

Sonderfälle

1. *Symmetrische Bogen* : $\zeta_x = J_c / J_x \cos \varphi$,

Es gilt : $\gamma_L^R = \gamma_R^L = \gamma, \quad \lambda_L = \lambda_R = \lambda.$

$$\Delta H = -\lambda \frac{\Delta M_L - \Delta M_R}{(1 + \gamma)}$$

a) Einseitig fest eingespannte Bogen :

Für Gelenk an den Kampfer L : $\Delta M_L = -M_L, \quad \Delta M_R = \gamma M_L$

$$\Delta H = \lambda M_L$$

Für Gelenk an den Kampfer R : $\Delta M_L = \gamma M_R, \quad \Delta M_R = -M_R$

$$\Delta H = -\lambda M_R.$$

b) Zweigelenkbogen : $\Delta M_L = -M_L, \quad \Delta M_R = -M_R$

$$\Delta H = \lambda \frac{M_L - M_R}{1 + \gamma}$$

2. *Symmetrische Bogen* : $\zeta = 1$

Es gilt : $\lambda = \frac{15}{2f(9 + 4\nu)}, \quad \frac{\lambda}{1 + \gamma} = \frac{15}{4f(6 + \nu)}$

a) Einseitig fest eingespannte Bogen : $\Delta H = \pm \frac{15}{2f(9 + 4\nu)} M_{L,R}.$

$$\nu = 0 \quad \Delta H = \pm \frac{5}{6f} M_{L,R}$$

b) Zweigelenkbogen : $\Delta H = \frac{15}{4f(6 + \nu)} (M_L - M_R)$

$$\nu = 0 \quad \Delta H = \frac{5}{8f} (M_L - M_R).$$

ΕΦΗΡΜΟΣΜΕΝΗ ΣΤΑΤΙΚΗ. — Περί τοῦ ἔμβαδοῦ τοῦ ἀνηγμένου δια-
γραμματος τῶν ροπῶν κάμψεως ἀμφιερείστου δοκοῦ μεταβλητῆς δια-
τομῆς, ὑπὸ Ἀχιλλ. Π. Σιμοπούλου*. Ἀνεκοινώθη ὑπὸ τοῦ κ. Βασ. Αἰγινίτου.

Κατὰ τὴν ἔκφρασιν τῶν νόμων τῆς ἐλαστικῆς λειτουργίας τῶν φοριζομένων
δοκῶν ἀπαιτεῖται ἡ τιμὴ τοῦ ἔμβαδοῦ τοῦ ἀνηγμένου διαγράμματος τῶν ροπῶν

* ACHILL. P. SIMOPOULOS: Inhalt der verzerrte Momentenfläche des frei aufliegen-
den Trägers veränderlichen Querschnitts.

κάμψεως, ὡς καὶ ἡ στατική ροπή τούτου, ὡς πρὸς ἓν στήριγμα τῆς δοκοῦ. Ἦτοι ὁ ὑπολογισμὸς ἐκάστοτε τῆς τιμῆς τῶν ὀλοκληρωμάτων.

$$E_{av} = \frac{1}{EJ_c} \cdot \int_0^l \zeta_x M_x dx, \quad E_{av} \cdot \alpha_1 = \frac{1}{EJ_c} \int_0^l \zeta_x M_x x dx.$$

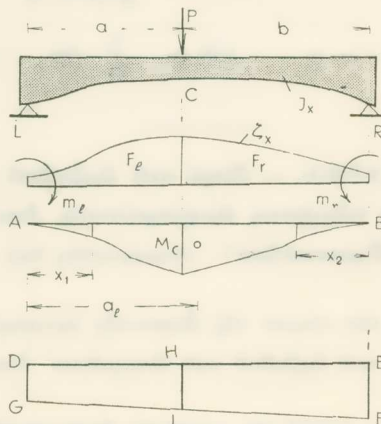
Ὁ ὑπολογισμὸς τῶν ὀλοκληρωμάτων τούτων ἐν τῇ περιπτώσει δοκῶν μεταβλητῆς διατομῆς, ἔνθα $\zeta_x =$ μεταβλητόν, ἐμφανίζει πολλὰς δυσκολίας συνεπεία τοῦ καταμερισμοῦ τῶν ὀλοκληρώσεων εἰς πολλὰς περιοχάς. Δι' ὃ δὲν ὑπάρχουσιν ἐν τῇ σχετικῇ τεχνικῇ βιβλιογραφίᾳ κλειστοὶ τύποι ὑπολογισμοῦ αὐτῶν.

