ΜΑΘΗΜΑΤΙΚΑ. – **Density discontinuities and local energy decay in acoustic wave propagation,** *by George Dassios***. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

## 1. INTRODUCTION

The energy of an evolving physical system is a positive functional that depends on the state of the system at each moment. If the system is conservative then the energy functional is independent of time and is therefore a functional defined on the space of all permissible initial states, represented by the elements of an appropriate function space.

In what follows we will confine ourselves to physical systems that are governed by the scalar wave equation.

It is well known that, no matter what the number of dimensions is, the energy contained in a sphere of some fixed radius (the local energy) decays to zero as the time tends to infinity, as long as the initial energy is finite and the system is conservative. In Euclidean spaces with odd number of dimensions Huygen's principle assures that if the support of the initial disturbance is compact then the local energy becomes zero in finite time.

A lot of interest is concentrated on the way the solutions evolve in time in the case of unbounded domains with finite or infinite boundaries. These problems, where the solutions are prescribed on certain boundaries, form a huge area of research known as Scattering Theory. Results on local energy decay for scattering theory problems have been obtain by Morawetz [**6**, **7**, **8**, **9**], Lax and Phillips [**2**, **3**, **4**], Strauss [**12**, **13**], Ralston [**10**], Wilcox [**14**] and others. The standard method in getting decay theorems is the a, b, c - method due to Friedrich [**1**] where the equation is multiplied by a first degree differential form of the solution and then integrated over an appropriate space-time domain. Then the divergence theorem allows the application of the prescribed boundary conditions.

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In this paper we are concerned with the problem of energy decay for the scalar wave equation exterior to a finite region $B$ bounded by a smooth surface $\partial B$. The boundary conditions are such that the field penetrates the region $B$ which forms a discontinuity of the medium of propagation. In Acoustic terminology the solution represents the excess pressure field. The region $B$ forms a disturbance in the medium characterized by different values of mass density and compressibility (inverse bulk modulus of elasticity) from those of the surrounding medium. It is proved that the energy of the system is conserved when the elastic properties of the two media are the same and they only differ in their mass densities. Using a modified version of the multiplier that comes from the Kelvin transformation we were able to show that under certain conditions on the physical and geometrical characteristics of the scatterer the local energy of the system decays to zero as time tends to infinity.

## 2. CONSERVATION OF ENERGY

Consider the space $\mathbf{R}^3$ and the domain (open and connected set) $B \subset \mathbf{R}^3$. Assume that the domain $B$ is finite and that its boundary $\partial B$ is a $C^1$ closed surface. We will refer to $B$ as the «body» or the «scatterer» and to $\partial B$ as its surface. Let $u(\mathbf{x}, t)$ be a scalar function of class $C^2$ defined on $\mathbf{R}^3 \times [0, \infty]$. In Acoustic terminology $u(\mathbf{x}, t)$ will represent the excess pressure field evaluated at the space point $\mathbf{x}$ and the time moment t. Let $\varrho_+ (\varrho_-)$ and $M_+ (M_-)$ be the values of the mass density and the bulk modulus of elasticity in the exterior space $(\overline{B})^c$ (interior space $B$), where $\overline{B}$ indicates the topological closure of the scattering region $B$.

The field u has to satisfy the following initial-boundary value problem (I.B.V.P.) [**5**]

$$u_{tt}^{+} - c_{+}^{2} \Delta u^{+} = 0, \quad \mathbf{x} \in (\overline{B})^c, \quad t \geqslant 0 \tag{1}$$

$$u_{tt}^{-} - c_{-}^{2} \Delta u^{-} = 0, \quad \mathbf{x} \in B, \quad t \geqslant 0 \tag{2}$$

$$u^{+} = u^{-}, \quad \mathbf{x} \in \partial B, \quad t \geqslant 0 \tag{3}$$

$$\partial_n u^+ = \frac{\varrho_+}{\varrho_-}\partial_n u^-, \qquad \mathbf{x} \in \partial B, \qquad t \geqslant 0 \tag{4}$$

$$u^{\pm}(\mathbf{x}, 0) = f(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R}^3 \tag{5}$$

$$u_t^{\pm}(\mathbf{x}, 0) = g(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R}^3 \tag{6}$$

where the superscript $+$ $(-)$ denotes the field u exterior (interior) to the scatterer, the subscript t denotes partial differentiation with respect to time, $\partial_n$ denotes interior normal derivative on the surface $\partial B$, $\Delta$ denotes the Laplace's operator in $\mathbf{R}^3$ and f, g are the Cauchy data which are assumed to be in the class $C_0^\infty(B \cup B^c)$, i.e. infinitely differentiable functions with compact support. The problem has four physical parameters (the value of $\varrho_+$, $\varrho_-$, $M_+$, $M_-$) which are related by the formula [**11**]

$$\frac{c_+^2}{c_-^2} = \frac{M_+}{M_-} \cdot \frac{\varrho_-}{\varrho_+}. \tag{7}$$

