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MECHANICS.– **(Micro) structures with variable poisson's ratio design concepts,**

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ABSTRACT

Materials and structures weak in shear strength are known to be able to exhibit negative Poisson's ratio. This fact has been shown to be valid for certain mechanisms, composites with voids and frameworks and has recently been verified for micro-structures optimally designed by the homogenization approach. For micro-structures composed of beams it has been postulated that non-convex shapes (with re-entrant corners) are responsible for this effect. In this paper it is numerically shown that mainly the shape, but also the ratio of shear to bending rigidity of the beams do influence the apparent (phenomenological) Poisson's ratio. The same is valid for continua with voids, or for composites with irregular shapes of inclusions, even if the constituents are quite usual materials, provided that their porosity is strongly manifested. Elements of the numerical homogenization theory and first attempts towards an optimal design theory are presented in this paper and applied for a numerical investigation of such types of materials.

1. INTRODUCTION

Modern structures and machine components working under extreme conditions are required to have optimal performance. Advances in computational modelling and mechanics and in the manufacturing technology nowadays render feasible an optimisation of such materials and structural designs. On

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the other hand, nature assesses that monolithic structural systems, made of isotropic and homogeneous materials, are far away from such an optimal design: one needs only see the form and the inner structure of a bone to realize how an optimal material design problem that has emerged out from centuries of evolution, looks like. Composite structures and materials made of microstructures allow to apply to some extent, optimal design concepts to tailor a structural system, such as to fit with certain requirements. Structures with even negative Poisson's ratios, which at first glance would appear to be exotic, are also feasible. It is the purpose of this paper to exploit the effect of a controllable, possibly negative, Poisson's ratio for structures and microstructures, and to study this effect numerically either for composite materials, or for materials with reentrant corner microstructures, by means of a numerical homogenization method and finite-element techniques.

Composite materials, like human bones present a certain, clearly not homogeneous and isotropic microstructure. Even if at the structure's level continuum mechanics' techniques are used for structural modelling purposes, it is the microstructure of the material, the appropriate property which allows us to control according to our wishes its overall mechanical properties. By applying modern techniques of optimal topology design by homogenization [1-3] the detailed material design (i.e. the choice of the appropriate microstructure, the constituents etc) is treated for a characteristic cell of the structure. In turn, classical modelling techiques at the structure's level with appropriate elasticity laws are used, as they result from the overall behaviour of the characteristic cell. The origin of this micro-macro approach can be traced back to the elastic framework modelling of continuous structures [4-9]. In general one gets orthotropic elastic moduli which are related to the bending stiffness of the members, composing the cell walls, or, in a more general context, with the topology of the microstructure (cell). Moreover, elastic collapse is associated with elastic buckling of these members, plastic collapse is modelled by plastic hinges formed inside these members and failure response including the transition from a compression-strong to a tension-strong behaviour with increased porosity, may also be explained by these models [10].

The effect of negative Poisson's ratio has also been explained by elastic framework of mechanism models [5-7] and [11-15]. One should recall here that anisotropy and variable Poisson's ratio may have a beneficial effect, among others, on the strength characteristics, on the stress concentration factors and on the failure response of the structures composed of these materials [10, 17-21]. In particular, the advantages of using materials with negative Poisson's ratio have already been reviewed and appreciated [12, 22]. They permit, among others, the reduction of stress concentration factors and the production of layered composite panels and beams, which allow for a smooth treatment by cold metal forming techniques.

Concerning the technological feasibility of manufacturing materials with «strange» microstructures, so that to exhibit negative Poisson's ratios, one should note that there exist already materials with this kind of microstructure, which have the aforementioned property. Let us recall here some anisotropic crystallic materials like the hexagonal cadmium [23] and the cubic pyrite [24]. Furthermore, the pyrolytic graphite, with a lattice structure, presents a value of Poisson's ratio equal to $\nu=0.21$ [25], the spongy parts of bones, with a lattice-like structure, reported in [12] and the granular materials [26]. Moreover, thermomechanical techniques have been developed, which transform conventional low density open-cell thermoplastic polymer foams to form materials with negative Poisson's ratio, up to a value of $\nu=-0.4$ for some loading directions [22]. More sophisticated techniques are also mentioned, without re-entrant corners and nonconvex shaped, two-dimensional cellular (micro) concrete examples for voided ceramics or even metals. In this paper the evidence of, and the ability to design for negative Poisson's ratio will be shown, by means of finite-element based unmerical examples, for porous materials with re-entrant corners and nonconvex shaped, two-dimensional cellular (micro) structures, as well as for composite materials with analogous, re-entrant corner inclusions.

Design concepts and inverse design (tailoring) aspects will be discussed and demostrated by numerical examples. Analogous effects also appear in two-dimensional cells of matrix-fiber reinforced composite materials, as it will be shown by means of typical numerical examples.

A concise survey of existing published results on structures, materials and mechanisms with variable and in particular negative Poisson's ratios and practical results of this effect are given is section 2. Continuous and discrete modelling aspects are discussed in sections three and four respectively. Elements of numerical homogenization theory, as they are used in the numerical investigation in this paper, are presented in section five. The inverse homogenization (optimal design) problem is briefly discussed in section six. Numerical examples and discussion are presented in the last section.

2. BODIES WITH VARIABLE AND NEGATIVE POISSON'S RATIOS

The admissible limits of variation for Poisson's ratio for homogeneous and isotropic materials lie in the interval [-1, + 1/2]. The right-side limit corresponds to incompressible materials, like the rubbers, while negative values correspond to dilatational materials in general, with weak bulk modulus and strong shear modulus. For anisotropic materials and in particular for orthotropic ones a condition has been given by the first author [20], which guarantees that they behave like isotropic materials with positive values of Poisson's ratio. In all other cases the possibility of the appearance of a negative Poisson's ratio, at least in one orientation of the anisotropic body is not excluded a priori from the theory of general anisotropic elasticity.

Several single crystals of a polygonal structure at the atomic level are reported to have negative Poisson's ratio along some directions of loading (see for example tests with cadmium crystals [23], the single-crystal pyrite [24] and the lattice structured pyrolytic graphite [25]. Thermomechanically treated low density open-cell thermoplastic polymeric foams are also meterials, which exhibit negative Poisson's ratios [15, 22, 27, 28]. These materials are usually porous and have a spongy nature with a lot of voids and a complicated microstructure.

From the microstructural picture of these materials, which exhibits non-convex cells with re-entrant corners, a number of micro-structures and mechanisms have been proposed for the explanation and the study of this effect. These examples are not actually proper materials, which can be found in nature, but, as manufacturing technology and micromechanics attain actually a higher level of development, the possibility of constructing materials with these microstructures as prototypes continuously grows.

Cellular microstructures composed of beams have been successfully used for the modelling of linear and nonlinear elastic properties of two-dimensional and three-dimensional cellular materials or honeycombs; the results correlated well with experimental measurements [5-7]. In the classical case for materials with positive Poisson's ratio, polygonal, convex cells are appropriate for their modelling. The same tools with non-convex cells, with re-entrant corners are able to predict an overall mechanical behaviour with negative Poisson's ratio [10, 12, 22]. Notice here that spongy parts of bones have lattice-like structures, a fact that indicates the significance of this work for applications in biomechanics.

Random isotropic granular materials with a ratio of interparticle tangential to normal stiffness greater than unity has theoretically negative Poisson's ratios [26], although one should accept that this case is physically rather unlikely to occur for particles made of natural materials.

Moreover, certain mechanisms formed by microstructures composed of springs and sliding collars, containing also re-entrant corners, are able to exhibit negative Poisson's ratios [11,29]. Mechanisms composed of rigit bars and folded nets constituting tensile networks, have also been proposed, for achieving the same Poisson's ratio effect [13]. It is of interest to observe here that optimal microstructures, that are produced by homogenization-based topology optimization techniques, have an analogous structure with re-entrant corners [2, 3]. Furthermore, for completeness, it should be mentioned that unilateral Poisson's ratio, i.e. different ratio in tension (greater) than in compression has also been reported for several low-density rigid plastic foams under conditions yielding pseudo-elastic values, as for example polystyrene-beam foams and polyurethane foams [30].

Finally, microstructures which appear in the course of the homegenization studies for composite materials exhibit the same effect. The elastic chessboard composites of ref.[31] belong to this class of structures. The appearance of negative Poisson's ratio for composites made of classical, with positive Poisson's ratio, composites is also not excluded, as the estimates provided in ref.[32] clearly show. A detailed study of composite materials with negative Poisson's ratio, which includes two-dimensional two-phase composites with hexagonal symmetry has been undertaken in ref. [14]. The same behaviour can also be produced by multiscale laminates.

The importance of having materials with negative Poisson's ratio has been early recognized with respect to modern structural analysis applications especially in the aerospace industry. These materials should have a high shear modulus relatively to the bulk modulus. This correlation of the moduli is especially appreciated, if the material is used in sheet or beam form, as it is actually the case in most structural applications [12]. Moreover, the deformation pattern of elastic structures made of such materials generally differs from the ones made of classical materials [15]. This latter effect requires a new way of thought for the design of structural elements or structures, but, at the same time, it opens new possibilities for novel applications. For example, a sandwich panel or beam with core made of this new material will exhibit a domelike double curvature in figure. Thus, it will allow a cold-metalforming treatment for the production of the shell from initially plane panels. The last advantage, as mentioned here, results in a reduction of the stress concentration factors in the shell, which, in turn, it enhances the crack and fatigue strength of the structure.

3. LIMITS OF POISSON'S RATIOS IN ELASTIC CONTINUA

In the framework of the theory of elasticity for isotropic materials the mechanical behaviour is described by the three material constants: the elastic modulus E, the shear modulus G and the Poisson ratio ν, which are given by the well known formulas:

$$G = \frac{E}{2(1+\nu)} \tag{1}$$

$$\theta = (\varepsilon_x + \varepsilon_y + \varepsilon_z) = \frac{p}{\frac{E}{3(1-2\nu)}} = \frac{p}{K} \tag{2}$$

where K is the bulk modulus of the material and θ and p express the volumetric strain and the applied hydrostatic pressure, respectively.