Ἡ μαθηματικὴ διερεύνησις τοῦ θέματος τούτου κατέληξεν εἰς τὸ κάτωθι θεώρημα, δι' ἐφαρμογῆς τοῦ ὁποίου προσδιορίσθησαν ἐν συνεχείᾳ οἱ κλειστοὶ τύποι ὑπολογισμοῦ τῶν μεγεθῶν E_{av} , καὶ $E_{av} \cdot \alpha_1$ ὠρισμένων περιπτώσεων φορτιζομένων δοκῶν.

Θεώρημα. Ἡ εἰς θέσιν C ἐπὶ ἀμφιερείστου δοκοῦ δρῶσα δύναμις P διαχωρίζει τὴν ἐπιφάνειαν τῆς καμπύλης ζ_x τῆς δοκοῦ εἰς δύο μέρη, ὧν αἱ ροπαὶ ὡς πρὸς τὰ πλησίον κείμενα στηρίγματα, ἐὰν ληφθῶσιν ὡς ροπαὶ στηρίξεως τῆς δοκοῦ, ὁρίζουσιν ἰδεατὸν τραπέζιον ροπῶν, οὗ τὸ $\frac{P}{lJE_c}$ πλάσιον τῆς εἰς C τεταγμένης ἰσοῦται πρὸς τὸ ἔμβαδόν τοῦ συνεπείᾳ P ἀνηγμένου διαγράμματος τῆς δοκοῦ.

Ἀπόδειξις.

Ἐστω δοκὸς μεταβλητῆς ἀδρανείας ἐφ' ἧς ἐνεργεῖ δύναμις P (α, β). Τὸ ἔμβαδόν τοῦ ἀνηγμένου διαγράμματος θὰ ἰσοῦται :



(Σχ. 1)

$$E_{av} = \frac{1}{EJ_c} \int_0^l \zeta_x M_x dx = \frac{1}{EJ_c} \left[\int_0^a \zeta_x \frac{M_c}{a} x_1 dx_1 + \int_0^b \zeta_x \frac{M_c}{b} x_2 dx_2 \right] \quad (1)$$

Ἡ σχέσης αὕτη συναρτήσῃ τῆς $M_c = P ab/l$ γράφεται

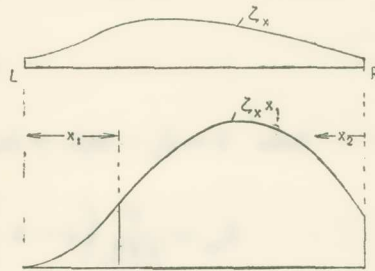
$$E_{av} = \frac{P}{lEJ_c} \left[b \int_0^a \zeta_x x_1 dx_1 + a \int_0^b \zeta_x x_2 dx_2 \right] \quad (2)$$

Ἐν τῇ σχέσει ταύτῃ παρατηροῦμεν πράγματι ὅτι ἡ ἐντὸς παρενθέσεων παράστασις ἐκφράζει τὴν τεταγμένην HI τοῦ τραπεζίου $DEFGD$ (σχ. 1), οὗ αἱ βάσεις ἰσοῦνται ἀντιστοίχως πρὸς τὰς ροπάς:

$$m_a = DG = \int_0^a \zeta_x x_1 dx_1, \quad m_b = EF = \int_0^b \zeta_x x_2 dx_2$$

τῶν ἐπιφανειῶν F_1, F_2 τῆς καμπύλης ζ_x , ὡς πρὸς τὰ γειτονικὰ στηρίγματα L, R τῆς δοκοῦ, ἀποδεικνυομένου οὕτω τοῦ θεωρήματος.

Ὡς πρὸς τὴν τιμὴν τῆς στατικῆς ροπῆς τοῦ ἀνηγμένου διαγράμματος ἰσχύουσι τὰ αὐτά, ἀλλὰ διὰ καμπύλην $\zeta_x x_1$ (σχ. 2).



(Σχ. 2)

Συμβολίζοντες συνεπῶς δι' αἱ τὴν ἀπόστασιν τοῦ $K. B.$ τῆς ἐπιφανείας E_{av} ἀπὸ τοῦ στηρίγματος L τῆς δοκοῦ θὰ ἔχωμεν τὸν τύπον.