It is easy to see from (4) and (7) that one can only consider the two physical parameters $\frac{M_+}{M_-}$ and $\frac{\varrho_+}{\varrho_-}$, i.e. the ratios of the two bulk modulei of elasticity and the two mass densities. The two phase velocities $c_\pm$ are given by [**11**]

$$c_\pm^2 = \frac{M_\pm}{\varrho_\pm}. \tag{8}$$

It is well known that the above formulated problem is well-posed and that its solution is a smooth function. If one allows the Cauchy data to be functions in the Sobolev space $W^{1,2} \equiv H^1$ with the energy norm

$$\| u(t) \|^2 = E(t) = \frac{1}{2}\int_B \left( |u_t^-|^2 + c_-^2 |\nabla u^-|^2 \right) dx + \frac{1}{2}\int_{B^c} \left( |u_t^+|^2 + c_+^2 |\nabla u^+|^2 \right) dx \tag{9}$$

then it can be showed (using the density of $C_0^\infty$ in $H^1$) that the problem remains well-posed. The integration in (9) is taken over $\mathbf{R}^3 - \{\partial B\}$ but this is equivalent to integration over the full space $\mathbf{R}^3$, since the Lebesque measure of $\partial B$ is zero.

The norm (9) gives the energy $E(t)$ of the acoustical disturbance at time t.

We are now in a position to prove the following theorem concerning the conservation of energy for such a system.

T h e o r e m 1: Suppose that u is a solution of the I.B.V.P. (1)-(6). If $M_+ = M_-$ then the energy $E(t)$, given by (9) is a constant independent of time. In other words if the exterior and interior media have the some elastic properties then the energy is conserved.

P r o o f: The proof is based on the well know identity

$$(u_{tt} - c^2 \Delta u) u_t = \nabla \cdot [-c^2 u_t \nabla u] + \frac{\partial}{\partial t} \left[ \frac{1}{2} \left( |u_t|^2 + c^2 |\nabla u|^2 \right) \right] \quad (10)$$

which is obtained by multiplying the wave equation by $u_t$.

Since u is a solution of the wave equation, (10) becomes

$$(\nabla, \partial_t) \cdot \left( -c^2 u_t \nabla u, \ \frac{1}{2} |u_t|^2 + \frac{c^2}{2} |\nabla u|^2 \right) = 0 \quad (11)$$

where $(\nabla, \partial_t)$ stands for the four dimensional (space-time) gradient.

Integrating (11) over the space-time region $B^c \times [0, t]$ and using the divergence theorem one obtains

$$\int_{S^+} \left( -c_+^2 u_t^+ \nabla u^+, \ \frac{1}{2} |u_t^+|^2 + \frac{c_+^2}{2} |\nabla u^+|^2 \right) \cdot (\mathbf{n}_x, n_t) \, ds = 0 \quad (12)$$

where $S^+ = (B^c \times \{0\}) \cup (\partial B \times [0, t]) \cup (B^c \times \{t\})$ and $(\mathbf{n}_x, n_t)$ is the inward (to the body) unit normal.

A similar integration over $B \times [0, t]$ gives

$$\int_{S^-} \left( -c_-^2 u_t^- \nabla u^-, \ \frac{1}{2} |u_t^-|^2 + \frac{c_-^2}{2} |\nabla u^-|^2 \right) \cdot (-\mathbf{n}_x, n_t) \, ds = 0 \quad (13)$$

where $S^- = (B \times \{0\}) \cup (\partial B \times [0, t]) \cup (B \times \{t\})$.

On the different part of $S^+$ and $S^-$ the unit normal becomes

$$(\mathbf{n}_x, n_t) = \begin{cases} (\mathbf{0}, -1), & \text{on } (B\times\{0\}) \cup (B^c\times\{0\}) \\ (\mathbf{n}_x, 0), & \text{on } \partial B\times[0,t] \\ (\mathbf{0}, 1), & \text{on } (B\times\{t\}) \cup (B^c\times\{t\}) \end{cases} \tag{14}$$

The sum of (12) and (13) gives

$$-\frac{1}{2}\int_B \left(|u_t^-|^2 + c_-^2 |\nabla u^+|^2\right)\Big|_{t=0} dx - \frac{1}{2}\int_{B^c} \left(|u_t^+|^2 + c_+^2 |\nabla u^+|^2\right)\Big|_{t=0} dx +$$

$$+\frac{1}{2}\int_B \left(|u_t^-|^2 + c_-^2 |\nabla u^-|^2\right)\Big|_{t} dx + \frac{1}{2}\int_{B^c} \left(|u_t^+|^2 + c_+^2 |\nabla u^+|^2\right)\Big|_{t} dx -$$

$$-\int_{\partial B\times[0,t]} \left(c_+^2\, u_t^+ \nabla u^+ - c_-^2\, u_t^- \nabla u^-\right)\cdot \mathbf{n}_x\, ds = 0\,. \tag{15}$$