From thermodynamic reasons, implying that E, G should be positive and that K is positive, the lower and the upper bounds for the Poisson ratio read as follows [33]:

$$-1 < \nu < \frac{1}{2} \tag{3}$$

For orthotropic and generally anisotropic materials the elasticity relations are more complicated and involve a greater number of material constants. The basic thermodynamic requirement that the work done by a given stress must remain always positive, when applied to the relations of the orthotropic or general anisotropic elasticity, yields bounds on the admissible Poisson's ratio. However, in ref. [33] a value f_{or} $\nu_{12}=1.92$ was reported for a composite material, as it is derived from the one of the limits established in Eqs. (4.1) This value for ν_{12} is considered as excesive. However, more stringent values for the limits of Poisson's ratio have been established in ref. [34], which are expressed by either of the inequalities:

$$|\nu_{12}| < \left\{ (1-\nu_{23}) \frac{E_{11}}{2E_{22}} \right\} \tag{4.1}$$

or

$$2\nu_{12}\,\nu_{23}\,\nu_{13}\,\frac{E_{22}}{E_{11}} < 1 - \nu_{12}^2\,\frac{E_{22}}{E_{11}} - \nu_{23}^2\,\frac{E_{33}}{E_{22}} - \nu_{13}^2\,\frac{E_{33}}{E_{11}} \tag{4.2}$$

where the double indices denote the values of the respective quantities along the axes of anisotropy. Applying these constraints for the Poisson ν_{ij}, reasonable values for these quantities have been established [34-36].

4. CELLULAR BODIES

While materials with extreme values for Poisson's ratio are not usual, the overall mechanical behaviour of microstructures may attain a variety of phenomenological Poisson's ratio values. Experimental evidence with foams containing cells with re-entrant corners [22] and from the relevant results applying the numerical homogenization theory [2], we may construct microstructures with an adjustable mechanical behaviour, which exhibit positive or negative Poisson's ratios. For the study of the overall mechanical properties of these materials we assume that they are periodic, i.e. created by arrays of a representative unit cell. Moreover, we assume that the overall mechanical behaviour of the material can be described by the classical elasticity relations.

In this framework a homogenization problem is posed as follows: find the elasticity constants of the continuous model, which lead to the same mechanical behaviour as the one of the material with the periodic microstructure. To this end a detailed analysis of a representative unit-cell is performed and a best fit method is followed, as it will be shown in the numerical examples in this paper. This technique is also valid for continuous structures with periodic inhomogeneities (i.e. composite materials, etc) and will be studied in the next section.

The possibility to adjust the overall mechanical properties by changing the geometric or material properties of the microstructure constitutes the following inverse (optimal) design problem: «*find a microstructure for which the material has a given (or optimal in some particular sense) mechanical beha-*

viour». In the following section numerical examples with two-dimensional materials, made of periodic, star-shaped beams as microstructures, are presented, followed by a mathematical and numerical approach of the homogenization method in different types of bodies. The mechanical properties of these materials will be presented.

5. THE NUMERICAL HOMOGENIZATION METHOD

For both a discrete (beam-like) and a continuous (composite), periodic structure, an equivalent homogeneous model can be constructed by using the *homogenization technique*. An appropriate numerical method for the application of this technique is developed here. It is based on the use of finite-element modelling for the real cell of the structure and on, optimality-criteria based, numerical homogenization concepts. The method developed will be applied for the numerical treatment of the examples presented in the last section.

Let us consider a representative unit-cell of the periodic structure, which, for simplicity, is assumed to be two-dimensional (Fig. 1). Let the unit cell be orthogonal with dimensions equal to l_1 and l_2 along the two coordinate axes and let it occupy the area Ω with boundary Γ. The boundary Γ is composed of the complementary and nonoverlapping parts Γ_1. Γ_2. Γ'_1 and Γ'_2 (i.e. $\Gamma_1 \cup \Gamma_2 \cup \Gamma'_1 \cup \Gamma'_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \varnothing$ etc). A *unit cell* of the real structure (case II in Fig. 1) and a unit cell with the same dimensions of the sought homogeneous structure (case I in Fig. 1) are considered. The cells I and II are subjected to the three unit stresses, respectively:

$$\begin{array}{llll} \text{problem (1):} & \sigma_1 = 1, & \sigma_2 = 0, & \sigma_3 = \tau_{12} = \tau_{21} = 0, \\ \text{problem (2):} & \sigma_1 = 0, & \sigma_2 = 1, & \sigma_3 = \tau_{12} = \tau_{21} = 0, \\ \text{problem (3):} & \sigma_1 = 0, & \sigma_2 = 0, & \sigma_3 = \tau_{12} = \tau_{21} = 1 \end{array} \qquad (5)$$

as it is shown in Fig. 1.

The solution of cell I for these loading modes can be based on simple engineering mechanics relations, due to the assumption that the dimensions of the periodic cell are small with respect to the dimensions of the structure.

For the cell II a finite-element method is employed for the solution of the above static-analysis problems. Moreover, the following periodicity restraints are taken into account (as multipoint constraints) in the above described problems: for problems 1 and 2 displacements on boundaries Γ_1. Γ'_1 along

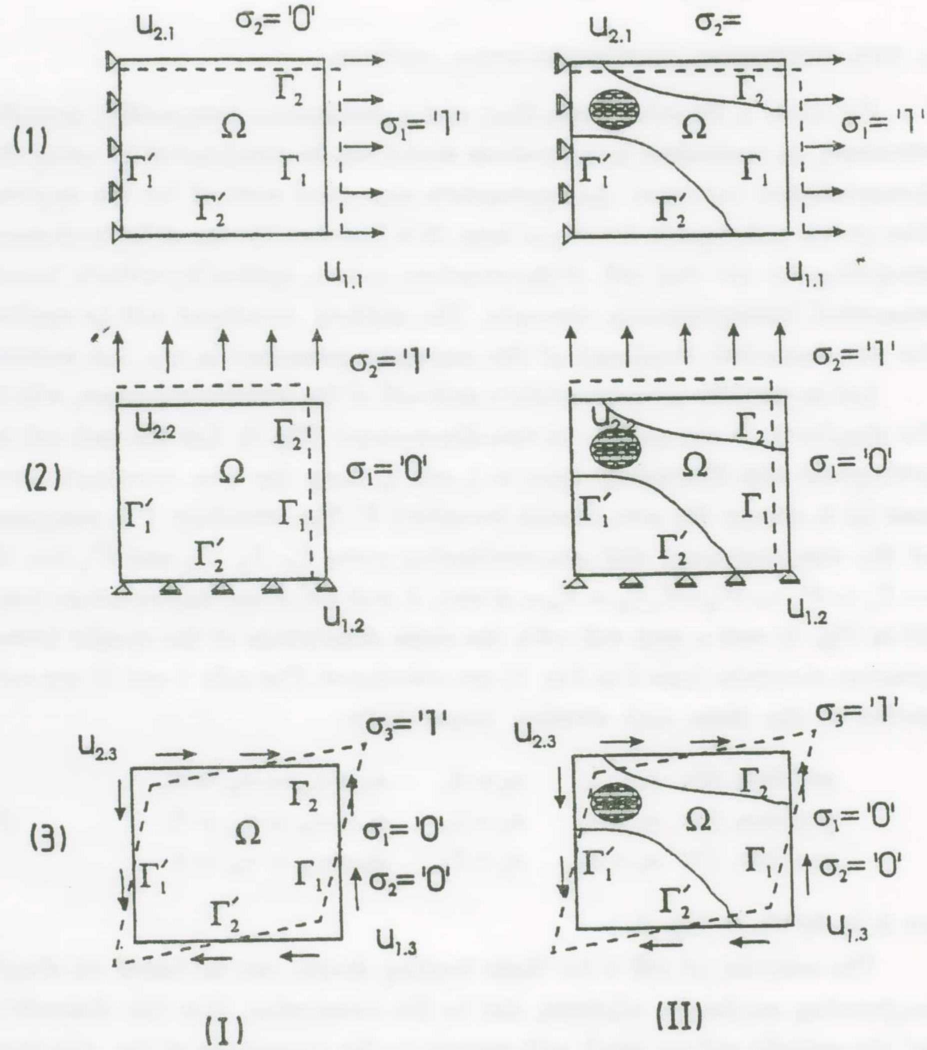


Fig. 1. Elements of the numerical homogenization technique for a cell. Case I: homogeneous cell, case II: real structure cell, cases (1), (2), (3): unit prestresses.

the horizontal direction 1 are the same, for problems 1 and 2 displacements on boundaries Γ_2, Γ'_2 along the vertical direction 2 are the same and for problem 3, boundaries Γ_1. Γ'_1. Γ_2 and Γ'_2 remain straight lines after deformation.

The essence of the energy-based numerical homogenization method is that the parameters of the homogeneous cell I are appropriately chosen, so that this cell has the same deformation energy with the cell of the real structure (cell II), if both are subjected to the same deformation patterns, which respect the periodicity assumptions, i.e. they are periodic for the whole structure.

If the parameters which define the mechanical behaviour of cell I (e.g. the elasticity constants) are gathered up in the design vector $\boldsymbol{\alpha}$, the numerical homogenization method can be described by the following identification problem:

Find $\boldsymbol{\alpha}$ *as a solution of the optimization problem*:

$$\min_{\alpha \in A_{ad}} \frac{1}{2} \sum_{i=1}^{3} w_i \left\{ \prod_{in}^{I(i)} (e^{(i)}, \alpha) - \prod_{in}^{II(i)} (e^{(i)}) \right\}^2 \tag{6}$$

Here A_{ad} is the admissible set for the material parameters of the homogenized cell, i runs over all independent periodic deformation patterns $e^{(i)}$, which are considered, w_i are appropriate weights, which transform the multiobjective optimization problem into a classical one, with a cost function as in (6), superscript I or II stands for the quantities of cell I or II respectively and Π_{in} is the internal energy of the considered structure.

The identification problem (6) can be solved either by classical numerical optimization techiques, or by neural-network based methods [37, 38]. In the sequel we will describe a simple procedure, which is based on the optimality criteria method for the solution of a certain class of problems (6). This method avoids the formulation and the solution of large scale optimization problems, and if it can be used, it is considered to be suitable for structural analysis applications [1].

Let us assume for simplicity here that all w's are equal to one. Moreover, we assume that the homogenized unit cell I obeys the classical isotropic elasticity relations:

$$e = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1-\nu)}{E} \end{Bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = K_0 \sigma \tag{7}$$

The design vector $\boldsymbol{\alpha}$ is chosen as: $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]^T = [^1/E, -^\nu/E]^T$.