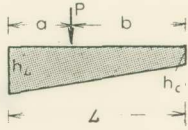
$$E_{av} \alpha'_i = \frac{P}{lEJ_c} \left[b \int_0^a \zeta_x x_1^2 dx_1 + a \int_0^b \zeta_x x_1 x_2 dx_2 \right] \quad (3)$$

ΕΞΑΓΟΜΕΝΑ ΕΦΑΡΜΟΓΗΣ ΤΩΝ ΤΥΠΩΝ (2), (3)

Κατωτέρω δίδομεν τὰ ἐξαγόμενα τῆς ἐφαρμογῆς τῶν τύπων (2), (3) ἐπὶ 24 περιπτώσεων φορτιζομένων δοκῶν.

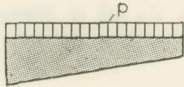
1. ΕΥΘΥΓΡΑΜΜΟΙ ΕΝΙΣΧΥΣΕΙΣ

$$q = h_c/h_L, \quad n = q^3, \quad k = 1 - q$$



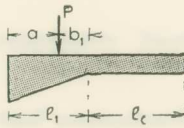
$$E_{av} = \frac{q^3 L P}{2 E J_c} \left(\frac{ab}{L - ka} \right)$$

$$E_{av} \alpha'_1 = \frac{n L^2 P}{2 k^2 E J_c} \left[\frac{kab}{q(L - ka)} + \frac{2}{k} \left(a \ln q - L \ln \frac{L - ka}{L} \right) \right]$$



$$E_{av} = \frac{p L^3 n}{2 k^2 E J_c} \left(\frac{1 + q}{2q} + \frac{\ln q}{k} \right)$$

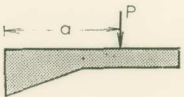
$$E_{av} \alpha'_1 = \frac{n L^4 p}{2 k^2 E J_c} \left[\frac{1 + 5q}{2kq} + \frac{2 + q}{k^2} \ln q \right]$$



$$E_{av} = \frac{P}{2 E J_c} \left[q_1 l \left(\frac{L - l_1}{k} - l_1 \right) \frac{\alpha}{L} - \frac{n l_1^2}{k} \frac{\alpha}{(l_1 - ka)} + \frac{l_c^2}{L} \alpha \right]$$

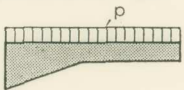
$$E_{av} \alpha'_1 = \frac{P}{E J_c} \left\{ \frac{n l_1^2}{2 k^2} \delta \alpha - \frac{n l_1^3}{2 k^2} \left[\frac{\alpha}{l_1 - ka} + \frac{2}{k} \left(\ln \frac{l_1 - ka}{l_1} - \frac{\alpha}{L} \ln q \right) \right] + \frac{(L^3 - 3L l_1^2 + 2l_1^3)}{6L} \alpha \right\}$$

$$\delta \alpha = (l_c - 2qL + 3ql_1) / q^2 L$$



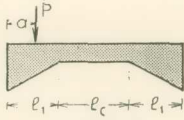
$$E_{av} = \frac{P}{2 E J_c} \left(\alpha - k \frac{l_1^2}{L} \right) b$$

$$E_{av} \alpha'_1 = \frac{P}{E J_c} \left[\frac{n l_1^3}{k^3} \left(\frac{k^2}{2q^2} - \frac{k}{q} - \ln q \right) \frac{b}{L} + \frac{(L + \alpha)ab}{6} - \frac{l_1^3 b}{3L} \right]$$



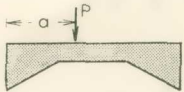
$$E_{av} = \frac{p}{2 E J_c} \left[\frac{n l_1^3}{k^2} \left(1 + \frac{\ln q}{k} \right) + \frac{q l_1^3}{2k} + \frac{q l_1^2}{2} (l_c - l_1) + \frac{L^3 - 3L l_1^2 + 2l_1^3}{6} \right]$$

$$E_{av} \alpha'_1 = \frac{p}{E J_c} \left\{ \frac{n l_1^4}{2 k^3} \left[3 + \frac{(1 - 2q)k}{2q^2} + \frac{(1 - 3q)k}{2q^2} \frac{l_c - l_1}{l_1} + \left(\frac{1 + 2q}{k} - \frac{l_c - l_1}{l_1} \right) \ln q \right] + \frac{L^4 - 4L l_1^3 + 3l_1^4}{24} \right\}$$