Using the boundary conditions (3), (4) and the expression (9) for the energy one can rewrite (15) as

$$-E(0) + E(t) - \int_{\partial B\times[0,t]} \left(c_+^2 \frac{\varrho_+}{\varrho_-} - c_-^2\right) u_t^- \partial_n u^- \, ds = 0 \tag{16}$$

and since $M_+ = M_-$ the surface integral bocomes zero.

Therefore for each $t \geqslant 0$ we obtain

$$E(t) = E(0) = \frac{1}{2}\int_{\mathbf{R}^3} \left(|g|^2 + c_\pm^2 |\nabla f|^2\right) dx \tag{17}$$

which shows that the energy is a constant independent of time. This completes the proof of Theorem 1.

In what follows it is assumed that the elastic properties of the regions $B$ and $B^c$ are identical and therefore the energy is conserved.

## 3. LOCAL ENERGY DECAY

Our main result in this section is Theomem 2 which asserts that under certain conditions the energy inside a fixed sphere becays at least as fast as $t^{-2}$ when $t \to +\infty$.

In order to free the proof from computations and lengthy arguments we split it into a set of Lemmas.

It is well known that the Laplace's operator is invariant under the conformal group on $\mathbf{R}^n$, the group of transformations on $\mathbf{R}^n$ which preserves angles. The conformal group of $\mathbf{R}^n$ consists of four types of transformations: translations, rotations, dilations and inversions. Since trere are 4 independent translations, 6 rotations, 1 dilation and 4 inversions in $\mathbf{R}^4$, the dimension of the conformal group in $\mathbf{R}^4$ is 15. Therefore the Kelvin inversion

$$\bar{\mathbf{x}} = \frac{\mathbf{x}}{r^2 - c^2 t^2}, \quad \bar{t} = \frac{t}{r^2 - c^2 t^2}, \quad \bar{r}\bar{u} = ru \tag{18}$$

preserves the 4-dimensional Laplace's operator

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial (ict)^2} \tag{19}$$

which coinsides with the D'Alemberts operator $\square$. Therefore the wave equation is invariant under the transformation (18), i. e. if u is a solution of the wave equation in the $(\mathbf{x}, t)$-space then $\bar{u}$ (as it is given by (18)) is a solution of the have equation in the inverted $(\bar{\mathbf{x}}, \bar{t})$-space.

The Kelvin inversion (18) gives rise to the multiplier

$$(r^2 + c^2t^2)\, u_t + 2c^2 tr\, u_r + 2c^2 tu \tag{20}$$

which is obtained by inverting the basic energy multiplier $u_t$.

With this motivation in mind one has the following.

L e m m a 1 : For any smooth function u the following identity holds

$$(u_{tt} - c^2 \Delta u)\left[(r^2 + c^2 t^2)\, u_t + 2c^2 tr\, u_r + 2c^2 tu\right] = \nabla \cdot \mathbf{Y} + \partial_t X \tag{21}$$

where

$$\begin{aligned} \mathbf{Y} = &- c^2 t u_t^2 \mathbf{x} - 2c^4 tru_r \nabla u + c^4 t\, |\nabla u|^2 \mathbf{x} - \\ &- c^2 (r^2 + c^2 t^2)\, u_t \nabla u - 2c^4 tu \nabla u - \\ &- \frac{c^2}{r^2}\left[tc^2 u^2 + (r^2 + c^2 t^2)\, uu_t\right] \mathbf{x} \end{aligned} \tag{22}$$

$$X = 2c^2 t r u_r u_t + \frac{1}{2}(r^2 + c^2 t^2)(u_t^2 + c^2 |\nabla u|^2) +$$

$$+ 2c^2 t u u_t + \frac{c^2}{r^2}(r^2 + c^2 t^2)\left(r u u_r + \frac{u^2}{2}\right). \tag{23}$$

Proof: The proof is a tedious calculation based on standard differential vector identities and the commutation law

$$(\nabla u)_r - (\nabla u_r) = \frac{\mathbf{x}}{r^2} u_r - \frac{1}{r} \nabla u . \tag{24}$$

Lemma 2: Let $\mathbf{Y}^+$ and $\mathbf{Y}^-$ be as in (22) when $u^+$, $c_+$ and $u^-$, $c_-$ are used, respectively.