The internal energy is expressed by $\prod_{in}^{i(j)} = \int_\Omega \sigma^{i(j)T} e^{i(j)} \, d\Omega$ for all $i = I, II$, where Ω is the area of the considered cell. For simplicity we assume here $j = 1,2,3$, that $A_{ad} = R^2$.

Under the above assumptions problem (6) reads:

$$\min_{\alpha \in R^2} \left\{ \frac{1}{2} \int_\Omega \left\{ \begin{array}{l} (\sigma^{I(1)T}(\alpha) e^{I(1)}(\alpha) - \sigma^{II(1)T} e^{II(1)})^2 + (\sigma^{I(2)T}(\alpha) e^{I(2)}(\alpha) - \sigma^{II(2)T} e^{II(2)})^2 + \\ + (\sigma^{I(3)T}(\alpha) e^{I(3)}(\alpha) - \sigma^{II(3)T} e^{II(3)})^2 \end{array} \right\} d\Omega \right\}$$

For the assumed unit stresses (5) and the elasticity relations (7) we get for the unit cell I that:

$$e_1^{I(1)} = \alpha_1 \sigma_1^{I(1)} = \alpha_1 \qquad e_2^{I(1)} = \alpha_2 \sigma_1^{I(1)} = \alpha_2$$

$$e_1^{I(2)} = \alpha_2 \sigma_2^{I(2)} = \alpha_2 \qquad e_2^{I(2)} = \alpha_1 \sigma_2^{I(2)} = \alpha_1 \tag{9}$$

$$e_3^{I(3)} = 2(\alpha_1 - \alpha_2) \sigma_3^{I(3)} = 2(\alpha_1 - \alpha_2)$$

with all other components equal to zero.

Moreover, the virtual work equality for the cell II reads:

$$\int_\Omega \sigma^{II(j)T} e^{II(j)} \, d\Omega = \int_\Omega S^{II(j)T} u^{II(j)} \, d\Gamma, \qquad j = 1, 2, 3 \tag{10}$$

for all given unit stresses of (5) (i.e. $S^{II(1)} = 1$ on Γ_1 $S^{II(1)} = 0$ on $\Gamma_2 \Gamma'_2$ etc.).

Finally, the optimality conditions for (8) are written by means of (10) as follows:

«*Find a_1, a_2, such that*:

$$\int_\Omega \left(\alpha_1 - \sigma^{II(2)T} e^{II(2)} \right) d\Omega \frac{\partial \alpha_1}{\partial \alpha_1} + \int_\Omega \left(2(\alpha_1 - \alpha_2) - \sigma^{II(3)T} e^{II(3)} \right) d\Omega \frac{\partial \{2(\alpha_1 - \alpha_2)\}}{\partial \alpha_2} = 0 \tag{11}$$

$$\int_\Omega \left(2(\alpha_1 - \alpha_2) - \sigma^{II(3)T} e^{II(3)} \right) d\Omega \frac{\partial \{2(\alpha_1 - \alpha_2)\}}{\partial \alpha_2} = 0 \tag{12}$$

By using (10) the area integrals are transformed into boundary integrals. Thus, we get the following optimality conditions:

«*Find* α_1, α_2 *such that:*

$$\left(\alpha_1 \zeta_1 \zeta_2 - u_1^{II(1)} \zeta_2 \right) 1 + \left(\alpha_1 \zeta_1 \zeta_2 - u_2^{II(2)} \zeta_1 \right) 1 + \left(2 (\alpha_1 - \alpha_2) \zeta_1 \zeta_2 - \int_\Omega \sigma^{II(3)T} e^{II(3)} \right) 2 = 0 \tag{13}$$

$$\left(\frac{2(\alpha_1 - \alpha_2) \zeta_1 \zeta_2 \cdot \int_\Omega \sigma^{II(3)T} e^{II(3)}}{9} \right) (-2) = 0 \tag{14}$$

Variable α_1 (the elacticity modulus E) results from (13) and (14) νas follows:

(15)

Variable α_1 (the Poisson ratio ν) may now be calculated either from (15), or from the elasticity relations (1), which have been assumed to hold true.

Analogous relations can be extracted for the more general case, where the homogeneous model I is assumed to obey the orthotropic elasticity relation or the general anisotropic elasticity relations [2].

6. OPTIMAL DESIGN CONCEPTS

The inverse homogenization problem can be formulated analogously to the direct problem of the previous section. We make here the same assumptions and we consider again the cells I and II and the unit stresses (1) : to (3) of Fig. 1.

Now the homogeneous cell I is given, that is the corresponding material constants (7) are known and constitute the goal of the optimal design problem. On the other hand, the *real* cell II may now be modified by means of a certain number of design parameters, which are summed up in the design-vector $\boldsymbol{\beta}$. For instance, either elasticity constants of the various constituents in a composite structure, or the shape of the inclusions in a reinforced composite, or the type and the shape of the microstructure, may be considered as design variables by an appropriate choice of the elements of vector $\boldsymbol{\beta}$.

By an analogous reasoning to the one used in the previous section, the optimal design problem reads:

Find $\boldsymbol{\beta}$ *as a solution of the optimization problem*:

$$\min_{\beta \in B_{ad}} \frac{1}{2} \sum_{i=1}^{3} w_i \left\{ \prod_{in}^{I(i)} (e^{(i)}) - \prod_{in}^{II(i)} (e^{(i)}, \boldsymbol{\beta}) \right\}^2 \tag{16}$$

Here B_{ad} is the admissible set for the design variables $\boldsymbol{\beta}$ and all other quantities are defined after problem (6).

As with problem (6), problem (16) can be solved by means of various methods. A detailed study of this problem is not included here. The reader may consult refs. [2, 3], among others, for analogous recent studies. The parametric investigation of the next section may be used to help the formulation and the study of the above outlined problem.

7. APPLICATIONS OF THE HOMOGENIZATION METHOD

As a first example, a fiber-reinforced composite material is considered. A cross section perpendicular to the direction of the fibers is shown in Fig. 2, along with the dimensions of the problem and the representative unit cell ABCD. All quantities here and in the sequel are assumed to be measured in compatible units. The finite element discretization of the analysed part, which is considered as a two-dimensional plane stress problem, is shown in Fig. 3, where the initial and the deformed configurations for a unit loading at the boundary AB are shown. For the application of the numerical homogenization technique and the determination of the overall mechanical properties of the composite material, support conditions, at the boundaries BC (resp. AD) which prescribe zero vertical (resp. horizontal) displacements are considered. Moreover, at the boundary BC (resp. AB) the horizontal (resp. the vertical) displacements are forced to be equal by means of the multipoint constraint strategy of the finite-element method.

For the matrix, which occupies the region Ω_2 in the finite element model of Fig. 2b, an anisotropic elastic material is considered with elastic modulus $E_1=10^2$ and Poisson's ratio $\nu=0.30$. For the fibers, which occupy the region Ω_1, an isotropic material is also considered with $\nu_2=0.30$ and values of E_2 between 10 and 10^3. In this way a parametric investigation of the considered composite is performed.

The values of the elastic modulus E and the Poisson ratio for the composite, as they are calculated by the numerical homogenization method, are shown in Figs 3 and 4 respectively. In the same figures the values of E, as predicted by the analytical method based on the Hashin and Rosen model (HR), presented in ref. [39], are also plotted, as well as by the unfolding model, introducing the concept of mesophase between phases, and developed by Theo-

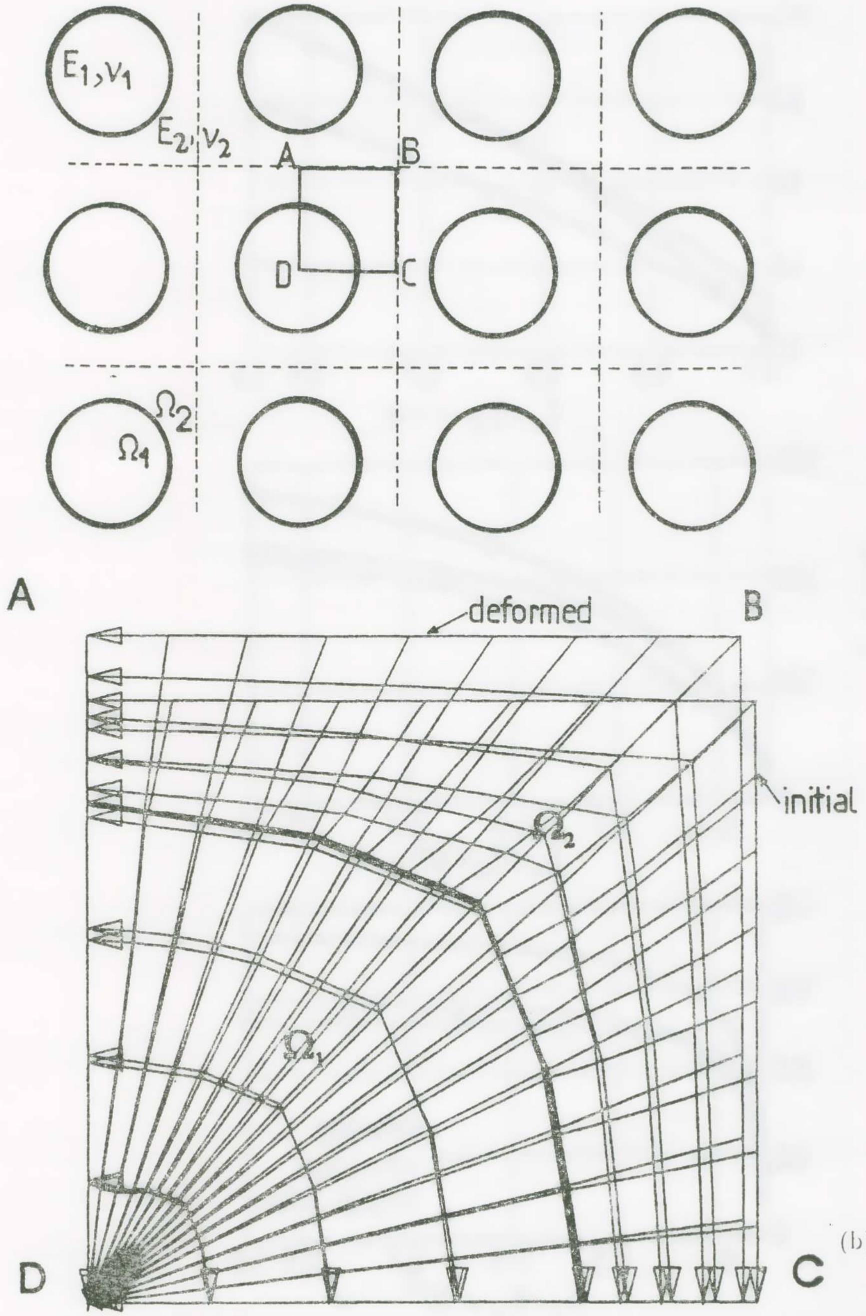


Fig. 2. a) A periodic fiber-reinforced composite, b) Finite element model of the analysed unit-cell.