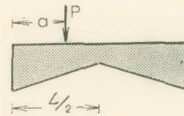
$$E_{av} = \frac{P}{2EJ_c} \left(\frac{ql_1}{k} \alpha - \frac{nl_1^2}{k} \frac{\alpha}{l_1 - k\alpha} + lc\alpha \right)$$

$$E_{av}\alpha'_1 = \frac{P}{EJ_c} \left\{ \frac{2nl_1^3}{k^3} \left(\frac{k}{q} - \frac{k^2}{2q^2} + \ln q \right) \frac{\alpha}{L} + \right. \\ \left. + nl_1^2 \left(\frac{1}{q^2} - \frac{1}{2k^2} - \frac{l_1}{2k^2(l_1 - k\alpha)} \right) \alpha - \frac{nl_1^3}{k^3} \ln \frac{l_1 - k\alpha}{l_1} + \frac{L^3 - 6Ll_1^2 + 4l_1^3}{6L} \alpha \right\}$$



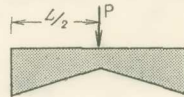
$$E_{av} = \frac{P}{2EJ_c} (ab - kl_1^2)$$

$$E_{av}\alpha'_1 = \frac{P}{EJ_c} \left\{ \frac{nl_1^3}{k^3} \left[\frac{k^2}{2q^2} - \frac{k}{q} - \ln q \right] \frac{L - 2a}{L} + \frac{nl_1^2 \alpha}{2q^3} + \right. \\ \left. + \frac{(4l_1^3 - 3l_1^2 L + L^2) \alpha - L \alpha^3 - 2l_1^2 L}{6L} \right\}$$

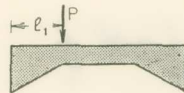


$$E_{av} = \frac{PL^2}{4kJ_c} \left(\frac{\alpha}{qL} - \frac{n}{2} \frac{\alpha}{(l_1 - k\alpha)} \right)$$

$$E_{av}\alpha'_1 = \frac{PnL^3}{8k^3EJ_c} \left\{ \left(\frac{2k}{q} - \frac{k^2}{q^2} + 2\ln q \right) \frac{\alpha}{L} + \left(\frac{2k^2}{q^2} - 1 \right) k \frac{\alpha}{L} - \right. \\ \left. - \frac{ka}{L - 2k\alpha} - \ln \frac{L - 2ka}{L} \right\}$$

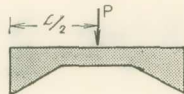


$$E_{av} = \frac{PL^2}{8EJ_c} q$$

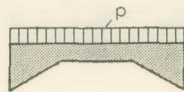


$$E_{av} = \frac{P}{2EJ_c} l_1(l_1 q + lc)$$

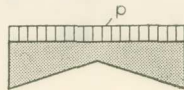
$$E_{av}\alpha'_1 = \frac{P}{EJ_c} \left\{ \frac{nl_1^3}{k^3} \left[\frac{k^2}{2q^2} - \frac{k}{q} - \ln q \right] \frac{lc}{L} + \frac{nl_1^2}{2q^2} + \frac{(L^3 - 6Ll_1^2 + 4l_1^3) l_1}{6L} \right\}$$



$$E_{av} = \frac{P}{8EJ_c} (L^2 - 4kl_1^2)$$



$$E_{av} = \frac{p}{2EJ_c} \left\{ \frac{2nl_1^3}{k^2} \left(1 + \frac{\ln q}{k} \right) + \frac{ql_1^3}{k} + ql_1^2 lc + \frac{L^3 - 6Ll_1^2 + 4l_1^3}{6} \right\}$$

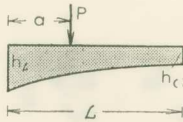


$$E_{av} = \frac{pL^3}{8kJ_c} \left[\frac{q}{2} + \frac{n}{k} \left(1 + \frac{1}{k} \ln q \right) \right]$$

2. ΠΑΡΑΒΟΛΙΚΑΙ ΕΝΙΣΧΥΣΕΙΣ

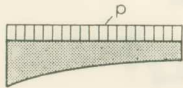
$$q = h_c/h_L, \quad k = 1 - q, \quad n = q^3, \quad \sqrt{\frac{q}{k}} \tau_{\text{οξ}} \varepsilon \varphi \sqrt{\frac{k}{q}} = \Phi$$

$$\sqrt{\frac{q}{k}} \tau_{\text{οξ}} \varepsilon \varphi \sqrt{\frac{k}{q}} \frac{b}{L} = \Phi_b, \quad \sqrt{\frac{q}{k}} \tau_{\text{οξ}} \varepsilon \varphi \sqrt{\frac{k}{q}} \frac{b_1}{l_1} = \Phi_{b_1}$$



