides contain ε as a common factor, then the approximations, and consequently their limits, are for small ε of order ε, that is:

$$|u_1-\bar{u}_1| = 0(\varepsilon), \qquad |u_2-\bar{u}_2| = 0(\varepsilon). \quad (9)$$

In (9) the $\{u_1, u_2\}$ and $\{\bar{u}_1, \bar{u}_2\}$ are solutions of (3) and (4) respectively, then any solution of (4) can be considered as an approximation of the solution of (3) of the first order in ε.

3. The constant solutions of (3) come when, in (1), εF_1 and εF_2 tend to zero then, since, the time t figures explicitly in F_1 and F_2, when $\varepsilon \to 0$. But in (4) the time t does not figure explicitly, then we can get constant solutions of (4), if ε is not necessarily zero, by taking proper initial conditions in the approximations (5), namely the initial conditions $\{\bar{u}_{1o}, \bar{u}_{2o}\}$ which fulfill the conditions:

$$A_o(\bar{u}_{1o}, \bar{u}_{2o})=0, \qquad C_o(\bar{u}_{1o}, \bar{u}_{2o})=0, \quad (10)$$

when the integrals in (5) are zero, and the solution $\{\bar{u}_1, \bar{u}_2\}$ of (4) is the constant $\{\bar{u}_{1o}, \bar{u}_{2o}\}$ for any ε, included $\varepsilon = 0$.

A constant solution $\{\bar{u}_1, \bar{u}_2\}$ of the approximate system (4), which fulfills (10), in considered as an approximate solution of the exact system (3) of first order in ε.

4. The above technique of replacing the system (3) where the time t figures explicity, by the approximate system (4), where the time t does not figure explicitly, consists essentially of substituting a function by its «*mean value*» over an interval[2], which is called «*moving average*» or «*sliding* *mean»*[4]. Since the *«moving average»* is, in general, smother than the original function, the above technique, which is known as the *«averaging principle»*[3], offers advantages in the study of the original system, and it is realized in practice with good results, say in economics, or in electrical problems, e. g. in the photoelectric reproduction of sound, in television images, etc.[4].

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ΠΕΡΙΛΗΨΙΣ

Ἡ ἐργασία αὕτη ἀναφέρεται ἐπὶ τῶν ἀνωμάλων σημείων τοῦ συστήματος (1), τῶν ὁποίων ἡ σπουδὴ γίνεται διὰ τῆς σπουδῆς τῶν ἀνωμάλων σημείων τοῦ συστήματος (4), τὰ ὁποῖα πληροῦν τὰς συνθήκας (2). Διὰ τῆς χρήσεως τῆς ἀνωτέρω μεθόδου, ἡ ὁποία εἶναι γνωστὴ ὡς *«ἀρχὴ τοῦ μέσου ὅρου»*, παρακάμπτονται μεγάλαι μαθηματικαὶ δυσκολίαι, αἱ δὲ λαμβανόμεναι κατὰ προσέγγισιν λύσεις εἶναι εἰς τὴν πρᾶξιν λίαν ἱκανοποιητικαί.

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