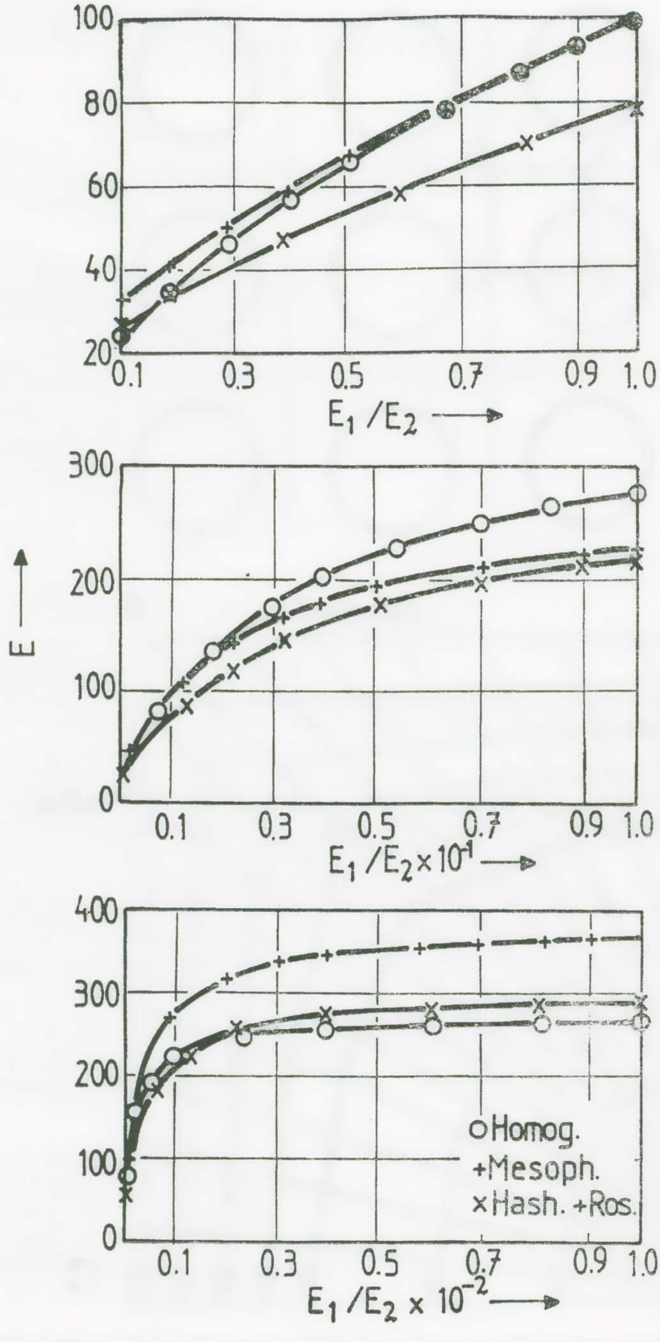
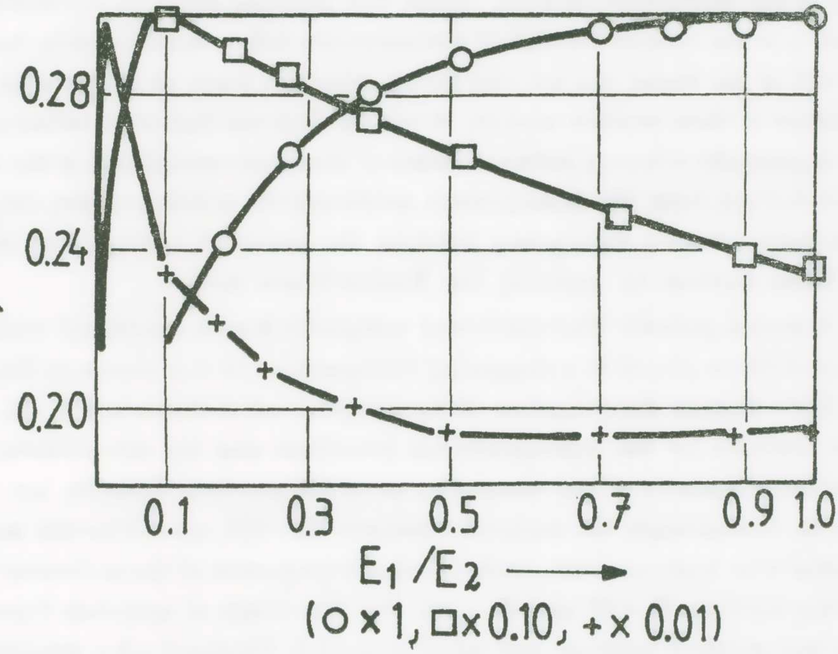


Fig. 3. Elastic modulus of the composite of Fig. 2 for various choices of the material constants.



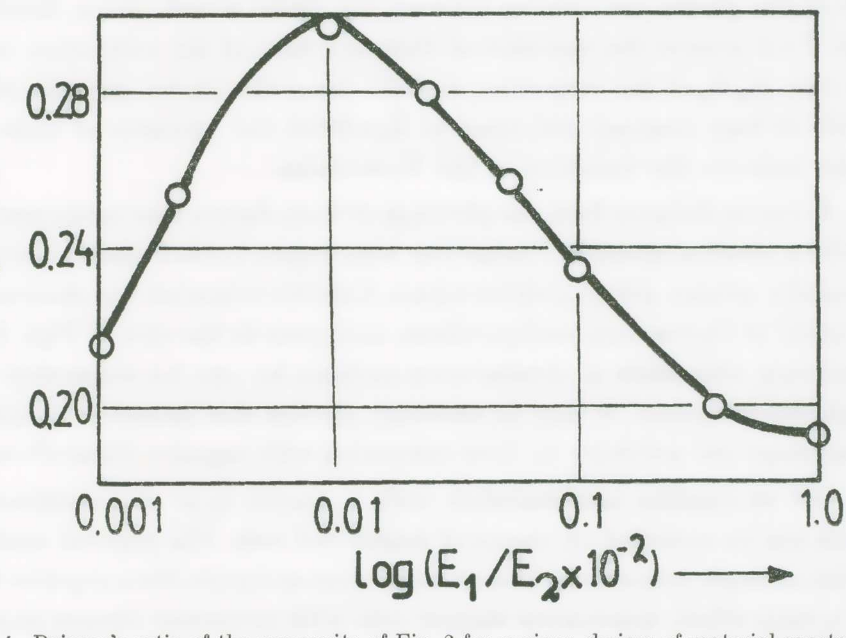


Fig. 4. Poisson's ratio of the composite of Fig. 2 for various choices of material constants.

caris [40, 41], designated by (IF). In this model the extent of the volume (specific) of the mesophase between phases was assumed equal to $v_1 = 0.001\ v_c$, where v_c is the volume fraction of the composite taken equal to unity for the unit-cell of the model (see ref. [40] for an extensive study of the validity and properties of these realistic models). It can be observed that an excellent accuracy is generally achieved between values of the elastic modulus E of the composite derived from the homogenized model and the unfoldig model, whereas there exists always a discrepancy between the values of homogenized model and those derived by applying the Hashin-Rosen model.

A second periodic fiber-reinforced composite is now considered with the arrays of fibers placed in a staggering configuration as it is shown in Fig. 5a. The finite element discretization of the analysed cell is shown in Fig. 5b. The same concepts for the homogenization procedure and the determination of the elastic constants of the composite, as in the previous example, are used here. In this example the material constants $E_2 = 100$, $\nu_2 = 0.3$ for the matrix material were kept constant, while the elastic properties of the inclusions were varying between $E_1 = 10$ and $E_1 = 10^3$. For this range of materials Poisson's ratio was assumed constant and equal to $\nu_1 = 0.1$. Figures 6 a,b,c present the variation of the elastic moduli of the composite, as the ratio between the moduli of the phases are varying between the limits stated above. Similarly, Figs. 7 a,b present the variation of Poisson's ratio of the composites versus the ratio E_1/E_2 of the respective moduli. Since the elastic modulus of the matrix is kept constant and equal to $E_2 = 100.0$ the abscissas of these diagrams indicate the variation of the E_1-modulus.

It can be deduced from the plottings of these figures that in the previous results a classical mechalical behaviour with respect to the Poisson ratio of the composite attains always positive values. A similar behaviour was observed for a variety of fiber-matrix configurations, analogous to the ones of Figs. 2 and 5 not only with fibers of circular cross-sections, by also for composites with ellipsoidal inclusions. It may be therefore derived that inclusions of complicated shape are necessary to form composites with negative Poisson's ratios.