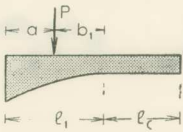
$$E_{\text{av}} = \frac{Pq^2L}{8kJ_c} \left\{ \frac{L^3}{qL^2 + kb^2} - \frac{a}{q} - b + 3 \frac{b}{q} (\Phi - \Phi_b) \right\}$$

$$E_{\text{av}} \alpha_1 = \frac{PL^2}{8EJ_c} \left\{ \frac{2k+1}{k} b \Phi - \frac{3kb+ql_1}{k} \Phi_b - \frac{qab^2}{ql_1^2 + kb^2} \right\}$$



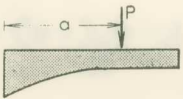
$$E_{\text{av}} = \frac{pqL^3}{16kJ_c} [1 + k - \Phi]$$

$$E_{\text{av}} \alpha'_c = \frac{pL^4}{8EJ_c} \frac{q}{k} (1 - \Phi)$$



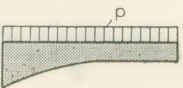
$$E_{\text{av}} = \frac{P}{8EJ_c} \left(\frac{q^2 l_1^2 (2l_1 - a) \alpha}{(q l_1^2 + k b_1^2)} + (4l_c^2 - q l_1^2) \frac{\alpha}{L} + 3(l_1^2 \frac{b}{L} \Phi - l_1 b_1 \Phi_{b_1}) \right)$$

$$E_{\text{av}} \alpha'_1 = \frac{P l_1^2}{8kJ_c} \left(-\frac{(L+l_c)q\alpha}{L} + \frac{q^2 l_1^2 \alpha}{q l_1^2 + k b_1^2} + \frac{l_1(2k+1)}{L} b \Phi - (3k b_1 + q l_1) \Phi_{b_1} \right) + \frac{P}{EJ_c} \frac{L^3 - 3L l_1^2 + 2l_1^3}{6} \frac{\alpha}{L}$$



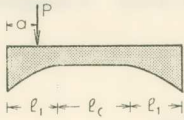
$$E_{\text{av}} = \frac{P}{2EJ_c} \left(ab - (k+3-3\Phi) \frac{l_1^2}{4} \frac{b}{L} \right)$$

$$E_{\text{av}} \alpha'_1 = \frac{P}{EJ_c} \left(-\frac{q l_1^3 b}{8kL} + \frac{l_1^3 b}{8L} \frac{2k+1}{k} \Phi + \frac{(L+a)ab}{6} - \frac{l_1^3 b}{3L} \right)$$



$$E_{\text{av}} = \frac{p}{16EJ_c} \left[\left(\frac{3l_c}{l_1} - \frac{q}{k} \right) l_1^3 \Phi + \frac{l_1^3}{k} - k l_1^2 L + \frac{4L^3 - 9L l_1^2 + 5l_1^3}{3} \right]$$

$$E_{\text{av}} \alpha'_1 = \frac{p}{EJ_c} \left\{ \frac{l_1^3}{16k} \left(q(2l_1 - l_c) + (3l_c - 2Lq)\Phi \right) + \frac{L^4 - 4L l_1^3 + 3l_1^4}{24} \right\}$$

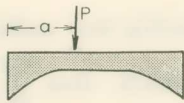


$$E_{av} = \frac{P}{8EJ_c} \left(\frac{(2l_1 - \alpha)\alpha}{ql_1^2 + kb_1^2} q^2 l_1^2 + 3l_1^2 \Phi - 3l_1 b_1 \Phi_{b_1} + 4l_c \alpha \right)$$

$$E_{av} \alpha_1 = \frac{P}{EJ_c} \left\{ -\frac{ql_1^2 \alpha}{8k} \left(\frac{L-2\alpha}{L} - \frac{ql_1^2}{ql_1^2 + kb_1^2} + q \right) + \right.$$

$$\left. + \frac{l_1^3}{8} \left(\frac{(2k+1)}{k} \frac{(L-2\alpha)}{L} l_1 + 3\alpha \right) \Phi - \frac{l_1}{8} \left(3l_1 b_1 + \frac{q}{k} l_1^2 \right) \Phi_{b_1} + \right.$$