If $u^+$, $u^-$ satisfy the boundary conditions (3), (4) on $\partial B$ and the matrix

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & 0 \\ A_{12} & A_{22} & 0 & A_{24} & 0 \\ 0 & 0 & 0 & A_{34} & 0 \\ A_{14} & A_{24} & A_{34} & A_{44} & 0 \\ 0 & 0 & 0 & 0 & A_{55} \end{bmatrix} \tag{25}$$

where the entries $A_{ij}$ are given by (34)-(41), is positive semidefinite, then

$$\hat{\mathbf{n}} \cdot (\mathbf{Y}^+ - \mathbf{Y}^-) \geqslant 0 . \tag{26}$$

Proof: The boundary conditions on $\partial B$ assume continuity of the field and discontinuity of the normal derivative. Therefore the tangential derivatives of the field on $\partial B$ are continuous and the same is true for the time derivative. For the gradient field one has

$$\nabla u^+ = \nabla u^- + \hat{\mathbf{n}}\left(\frac{\varrho_+}{\varrho_-} - 1\right)\partial_n u^- . \tag{27}$$

Hence

$$u_r^+ = \hat{\mathbf{x}} \cdot \nabla u^+ = \hat{\mathbf{x}} \cdot \nabla u^- + (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}})\left(\frac{\varrho_+}{\varrho_-} - 1\right)\partial_n u^- =$$

$$= u_r^- + (\hat{\mathbf{x}} \cdot \hat{\mathbf{n}})\left(\frac{\varrho_+}{\varrho_-} - 1\right)\partial_n u^- . \tag{28}$$

Similarly

$$|\nabla u^+|^2 = (\nabla u^+) \cdot (\nabla u^+) =$$

$$= \left[\nabla u^- + \hat{\mathbf{n}}\left(\frac{\varrho_+}{\varrho_-} - 1\right)\partial_n u^-\right] \cdot \left[\nabla u^- + \hat{\mathbf{n}}\left(\frac{\varrho_+}{\varrho_-} - 1\right)\partial_n u^-\right] =$$

$$= |\nabla u^-|^2 + 2\left(\frac{\varrho_+}{\varrho_-} - 1\right)(\partial_n u^-)^2 + \left(\frac{\varrho_+}{\varrho_-} - 1\right)^2 (\partial_n u^-)^2 =$$

$$= |\nabla u^-|^2 + \left(\frac{\varrho_+^2}{\varrho_-^2} - 1\right)(\partial_n u^-)^2. \tag{29}$$

The relations (3), (4), (27), (28), (29), and

$$u_t^+ = u_t^- \tag{30}$$

connect the values of the fields $u^+$, $u^-$ and their derivatives on the scatterer's surface. Using these relations in (22) we obtain

$$\hat{\mathbf{n}} \cdot (\mathbf{Y}^+ - \mathbf{Y}^-) = -(c_+^4 - c_-^4)\frac{t}{r}(\hat{\mathbf{n}} \cdot \hat{\mathbf{x}})(u^-)^2 -$$

$$-\left[(c_+^2 - c_-^2)r^2 + (c_+^4 - c_-^4)t^2\right]\frac{1}{r}(\hat{\mathbf{n}} \cdot \hat{\mathbf{x}})\, u^- u_t^- -$$

$$- 2c_-^2 (c_+^2 - c_-^2)\, t u^- u_n^- - c_-^2 (c_+^2 - c_-^2)\, t^2 u_t^- u_n^- +$$

$$+ (c_+^4 - c_-^4)\, tr\, (\hat{\mathbf{n}} \cdot \hat{\mathbf{x}})\, |\nabla u^-|^2 - 2c_-^2 (c_+^2 - c_-^2)\, tr\, u_r^- u_n^- -$$

$$- (c_+^2 - c_-^2)\, tr\, (\hat{\mathbf{n}} \cdot \hat{\mathbf{x}})(u_t^-)^2 - (c_+^2 - c_-^2)^2\, tr\, (\hat{\mathbf{n}} \cdot \hat{\mathbf{x}})(u_n^-)^2 \tag{31}$$

where all the subscripts indicate differentiation. The relation (31) is a quadratic form in the vector variable

$$\mathbf{v} = (u^-, u_t^-, u_r^-, u_n^-, |\nabla u^-|) \tag{32}$$

i. e.