Let us consider now materials with a specific type of microstructure, which can be modelled by means of beams and rods. The physical analogue of this example comes from foamed and porous materials. For a negative Poisson's ratio effect, non-convex shaped cells with re-entrant corners must be

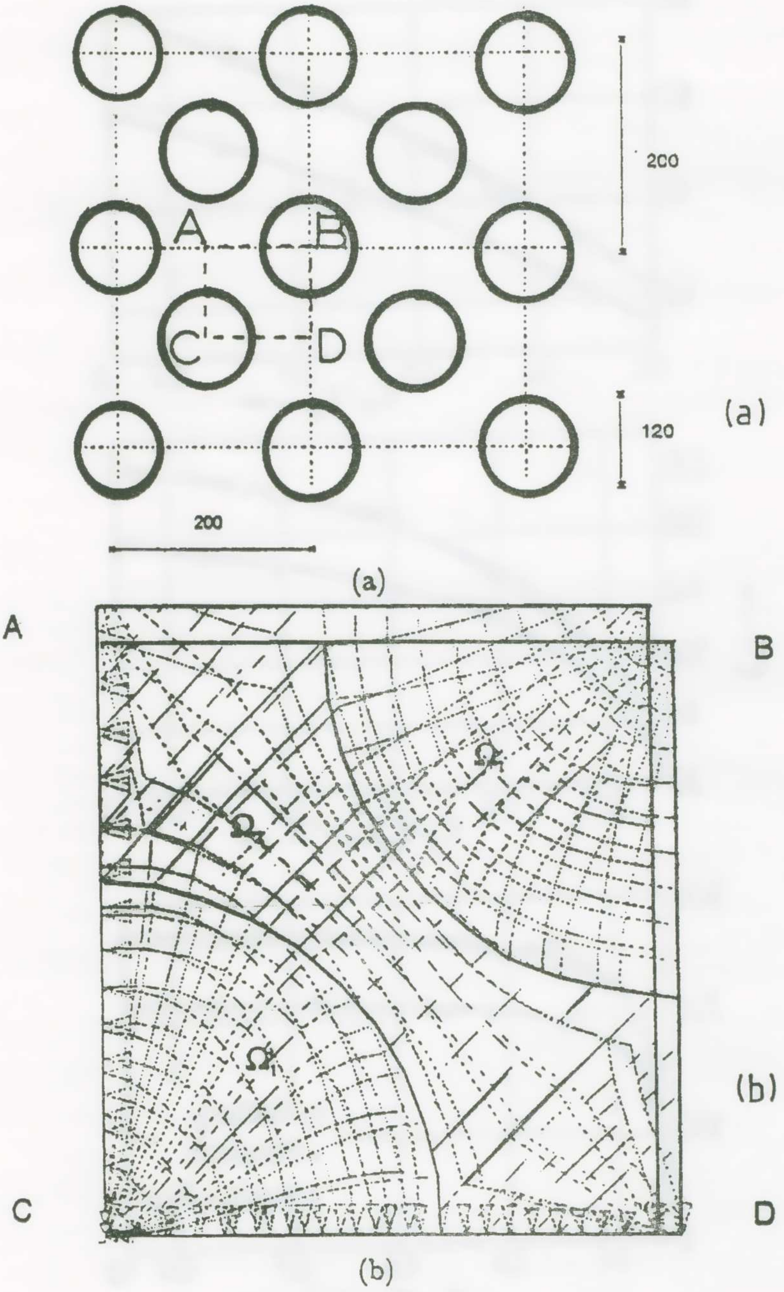


Fig. 5. a) Configuration of a fiber-reinforced composite, b) Finite element model of analysed unit-cell.

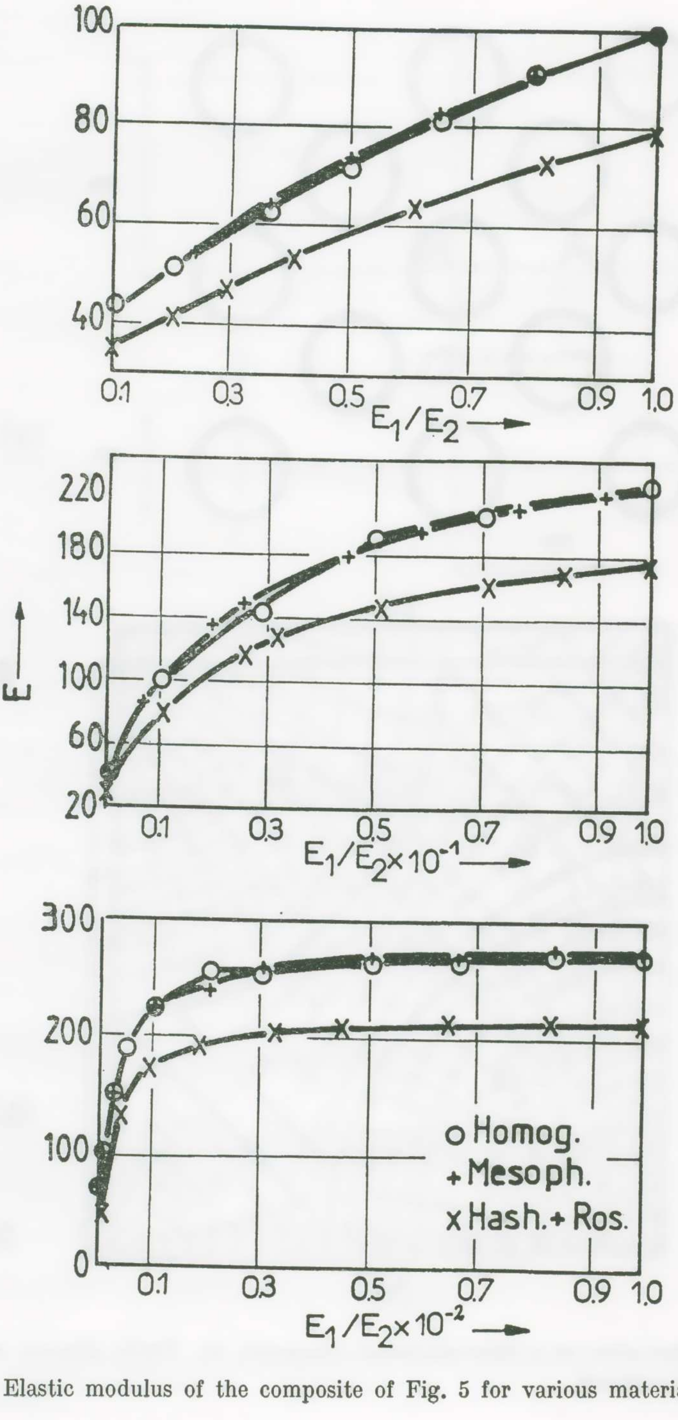
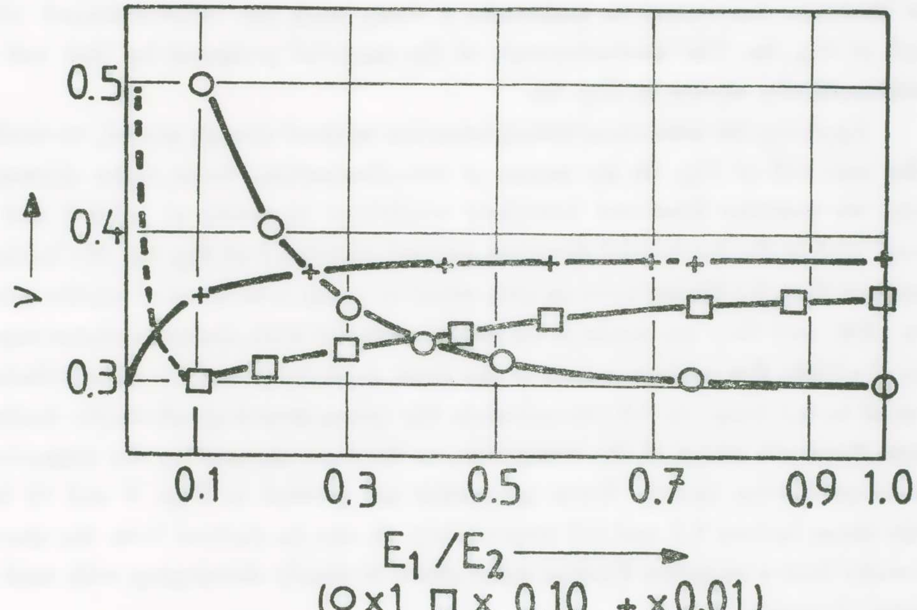


Fig. 6. Elastic modulus of the composite of Fig. 5 for various material constants.



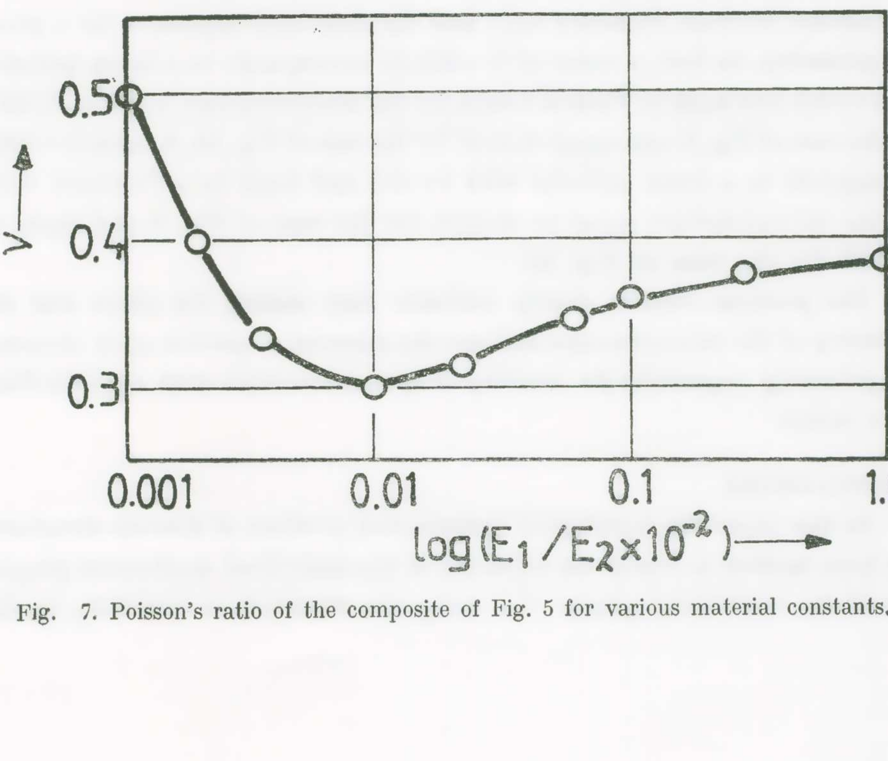


Fig. 7. Poisson's ratio of the composite of Fig. 5 for various material constants.

considered, as it is dictated from the results of refs. [2,3,6,11-13,22, 27-30]. It is therefore convenient to undertake a study with the convex-shaped unit cell of Fig. 8a. The microstructure of the material produced by this cell is schematically shown in Fig. 8b.

Applying the numerical homogenization method already stated, we model the unit cell of Fig. 8b by means of two-dimensional beam finite elements and we consider fixed-end boundary conditions (support) at point 1 and a unit load in the horizontal direction applied at point 7 of Fig. 8b. We further assume that the beams have an area equal to unity, a moment of inertia equal to 1000, and they are made of an elastic material with elastic modulus equal to $E=1000$. For various values of the shear modulus G and for a shear factor equal to 0.3 (resp. to 0.9) we calculate the (phenomenological) elastic moduli and Poisson's ratios of the composite, as they are derived by the numerical homogenization theory. These quantities are plotted in Figs. 9 and 10 for the shear factors 0.3 and 0.9 respectively. It can be derived from the above results that a negative Poisson ratio effect is clearly developing with such a type of composites.