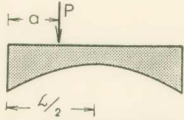
$$\left. + (L^3 - 6Ll_1^2 + 4l_1^3) \alpha / 6L \right\}$$



$$E_{av} = \frac{P}{8EJ_c} \left(ql_1^2 + 3l_1^2 \Phi + 4(ab - l_1^2) \right)$$

$$E_{av} \alpha_1 = \frac{P}{EJ_c} \left\{ \frac{ql_1^2}{8} \left(\alpha - \frac{l_1(L-2\alpha)}{kL} \right) + \frac{l_1^3}{8} \left(\frac{(2k+1)}{k} \frac{(L-2\alpha)}{L} l_1 + 3\alpha \right) \Phi \right.$$

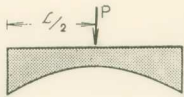
$$\left. + \frac{(4l_1^3 - 3l_1^2 L + L^3) \alpha - L\alpha^3 - 2l_1^3 L}{6L} \right\}$$



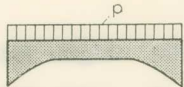
$$E_{av} = \frac{P}{8EJ_c} \left(ql_1^2 q \frac{b\alpha}{ql_1^2 + kb_1^2} + 3l_1^2 \Phi - 3l_1 b_1 \Phi_1 \right)$$

$$E_{av} \alpha_1 = \frac{PL^2}{32EJ_c} \left\{ \frac{q\alpha}{k} \left(\frac{qL^2}{qL^2 + 4kb_1^2} - 1 + \frac{2\alpha}{L} - q \right) + \right.$$

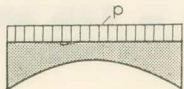
$$\left. + \left(\frac{(2k+1)}{2k} (L-2\alpha) + 3\alpha \right) \Phi - \left(3b_1 + \frac{qL}{2k} \right) \Phi_{b_1} \right\}$$



$$E_{av} = \frac{PL^2}{32EJ_c} (q + 3\Phi)$$



$$E_{av} = \frac{pl_1^2}{8EJ_c} \left\{ \left(q \frac{l_1}{k} + L \right) + (2L + lc \frac{l_1}{k}) \Phi + \frac{2L^3 - 6Ll_1^2 + 4l_1^3}{3l_1^2} \right\}$$



$$E_{av} = \frac{pL^3}{64EJ_c} \left\{ \frac{q}{k} (1 + 2k) + \frac{4k-1}{k} \Phi \right\}$$

ZUSAMMENFASSUNG

Bei der Berechnung des durchlaufenden Trägers mit veränderlichem Trägheitsmoment ist erforderlich die verzerrte Momentenfläche E_{av} sowie

ihr statisches Moment $E_{av}\alpha'_1$ bezüglich des linken b. w. rechten Auflagerstützes zu wissen. Es lautet :

$$E_{av} = \frac{1}{EJ_c} \int_0^l \zeta_x M_x dx, \quad \alpha'_1 E_{av} = \frac{1}{EJ_c} \int_0^l \zeta_x M_x x dx.$$

Mit J_c bezeichnen wir irgend ein festes, unveränderliches Trägheitsmoment und setzen : $\zeta_x = J_c / J_x$.

Die mathematische Untersuchung der obigen Formeln führte zum folgenden Satz (Abb, 1,2).

Der Inhalt der verzerrten Momentenfläche eines beiderseitig frei aufliegenden Balkens infolge einer Last im Punkte C, ist gleich dem $\frac{P}{lEJ_c}$ fache des Biegemomentes welches man im Punkt C bekommt wenn auf beiden Stützen die Momente

$$m_L = \int_0^a \zeta_x x_1 dx_1, \quad m_R = \int_0^b \zeta_x x_2 dx_2$$

wirken lässt.

Es bedeutet :

m_L, m_R : Das Moment der zwischen $\frac{l}{R}$ und P liegenden ζ_x Fläche bezogen auf den Auflager $\frac{l}{R}$.

Bezüglich des Momentes der verzerrten Momentenfläche bezogen auf den linken Auflager L, gilt die Kurve $\zeta_x x_1$.

Es ist (Abb. 2).

$$E_{av}\alpha'_1 = \frac{P}{lEJ_c} \left(b \int_0^a \zeta_x x_1^2 dx_1 + a \int_0^b \zeta_x x_1 x_2 dx_2 \right) \quad (3)$$

Durch Anwendung der Formeln (2) (3) ergeben sich die obigen Ausdrücke für die Berechnung der verzerrten Momentenfläche E_{av} sowie ihres Momentes $E_{av}\alpha'_1$ bezogen auf den linken Auflager L, von 24 Fälle belasteten Trägern.