$$\hat{\mathbf{n}} \cdot (\mathbf{Y}^+ - \mathbf{Y}^-) = \mathbf{v} \cdot \mathbf{A} \cdot \mathbf{v}^\perp \tag{33}$$

where

$$A_{11} = \alpha \frac{t}{r}(c_+^4 - c_-^4) \tag{34}$$

$$A_{22} = \alpha\, tr\, (c_+^2 - c_-^2) \tag{35}$$

$$A_{44} = \alpha\, tr\, (c_+^2 - c_-^2)^2 \tag{36}$$

$$A_{55} = -\alpha\, \mathrm{tr}\,(c_+^4 - c_-^4) \tag{37}$$

$$A_{12} = \alpha \frac{1}{2r} \left[(c_+^2 - c_-^2)\, r^2 + (c_+^4 - c_-^4)\, t^2\right] \tag{38}$$

$$A_{14} = -t\, c_-^2\, (c_+^2 - c_-^2) \tag{39}$$

$$A_{24} = -t^2 \frac{c_-^2}{2} (c_+^2 - c_-^2) \tag{40}$$

$$A_{34} = -\mathrm{tr}\, c_-^2\, (c_+^2 - c_-^2) \tag{41}$$

$$\alpha = -\hat{\mathbf{n}} \cdot \hat{\mathbf{x}} \tag{42}$$

Since **A** is a positive semidefinite matrix we obtain the inequality (26) and the Lemma is proved.

The matrix **A** is independent of the particular solution u. The positive definiteness of **A** provides a condition connecting the physical (through $c_+$, $c_-$) properties of the media to the geometry (through $\alpha$) of the scattering region.

Lemma 3: Each smooth solution of the I.B.V.P. (1)-(6) satisfy the inequality

$$\int_{B^c \times \{t\}} X^+ dx + \int_{B \times \{t\}} X^- dx \leqslant \int_{B^c \times \{0\}} X^+ dx + \int_{B \times \{0\}} X^- dx \tag{43}$$

where $X^+ (X^-)$ is given by (23) where $u^+$, $c_+$ ($u^-$, $c_-$) are used in place of u, c, respectively.

Proof: Since $u^+$ satisfies equation (1) in $B^c$ the identity (21) gives

$$(\nabla, \partial_t) \cdot (\mathbf{Y}^+, X^+) = 0 \tag{44}$$

Similarly in $B$ we have

$$(\nabla, \partial_t) \cdot (\mathbf{Y}^-, X^-) = 0 \tag{45}$$

Integrating (44) over the four-dimensional region swept out by $B^c$ as time varies from 0 to t and applying the divergence theorem one obtains

$$\int_{B^c \times \{t\}} X^+ dx - \int_{B^c \times \{0\}} X^+ dx + \int_{\partial B \times [0,t]} \mathbf{Y}^+ \cdot \hat{\mathbf{n}}\, ds = 0 \tag{46}$$

A similar integration of (45) over the space-time cylindrical section $B \times [0, t]$ gives

$$\int_{B \times \{t\}} X^{-} dx - \int_{B \times \{0\}} X^{-} dx - \int_{\partial B \times [0, t]} \mathbf{Y}^{-} \cdot \hat{\mathbf{n}} \, dx + 2\pi t^2 c_{-}^4 u^{-}(\mathbf{0}, t) = 0 \quad (47)$$

where the last term in (47) is due to the integrable singularity $r^{-2}$ that is contained in the expression (23) for $X^{-}$.

Adding up (46) and (47), using the result (26) of Lemma 2 and dropping the positive term $2\pi t^2 c_{-}^4 u^{-}(\mathbf{0}, t)$ we obtain the inequality (43). This completes the proof of the Lemma.

L e m m a 4:

$$X \geqslant 0 \quad (48)$$

P r o o f: We observe that the expression (23) can be put into the following form

$$X = \frac{c^2}{2} (r^2 + c^2 t^2) (|\nabla u|^2 - u_r^2) +$$

$$+ \frac{c^2}{4r^2} \left\{ (r + ct)^2 \left[ (ru)_r + \frac{1}{c} (ru)_t \right]^2 + (r - ct)^2 \left[ (ru)_r - \frac{1}{c} (ru)_t \right]^2 \right\} \quad (49)$$

which is nonenegative since the magnitude of the radial derivative $u_r$ is always less or equal to the magnitude to the gradient $\nabla u$. This complete the proof of Lemma 4.

L e m m a 5: If $u^{\pm}$ is a solution of the I. B.V. P. (1) - (6) then the

(i) the solution $u^{\pm}$ is unique

(ii) there are positive constants $M_1$, $M_2$ such that

$$\int_{B^c} (u^{+})^2 dx \leqslant M_1 \quad (50)$$

$$\int_{B} (u^{-})^2 dx \leqslant M_2 \quad (51)$$

P r o o f: (i) If $u_1^{\pm}$, $u_2^{\pm}$ are two solutions of (1) - (6) then the function $u^{\pm} = u_1^{\pm} - u_2^{\pm}$ will satisfy equations (1)-(4) with Cauchy data identical zero. Then by (17)

$$E(t) = E(0) = 0, \quad t \geqslant 0 \quad (52)$$

which imply that all partial derivatives of $u^+$ are zero. This is true only when $u^+$ is constant for $t \geqslant 0$, $\mathbf{x} \in \mathbf{R}^3$. Finally the initial conditions give $u^{\pm} \equiv 0$ and the uniqueness result follows.