One should nevertheless underline here that the above parametric investigation is extrapolated outside the range of mechanically admissible values for the material constants, in order to give a better overall picture of the sought dependence between Poisson's ratio and the structural constants for a given cell geometry. In fact, a value of $G=333.33$ corresponds to a beam material with $\nu=0.5$ and leads to Poisson's ratio for the microstructure equal to -0.2815 for the case of Fig. 9, and equal -0.1538 for the case of Fig. 10. A value $G=1000$ corresponds to a beam material with $\nu=-0.5$ and leads to a Poisson's ratio for the microstructure equal to -0.1524 for the case of Fig. 9 and equal to -0,0120 for the case of Fig. 10.

The previous results clearly indicate that mainly the shape and the geometry of the microstructure and not the material properties of its elements are primarily responsible for creating composite materials with negative Poisson's ratios.

8. CONCLUSIONS

In this paper the topological optimization problem of discrete structures has been applied to study the influence of the individual mechanical properties of the constituent phases of a composite material, or structure, on the

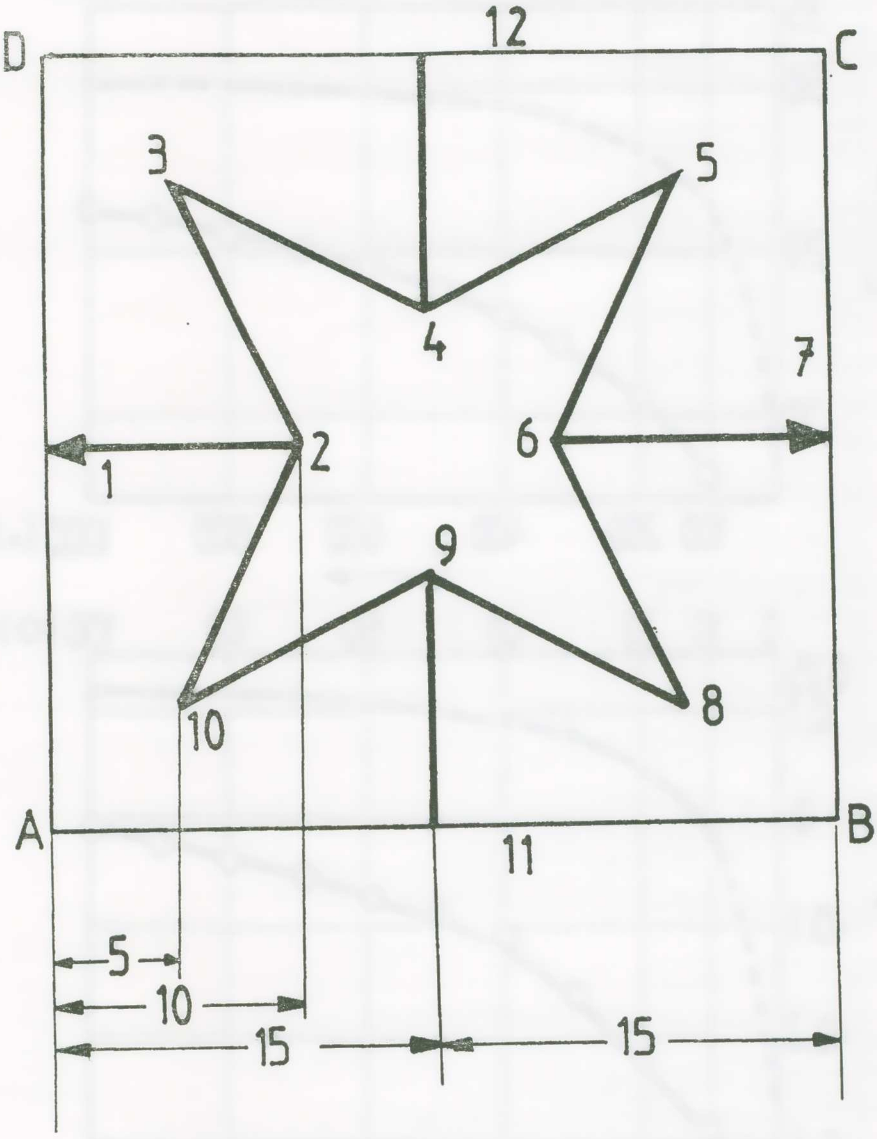


Fig. 8. A star-shaped two-dimensional beam cell with re-entrant corners. Finite element discretization and mode numbering.

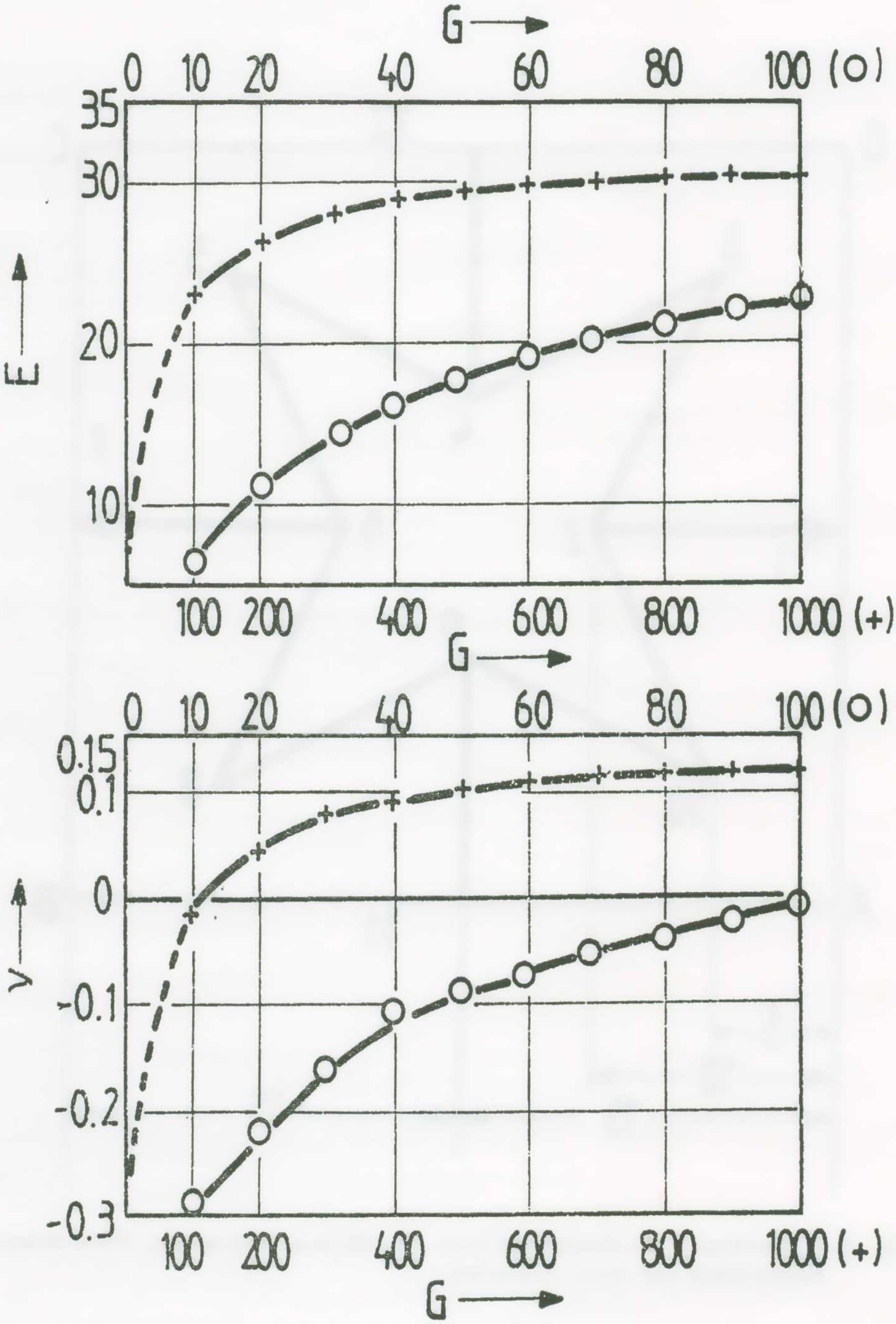
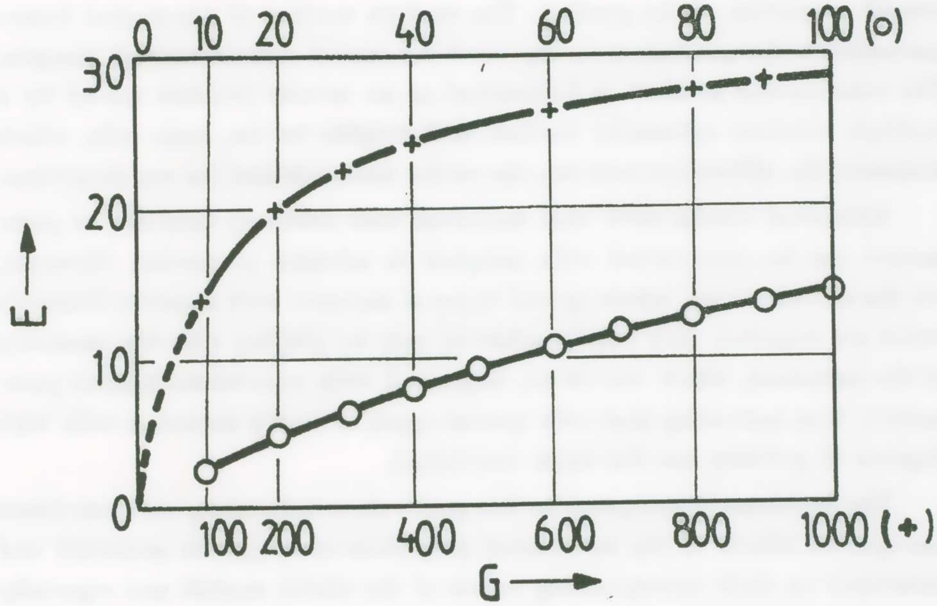


Fig. 9. Phenomenological elastic modulus (a) and Poisson's ratio (b) for a material with microstructure as in Fig. 8 possessing a shear factor equal to 0.3