(ii) Suppose that $\omega^+$ is the solution of (1)-(4) with the Cauchy data

$$\omega^+(\mathbf{x}, 0) = h^{\pm}(\mathbf{x}) \tag{53}$$

$$\omega_t^{\pm}(\mathbf{x}, 0) = u(\mathbf{x}, 0) = f(\mathbf{x}) \tag{54}$$

where $h^{\pm}(\mathbf{x})$ is a function that satisfies the following eqnations

$$c^2 \Delta h(\mathbf{x}) = g(\mathbf{x}) \quad , \qquad \mathbf{x} \in \mathbf{R}^3 \tag{55}$$

$$h^+(\mathbf{x}) = h^-(\mathbf{x}), \qquad \mathbf{x} \in \partial B \tag{56}$$

$$\partial_n h^+(\mathbf{x}) = \frac{\varrho_+}{\varrho_-} \partial_n h^-(\mathbf{x}), \qquad \mathbf{x} \in \partial B \tag{57}$$

The function $\omega_t^{\pm} - u$ will then satisfy equations (1)-(4) and the initial conditions

$$(\omega_t - u)\Big|_{t=0} = f - f = 0 \tag{58}$$

$$(\omega_t - u)_t\Big|_{t=0} = c^2 \Delta\omega\Big|_{t=0} - g = c^2 \Delta h - g = g - g = 0 \tag{59}$$

By part (1) we obtain

$$\omega_t \equiv u \tag{60}$$

Finally we have

$$\int_{B^c} (u^+)^2 dx + \int_B (u^-)^2 dx = \int_{B^c} (\omega_t^+)^2 dx + \int_B (\omega_t^-)^2 dx \leqslant$$

$$\leqslant \int_{B^c} (|\omega_t^+|^2 + c_+^2 |\nabla\omega^+|^2) dx + \int_B (|\omega_t^-|^2 + c_-^2 |\nabla\omega^-|^2) dx \leqslant 2E(0) \tag{61}$$

and since the two integrals in the firs part of (61) are nonenegative we obtain the bounds (50) and (51).

L e m m a 6 : If u is a solution of the I.B.V.P. (1)-(6) then

$$\int_{B^c} X^+ du + \int_B X^- dx \leqslant \varrho^2 E(0) + M(\varrho) \tag{62}$$

where $X^+$, $X^-$ are as in Lemma 3, $E(0)$ is given by (9), $\varrho$ is the radius of a sphere that contains the support of the Cauchy data f, g and M is a constant that depends on $\varrho$.

Proof: After some calculations we obtain

$$\int_{B^c} X^+ dx + \int_B X^- dx =$$

$$= \frac{1}{2} \int_{B^c} \left[ r^2 \left( |u_t^+|^2 + c_+^2 |\nabla u^+|^2 \right) + c_+^2 \left( (u^+)^2 + 2r\, u^+ u_r^+ \right) \right] dx +$$

$$+ \frac{1}{2} \int_B \left[ r^2 \left( |u_t^-|^2 + c^2 |\nabla u^-|^2 \right) + c^2 \left( (u^-)^2 + 2r\, u \;\; u_r \right) \right] dx \leqslant$$

$$\leqslant \frac{\varrho^2}{2} \left[ \int_{B^c} \left( |u_t^+|^2 + c_+^2 |\nabla u^+|^2 \right) dx + \int_B \left( |u_t^-|^2 + c^2 |\nabla u \;\; |^2 \right) dx \right] +$$

$$+ \frac{c_+^2}{2} \int_{B^c} \left[ (1 + r)\, (u^+)^2 + r\, (u_r^+)^2 \right] dx +$$

$$+ \frac{c_-^2}{2} \int_B \left[ (1 + r)\, (u \;\; )^2 + r\, (u_r^-)^2 \right] dx \tag{63}$$

where the inequality is based on the fact that u and all its derivatives are zero outside the sphrere of radius $\varrho$ as well as the trivial inequality

$$2\alpha\beta \leqslant \alpha^2 + \beta^2 \tag{64}$$

By Lemma 5 we have

$$\frac{c_+^2}{2} \int_{B^c} \left[ (1 + r)\, (u^+)^2 + r\, (u_r^+)^2 \right] dx +$$

$$+ \frac{c_-^2}{2} \int_B \left[ (1 + r)\, (u^-)^2 + r\, (u_r^-)^2 \right] dx \leqslant \frac{c_+^2}{2} (1 + \varrho) \int_{B^c} (u^+)^2 dx +$$

$$+ \frac{c_-^2}{2} (1 + \varrho) \int_B (u^-)^2 dx + \varrho\, E(0) \leqslant$$

$$\leqslant \frac{c_+^2}{2} (1 + \varrho)\, M_1 + \frac{c_-^2}{2} (1 + \varrho)\, M_2 + \varrho\, E(0) = M(\varrho) \tag{65}$$

Hence

$$\int_{B^c} X^+ \, dx + \int_B X^- \, dx \leqslant \varrho^2 E(0) + M(\varrho) \tag{66}$$

and this comletes the proof of Lemma 6.