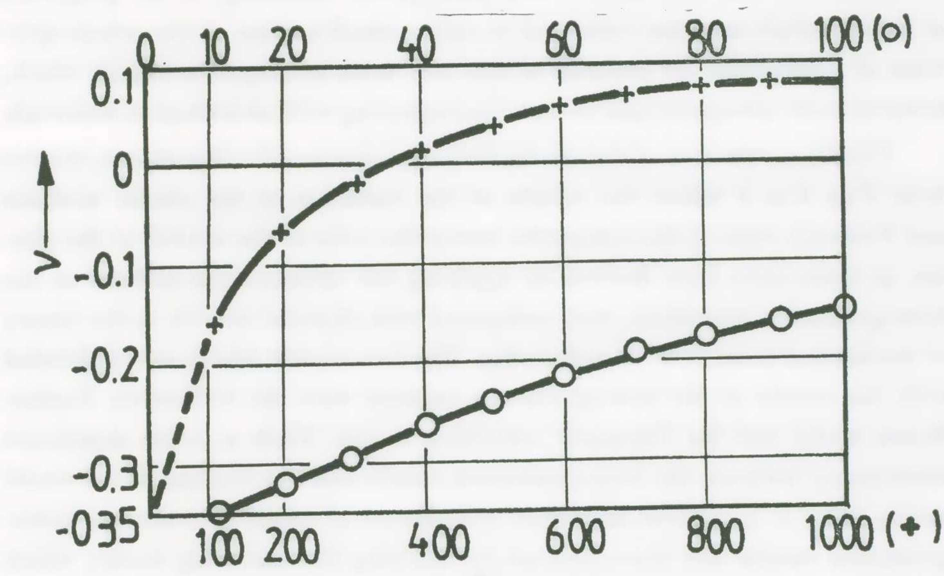


Fig. 10. Phenomenological elastic modulus (a) and Poisson's ratio (b) for a material with microstructure as in Fig. 8 possessing a shear factor equal to 0.9

overall properties of the product. The modern method of topological homogenization technique was used, expressed in terms of element mutual energies. The construction problem is formulated as an inverse problem solved by a multiple constant optimality method with weights for the basic cells, which minimize the differences between the model materials and the real structures.

Numerical results show that materials with arbitrary constitutive parameters can be constructed with assigned in advance properties. However, for the extreme cases, where special types of material with negative Poisson's ratios are required, they can be achieved only by playing with the geometry of the inclusions, which should be, in general, with reduced mechanical parameters, thus indicating that only special types of foamy materials with high degrees of porosity are the most convinient.

The problems investigated in this paper show interesting and sometimes unexpected effects of the mechenical properties of composite materials and structures on their corresponding values of the elastic moduli and especially on values of Poisson's ratio. Although these effects may contradict our intuition and long term experience, acquired by testing 'regular' materials in the constructions, they also suggest that perhaps our knowledge of the properties of the materials is rather restricted to only a small section of the whole spectrum of possibilities for creation of new and more intelligent meterials which, however,exist always around us from the beginning with all biological materials.

Finally, a side issue of this study, but of great scientific importance, derives from Figs. 6 to 8 where the results of the variation of the elastic modulus and Poisson's ratio of the composite versus the ratio of the moduli of the phases, as these have been derived by applying the optimization method on the homogenization procedure, were compared with classical models in the theory of mechanical behaviour of composites. The two models which are confronted with the results of the homogenization method were the wellknown Hashin-Rosen model and the Theocaris' unfolding model. While a rather significant discrepancy between the homogenization results and the Hashin-Rosen model exists, there is in general an almost coincidence of results between the homogenization results and those derived by applying the unfolding model, which takes into consideration the development of a mesophase as a boundary layer between the main phases of the composites. This is another proof of the soundness of these models.

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REFERENCES

1. L. J. Gibson, M. F. Ashby, F. R. S., G. S. Schajer, C. I. Robertson, The mechanics of two-dimensional cellular materials, *Proc. Royal Society London*, **A382** (1982), 25-42.

2. R. J. Barhurst, L. Rothenbur, Note on a random isotropic granular material with negative Poisson's ratio, *Int. jnl. Engng. Sciences*, **26(4)**, (1988), 373-383.

3. P. S. Theocaris, Th. P. Philippidis, True bounds on Poisson's ratios for transversely isotropic solids, Int. Jnl. of Strain Analysis, **27(1)**, (1992), 43-44.

4. P. S. Theocaris, *The mesophase concept in composites*, Springer Verlag, Berlin, Heidelberg, 1987.

5. P. S. Theocaris, Th. P. Philippidis, Stress distribution in orthotropic plates with coupled elastic properties, *Acta Mechanica*, **80(2)**, (1989), 95-111.

6. P. S. Theocaris, Failure criteria for weak-axis quasi-isotropic woven-fabric composites, *Acta Mechanica*, **95(1)**, (1992), 69-86.

7. A. P. Hrennikoff, A solution of problem in elasticity by the framework method, *Jnl. Appl. Mech.*, **8**, 4 (1941).

8. L. J. Gibson, M. F. Ashby, F. R. S, The mechanics of two-dimensional cellular materials, *Proc. Royal Society London*, **A382** (1982), 43-59.

9. O. Sigmund, Design of material structures using topology optimization, Danish Center for Applied Mathematics and Mechanics, Technical Univ. of Denmark, *Report No.* **502**, June 1995.

10. G. I. N. Rozvany, *Structural design via optimality criteria. The Prager approach to structural optimization*, Kluwer Academic, Dordrecht, 1989.

11. A. V. Cherkaev, L. V. Gibiansky, Coupled estimates for the bulk and shear moduli of a two-dimensional isotropic elastic composite, *Jnl. Mech. Phys. Solids*, **41(5)** (1993), 937-980.

12. K. Evans, Tensile network microstructures exhibiting negative Poisson's ratios, *Jnl. Phys. D: Appl. Phys.* **22** (1989), 1870-1876.

13. B. D. Caddock, K. E. Evans, Microporous materials with negative Poisson's: I Microstructure and mechanical properties, *Jnl. Phys. D: Appl. Phys.*, **22** (1989) 1877-1882.

14. P. S. Theocaris, The elliptic paraboloid failure criterion for cellular solids and brittle foams, *Acta Mechanica*, **89**(**2**), (1991), 93-121.

15. B. D. Caddock, K. E. Evans, Microporous materials with negative Poisson's ratios: II Mechanisms and interpretation, *Jnl. Phys. D: Appl. Phys.*, **22**(1989) 1883-1887.

16. P. S. Theocaris, On a family of quasi-isotropic fiber-reinforced composites, *Acta Mechanica*, **96** (**2**), (1993), 163-180.

17. G. W. Milton, Composite materials with Poisson's ratio close to -1. *Jnl. Mech. Phys. Solids*, **40**(**5**), (1992), 1105-1137.

18. A. G. Kolpakov, Determination of the average characteristics of elastic frameworks, *Prikl. Matem. Mekhan.*, **49**(**6**), (1985), 969-977.

19. K. Evans, Tailoring the negative Poisson's ratio, *Chemistry and Industry*, (Oct. 1990) 654-657.

20. A. K. Noor, M. S. Anderson, W. H. Green, Continuum models for beam-and platelike lattice structure, *AIAA Jnl.* **6**(**12**) (1978), 1219-1228.

21. J. A. Rinde, Poisson's ratio for rigid plastic foams, *Jnl. of Applied Polymer Science*, **14** (1970) 1913-1926.

22. R. M. Jones, *Mechanics of composite materials*, Scripta Book Co., Washington and McGraw Hill Intern., 1975.

23. E. A. Friis, R. S. Lakes, J. B. Park, Negative Poisson's ratio polymeric and metallic foams., *Jnl. of Material Science*, 23 (1988), 4406-4414.

24. P. S. Theocaris,Th. P. Philippidis, Variational bounds in the eigenvalue ω in transversely isotropic materials, *Acta Mechanica*, **85**(**1**), (1990), 13-26.

25. R. F. Almgren, An isotropic three-dimensional structure with Poisson's ratio =-1., *Jnl. of Elasticity*, **15** (1985), 427-430.

26. O. Sigmund, Material with prescribed constitutive parameters: an inverse homogenization problem. *Int. Jnl. Solids and Structures*, 31(**17**), (1994), 2313-2329.

27. L. V. Berlyand, S. M. Kozlov, Asymptotics of the homogenized moduli for the elastic chess-board composite, *Arch. Rat. Mech. Anal.*, **118**, (1992), 95-112.

28. P. S. Theocaris, P. D. Panagiotopoulos, Neural networks for computing in fracture mechanics. Methods and prospects of applications, *Computer methods in Applied Mechanics and Engineering*, 106(3), (1993), 213-228.

29. S. Kortesis, P. D. Panagiotopoulos, Neural networks for computing in structural analysis. Methods and prospects of applications, *Int. Jnl. of Numerical Methods in Engng.* (1993).

30. P. S. Theocaris, Failure modes of closed cell polyurethane foams, *Int. Jnl. of Fracture*, **4**, (1992), 353-375.

31. P. S. Theocaris, Th. P. Philippidis, Evaluation of transverse Poisson's ratios in thin orthotropic plates by the optical method of pseudocaustics *Acta Mechanica*, **43**(**4**), (1992), 369-378.

32. R. Lakes, Foam structures with a negative Poisson's ratio, *Science*, **235** (1987), 1038-1040.

33. R. S. Lakes, J. B. Park, E. A. Friis, Materials with negative Poisson's ratios: Dependence of properties on structure, *Proc. Am. Soc. of Composites, Third Tech. Conference Integrated Compositis Technology*, Technomic Publ. Co. Inc. Lancaster, Basel (1988), 527-533.

34. P. S. Theocaris, The beneficial influence of matrix anisotropy in fiber composites, *Acta Mechanica*, **97**(3-**4**), (1993), 127-139.

35. A. M. Garber, Pyrolitic materials for thermal protection systems, *Aerospace Engng.* (Jan. 1963), 126-137.

36. U. Lee, Equivalent continuum models of large platelike lattice structures, *Int. Jnl. Solids and Structures*, 3**1**(**4**), (1994), 457-467.

37. Υ. Li, The anistropic behaviour of Poisson's ratio, Υoung's modulus and shear modulus in hexagonal materials, Phys. Stat. Sol. (a), 3**8**, **171** (1976), 171-175.

38. A. E. H. Love, *A treatise of the mathematical theory of elasticity*, Dover, N. Υork, fourth edition, 1944.

39. Z. Hashin, B. W. Rosen, The elastic moduli of Fibre-Reinforced Materials, *Jnl. Appl. Mech.*, **31** (1964), 223-232.