L e m m a 7 : If $t \geqslant \frac{4r}{c}$ then

$$\frac{c^2 t^2}{4} \left[ c^2 |\nabla u|^2 + u_t^2 \right] + \frac{c^4 t^2}{4} \nabla \cdot \left( \frac{u^2}{r^2} \mathbf{x} \right) \leqslant X \tag{67}$$

where X is given by (23) or (49).

P r o o f : The inequality $t \geqslant \frac{4r}{c}$ implies that

$$\frac{c^2 t^2}{2} \leqslant (r - ct)^2 \tag{68}$$

It is also true that

$$\frac{c^2 t^2}{2} \leqslant r^2 + c^2 t^2 \leqslant (r + ct)^2 \tag{69}$$

Using (68) and (69) in (49) one obtains

$$X \geqslant \frac{c^4 t^2}{4} \left( |\nabla u|^2 - u_r^2 \right) + \frac{c^4 t^2}{4r^2} \left[ (ru)_r^2 + \frac{r^2}{c^2} u_t^2 \right] =$$

$$= \frac{c^2 t^2}{4r^2} \left[ r^2 c^2 |\nabla u|^2 + r^2 u_t^2 + c^2 u^2 + 2c^2 r u u_r \right] \tag{70}$$

Using the vector identity

$$\nabla \cdot \left( \frac{u^2}{r^2} \mathbf{x} \right) = \frac{u^2 + 2r u u_r}{r^2} \tag{71}$$

in (70) we obtain (67). The proof of Lemma 7 is then completed.

T h e o r e m 2 : If u is the solution of the I.B.V.P. (1) - (6) and $\alpha (c_+^4 - c^4) \leqslant 0$ then

$$E_\varrho(t) = O(t^{-2}), \qquad t \to +\infty \tag{72}$$

where $E_\varrho(t)$ denotes the energy in the interior of a sphere of radius $\varrho$, at time t.

Proof: By Lemma 4 we obtain

$$\int_{B^c} X^+ dx + \int_B X^- dx \geqslant \int_{\substack{|\mathbf{x}| \leqslant \varrho \\ \mathbf{x} \in B^c}} X^+ dx + \int_B X^- dx \tag{73}$$

where we have assumed that the scatterer $B$ is inside the sphere or radius $\varrho$.

By Lemmas 7 and 6 we have that

$$\frac{c_+^2 t^2}{4} \int_{\substack{|\mathbf{x}| \leqslant \varrho \\ \mathbf{x} \in B^c}} [|u_t^+|^2 + c_+^2 |\nabla u^+|^2] dx + \frac{c_-^2 t^2}{4} \int_B [|u_t^-|^2 + c_-^2 |\nabla u^-|^2] dx +$$

$$+ \frac{c_+^4 t^2}{4} \int_{\substack{|\mathbf{x}| \leqslant \varrho \\ \mathbf{x} \in B^c}} \nabla \cdot \left( \left( \frac{u^+}{r} \right)^2 \mathbf{x} \right) dx +$$

$$+ \frac{c_-^4 t^2}{4} \int_{\substack{|\mathbf{x}| \leqslant \varrho \\ \mathbf{x} \in B^c}} \nabla \cdot \left( \left( \frac{u}{r} \right)^2 \mathbf{x} \right) dx \leqslant \varrho^2 E(0) + M(\varrho) \tag{74}$$

Setting

$$c_0 = \min\{c_+, c_-\} \tag{75}$$

and using the divergence theorem in the third and fourth integrals we obtain

$$\frac{c_0^2 t^2}{2} E_\varrho(t) + \frac{t^2}{4} (c_+^4 - c_-^4) \int_{\partial B} \frac{u^2}{r^2} (\mathbf{x} \cdot \hat{\mathbf{n}}) ds +$$

$$+ \frac{c_+^4 t^2}{4} \int_{|\mathbf{x}| = \varrho} \frac{(u^+)^2}{r} ds \leqslant \varrho^2 E(0) + M(\varrho) \tag{76}$$

By our hypotheses both integrals are nonenegative and therefore

$$\frac{c_0^2 t^2}{2} E_\varrho(t) \leqslant \varrho^2 E(0) + M(\varrho) \tag{77}$$

which is equivalent to

$$E_\varrho(t) = O(t^{-2}), \qquad t \to \infty \tag{78}$$

This completes the proof of Theorem 2.