40. P. S. Theocaris, *The Mesophase Concept in Composites*, Polymers/Properties and Applications Series Vol. No 11, Springer Verlag, Berlin etc. (1987).

41. P. S. Theocaris, A. G. Varias, The Influence of Mesophase of the Transverse and longitudinal moduli and the major Poisson's ratio in fibrous composites, *Colloid and Polymer Science*, **264** (**7**), (1986), 561-569.

ΠΕΡΙΛΗΨΗ

(Μικρο)-κατασκευαὶ μεταβλητοῦ λόγου poisson: ἀρχαὶ σχεδιασμοῦ

Οἱ αὐξημένες ἀπαιτήσεις γιὰ τὴν βέλτιστη ἀπόδοση τῶν σύγχρονων κατασκευῶν καὶ στοιχείων μηχανῶν σὲ συνδυασμὸ μὲ τὶς ἐξελίξεις στὴν ὑπολογιστικὴ μηχανικὴ καὶ στὴν τεχνολογία τῶν κατασκευῶν, καθιστοῦν σήμερα ἐφικτὴ τὴν βελτιστοποίηση ὑλικῶν καὶ κατασκευῶν. Ἡ φύση ἐξάλλου μᾶς διδάσκει ὅτι μονολιθικὰ μηχανικὰ συστήματα ἀποτελούμενα ἀπὸ ἰσότροπα καὶ ὁμογενῆ ὑλικὰ ἀπέχουν πολὺ ἀπὸ τὸ νὰ εἶναι βέλτιστα: δὲν ἔχει κάποιος παρὰ νὰ παρατηρήσει τὴν μορφὴ καὶ τὴν ἐσωτερικὴ δομὴ τῶν ὀστῶν γιὰ νὰ συνειδητοποιήσει πῶς περίπου μοιάζει ἕνα βέλτιστα σχεδιασμένο ὑλικό, τὸ ὁποῖον προῆλθε ἀπὸ βιολογικὲς διαδικασίες ἐξελίξεως διάρκειας πολλῶν αἰώνων.

Οἱ σύνθετες κατασκευὲς καὶ τὰ ὑλικὰ ἐπιτρέπουν, σὲ κάποια ἔκταση, τὴν ἐφαρμογὴ τῶν ἀρχῶν τοῦ βελτίστου σχεδιασμοῦ γιὰ τὴν σύνθεση ἑνὸς συστήματος ποὺ ταιριάζει στὶς τιθέμενες ἀπαιτήσεις. Κατασκευὲς ποὺ ἐμφανίζουν μέχρι καὶ ἀρνητικὸ λόγο Poisson εἶναι ἐφικτές, παρόλο ποὺ σὲ πρώτη θεώρηση φαίνονται νὰ εἶναι ἐξωτικές. Σκοπὸς τῆς παρούσης ἐργασίας εἶναι ἡ διερεύνηση τοῦ ἐλεγχόμενου, πιθανῶς ἀρνητικοῦ λόγου Poisson γιὰ κατασκευὲς καὶ μικροκατασκευὲς καὶ ἡ μελέτη τοῦ φαινομένου δ' ἀριθμητικῶν μεθόδων εἰς σύνθετα ὑλικὰ ἢ εἰς ὑλικὰ μὲ μικροδομὴ εἰσερχουσῶν γωνιῶν, μὲ τὴν βοήθεια μεθόδων ἀριθμητικῆς βελτιστοποιήσεως καὶ τεχνικῶν προσομοιώσεως μὲ πεπερασμένα στοιχεῖα.

Τὰ σύνθετα ὑλικά, ὅπως ἀκριβῶς καὶ τὰ ἀνθρώπινα ὀστᾶ, δὲν ἔχουν ὁμογενῆ καὶ ἰσότροπη μικροδομή. Ἀκόμη καὶ ἂν στὸ ἐπίπεδο τῆς κατασκευῆς χρησιμοποιοῦνται μέθοδοι τῆς μηχανικῆς τῶν συνεχῶν σωμάτων γιὰ τὴν μηχανικὴ προσομοίωση, ἡ μικροδομὴ τοῦ ὑλικοῦ εἶναι ἡ ἰδιότητα ἐκείνη ἡ ὁποία ἐπιτρέπει τὸν ἔλεγχο τῶν μηχανικῶν χαρακτηριστικῶν τοῦ ὑλικοῦ καὶ τὴν τροποποίησή τους ἀνάλογα μὲ τὶς ἀνάγκες σχεδιασμοῦ. Μὲ τὴν χρήση συγχρόνων τεχνικῶν βέλτιστου τοπολογικοῦ σχεδιασμοῦ, μὲ χρήση ὁμογενοποιήσεως, ἀντιμετωπίζεται τὸ πρόβλημα τοῦ ἀναλυτικοῦ σχεδιασμοῦ τοῦ ὑλικοῦ (π.χ. ἡ κατάλληλη μικροδομή, τὰ ὑλικὰ κλπ.). Στὴν συνέχεια στὸ ἐπίπεδο τῆς κατασκευῆς χρησιμοποιοῦνται τεχνικὲς προσομοιώσεως μὲ κατάλληλους, ὁλικοὺς φαινομενολογικοὺς νόμους ἐλαστικότητος, ὅπως προκύπτουν ἀπὸ τὴν συνολικὴ μηχανικὴ συμπεριφορὰ τοῦ χαρακτηριστικοῦ κελύφους τοῦ ὑλικοῦ. Οἱ ἀρχὲς τῆς μικρο-μακρο-προσεγγίσεως μποροῦν νὰ ἀναζητηθοῦν στὴν

προσομοίωση συνεχῶν κατασκευῶν μὲ μοντέλα ἐλαστικῶν, πλαισιακῶν φορέων. Γενικῶς προκύπτουν ὀρθότροπες ἐλαστικὲς σταθερὲς ποὺ συσχετίζονται μὲ τὶς καμπτικὲς ἀκαμψίες τῶν στοιχείων ποὺ συνιστοῦν τὰ τοιχώματα τοῦ κελύφους ἤ, σὲ ἕνα πιὸ γενικὸ πλαίσιο, μὲ τὴν τοπολογία τῆς μικροδομῆς (κέλυφος). Ἐπιπλέον, ἡ ἐλαστικὴ ἀστοχία σχετίζεται μὲ τὸν ἐλαστικὸ λυγισμὸ τῶν μελῶν, ἡ πλαστικὴ ἀστοχία προσομοιάζεται μὲ πλαστικὲς ἀρθρώσεις ποὺ ἐμφανίζονται στὰ μέλη καὶ ἡ ἀστοχία, μαζὶ μὲ τὴν μετάβαση ἀπὸ ἰσχυρὰ σὲ θλίψη σὲ ἰσχυρὰ σὲ ἑλκυσμὸ συμπεριφορὰ μὲ αὐξανόμενο πορῶδες, ἐξηγοῦνται ἐπίσης μὲ παρόμοια μοντέλα.

Τὸ φαινόμενο τοῦ ἀρνητικοῦ λόγου Poisson ἔχει ἐπίσης ἐξηγηθεῖ μὲ ὁμοιώματα ἐλαστικῶν πλαισίων μὲ μηχανισμούς. Πρέπει ἐδῶ νὰ τονισθεῖ ὅτι ἡ ἀνισοτροπία καὶ ὁ μεταβλητὸς λόγος Poisson δύναται νὰ ἔχουν θετικὴ ἐπιρροή, μεταξὺ ἄλλων στὰ χαρακτηριστικὰ ἀντοχῆς, στοὺς συντελεστὲς συγκεντρώσεως τάσεων καὶ στὴν ἀπόκριση ἀστοχίας τῆς κατασκευῆς. Εἰδικώτερον, ὑλικὰ μὲ ἀρνητικὸν λόγον Poisson ἐπιτρέπουν τὴν μείωση τῶν συντελεστῶν συγκεντρώσεως τάσεων καὶ τὴν κατασκευὴ συνθέτων πλακῶν καὶ δοκῶν ποὺ ἐπιτρέπουν ὁμαλὴ ἐπεξεργασία τους μὲ τεχνικὲς ψυχρῆς διαμορφώσεως (πρέσα). Ὅπως φαίνεται καὶ ἀπὸ τὴν σύντομη ἐπισκόπηση τοῦ πεδίου ποὺ παρατίθεται στὴν ἐργασία, εἶναι ἤδη γνωστὰ στὴν φύση ὑλικὰ μὲ παρόμοια μηχανικὴ συμπεριφορά. Ἐπίσης κλασικὰ θερμοπλαστικά, ἀφρώδη πολυμερῆ χαμηλῆς πυκνότητος μὲ ἀνοιχτὰ κελύφη ὁδηγοῦν σὲ ὑλικὰ μὲ ἀρνητικὸν λόγον Poisson μετὰ ἀπὸ κατάλληλη θερμομηχανικὴ ἐπεξεργασία.

Στὴν παροῦσα ἐργασία ἡ ἐμφάνιση καὶ ἡ ἱκανότητα σχεδιασμοῦ ὑλικῶν μὲ ἀρνητικὸν λόγον Poisson ἀποδεικνύεται μὲ τὴν βοήθεια ἀριθμητικῶν παραδειγμάτων ποὺ ὑπελογίσθηκαν μὲ τὴν μέθοδο τῶν πεπερασμένων στοιχείων, γιὰ πορώδη ὑλικὰ μὲ εἰσερχόμενες γωνίες καὶ διδιάστατες κελυφωτὲς (μικρο)κατασκευὲς μὲ μὴ-κυρτὴ μορφή, ὅπως ἐπίσης καὶ γιὰ σύνθετα ὑλικὰ μὲ ἀνάλογα ἐγκλωβίσματα μὲ εἰσερχόμενες γωνίες. Μελετῶνται ἐπίσης διάφορες πτυχὲς τῆς βάσεως τοῦ σχεδιασμοῦ καὶ τοῦ ἀντιστρόφου προβλήματος σχεδιασμοῦ (διαμόρφωση). Ἡ μέθοδος τῆς ἀριθμητικῆς ὁμογενοποιήσεως μὲ πεπερασμένα στοιχεῖα ποὺ χρησιμοποιεῖται γιὰ τὰ ἀριθμητικὰ παραδείγματα περιγράφεται μὲ κάποια λεπτομέρεια ἐπίσης στὴν ἐργασία.