## ΠΕΡΙΛΗΨΙΣ

Ἡ ἐργασία αὐτὴ ἀναφέρεται στὴν ἐνεργειακὴ συμπεριφορὰ τῶν λύσεων τῆς βαθμωτῆς κυματικῆς ἐξισώσεως στὸν τριδιάστατο Εὐκλείδειο χῶρο. Τὸ φυσικὸ πρότυπο ποὺ ἀντιστοιχεῖ σ' αὐτὴν τὴν περίπτωση εἶναι ἡ διάδοση ἀκουστικῶν κυμάτων. Ὁ χῶρος διαδόσεως ἐμφανίζει μιὰ ἀσυνέχεια στὶς φυσικές του ἰδιότητες ποὺ ἀναφέρεται στὴν πυκνότητα μάζας καὶ τὴ συμπιεστικότητα (ἀντίστροφο τοῦ μακροσκοπικοῦ μέτρου ἐλαστικότητας). Ἡ ἀσυνέχεια αὐτὴ περιορίζεται γεωμετρικὰ σὲ ἕνα φραγμένο ὑποσύνολο μὲ λεῖο σύνορο. Ἡ διαταραχὴ αὐτὴ μέσα στὸ χῶρο διαδόσεως λειτουργεῖ σὰν διαπερατὸς σκεδαστής. Ἂν ἀντιστοιχίσουμε τὸ γραμμικοποιημένο πεδίο ὑπερπιέσεως στὴ λύση τότε οἱ κατάλληλες συνοριακὲς συνθῆκες ἀπαιτοῦν τὴ συνέχεια τοῦ πεδίου ὑπερπιέσεως καὶ τὴν ἀσυνέχεια (μὲ πεπερασμένο σταθερὸ πήδημα) τοῦ πεδίου ταχυτήτων ποὺ εἶναι ἀνάλογο τῆς ἀποκλίσεως τοῦ πεδίου ὑπερπιέσεως. Στὸ Θεώρημα 1 ἀποδεικνύεται ὅτι ἂν ἡ ἀσυνέχεια στὸ χῶρο τῆς διαδόσεως ὀφείλεται μόνο σὲ διαφορὰ πυκνοτήτων καὶ ὄχι στὶς ἐλαστικὲς ἰδιότητες τῶν δύο ὑλικῶν, τότε ἡ ὁλικὴ ἐνέργεια διατηρεῖται (συντηρητικὸ σύστημα), ἐφ' ὅσον εἶναι ἀρχικὰ πεπερασμένη. Στὸ Θεώρημα 2 μετὰ ἀπὸ μιὰ ἀλληλουχία 7 λημμάτων, ἀποδεικνύεται ὅτι ἂν τὰ ἀρχικὰ δεδομένα τοῦ Cauchy ἔχουν συμπαγὲς στήριγμα, καὶ ἂν ἰσχύει μιὰ συνθήκη μεταξὺ τῶν φυσικῶν παραμέτρων καὶ τῶν γεωμετρικῶν χαρακτηριστικῶν τοῦ προβλήματος, τότε ἡ ἐνέργεια ποὺ περιέχεται σὲ μιὰ σφαίρα τυχούσας ἀκτίνας ἐξασθενεῖ τουλάχιστον σὰν $t^{-2}$ καθὼς ὁ χρόνος $t \to +\infty$. Οἱ ἀρχικὲς συνθῆκες ὑπετέθησαν τόσο λεῖες ὅσο ἀπαιτεῖ ἡ ἁπλούστευση τῶν ἀποδείξεων. Ὁ περιορισμὸς ὅμως αὐτὸς δὲν περιορίζει τὴν ἰσχὺν τῶν ἀποτελεσμάτων δεδομένου ὅτι ἂν οἱ ἀρχικὲς συνθῆκες εἶναι στοιχεῖα ἑνὸς κατάλληλου χώρου Sobolev τότε μὲ τὰ γνωστὰ ἐπιχειρήματα πυκνῶν ὑποχώρων μπορεῖ κανεὶς νὰ μεταφέρει τὰ ἀποτελέσματα στοὺς γενικοὺς αὐτοὺς χώρους τοῦ Sobolev. Τὰ ὁλικὰ θεωρήματα ἐνέργειας (νόμοι ἐνεργειακῆς διατηρήσεως) καὶ κυρίως τὰ θεωρήματα τοπικῆς ἐνεργειακῆς ἐξασθενήσεως στὸ χρόνο, ἀποτελοῦν τὶς θεμελιώδεις ἐκτιμήσεις (ἀνισότητες) ἐπάνω στὶς ὁποῖες στηρίζεται ἡ ποιοτικὴ μελέτη τῆς σύγχρονης θεωρίας σκεδάσεως στὰ πλαίσια τῶν ἡμιομάδων γραμμικῶν τελεστῶν καὶ τῆς φασματικῆς των ἀναλύσεως.

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