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ΜΑΘΗΜΑΤΙΚΑ.— **On the problem of a local extension of the quantum formalism,** by *Thomas D. Angelidis*, διὰ τοῦ Ἀκαδημαϊκοῦ κ. Περικλέους Θεοχάρη.*

A B S T R A C T

The suggested impossibility of a consistent local extension of the quantum formalism is reviewed in the context of the Einstein - Podolsky - Rosen-Bohm (EPRB) ideal experiment, and a certain "impossibility proof" is shown to fall short of its stated goal. A consistent local theory Th (G) is proposed here, which shows that local action suffices to explain all that the quantum formalism predicts for the EPRB ideal experiment as well as some other results.

1. INTRODUCTION

Although perhaps largely forgotten now, almost all eminent physicists of von Neumann's generation were spellbound by his 1932 proof[1] of the suggested impossibility of a *consistent extension* of the quantum formalism (QF) by adjoining "hidden variables" to it. A "hidden variable" was postulated to be anything else not yet accounted for by the specification of the quantum state $|\psi>$ characterizing a physical system. Von Neumann's proof seemingly showed that the postulated existence of such "hidden variables" contradicts QF, which would have to be «objectively false»[1] in order that a finer specification of the state of a physical system could be possible than that stipulated by the quantum state $|\psi>$.

The issue was whether or not there were deeper layers of physical reality, such as those envisaged by de Broglie[2] and Einstein[3], not yet captured by the

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usual QF. Given that QF itself was not considered to be false, von Neumann's proof was hailed as ruling out the postulated existence of "hidden variables" and the de Broglie-Einstein fanciful notions of some physical reality extending beyond the horizon delimited by QF and its dominant interpretation advocated by Bohr and Heisenberg[4]. Nevertheless, on the issue of causality, von Neumann did not apparently attribute to his theorem the extraordinary claims attributed to it by others. For he clearly stated that «it would be an exaggeration to maintain that causality has thereby been done away with»[1].

In 1935 Grete Hermann[5] published a careful critique of the von Neumann theorem, and in particular of the von Neumann claim that one of the postulates of his proof, namely, the additivity postulate, was valid «under all circumstances»[1]. This claim, which with hindsight we may here call the von Neumann Universality Claim (cf. Section 2), asserted that the additivity postulate was valid for the class of all arbitrary states, which included both the class of all quantum states and the class of all "hidden-variable" states (the so-called "dispersion free" states). In her little appreciated essay (written in German) — it took until 1974 for her essay to be cited[6]—, Hermann argued that the additivity postulate could not be claimed to be valid for the class of all "hidden-variable" states as von Neumann had asserted. Thus, Hermann's argument essentially established the falsity of von Neumann's Universality Claim[7].

Also in 1935 Einstein, Podolsky, and Rosen[8] (EPR) put forward a brilliant (and now famous) argument which, without contradicting QF, cogently demonstrated the existence of "elements of reality", which had eluded the net of the usual specification of the quantum state $|\psi>$, namely, that a particle can at the same time possess a sharp position *and* momentum independently of any measurement. The EPR argument rested on the tacit but crucial assumption that there is no action at a distance. This crucial assumption was most reasonable in the light of Einstein's theory of special relativity, which prohibits any causal action or influence from propagating faster than light, and it was later explicitly formulated by Einstein[9] as the (weaker) *Principle of Local Action*. EPR showed that the quantum state $|\psi>$ does not provide a complete description of physical reality, but left open the question whether or not a finer description exists and concluded with the belief that such a theory is possible.

In 1952 Bohm[10] proposed an ingenious extension of de Broglie's[2] "pilot wave" theory showing explicitly how "hidden variables" could be consistently adjoined to QF, thereby circumventing von Neumann's impossibility proof, and how they could be interpreted as *definite* particle trajectories in the Galilean space-time underlying Bohm's (non-relativistic) theory. The manifest "elements of reality", in the shape of definite particle trajectories, were denied any existence in the Bohr-Heisenberg interpretation of QF.

Bohm's theory had indicated to some extent what paths to pursue (positive heuristic), and what paths to avoid (negative heuristic). The positive heuristic of Bohm's theory led to Bell's[11] praiseworthy critique, which essentially added to Hermann's critique the construction of a counterexample. More precisely, Bell demonstrated once again the falsity of von Neumann's Universality Claim (attributed to the additivity postulate) by exhibiting a counterexample showing that the additivity postulate was not satisfied for certain "hidden-variable" states (albeit of «no physical significance»[11]), which when averaged over gave results in agreement with QF. In the presence of this counterexample and of Bohm's theory, which both circumvented the no-hidden variable theorem, the von Neumann theorem was gradually laid to rest. And like the EPR paper, Bell's paper left open the question whether or not QF could be consistently extended by adjoining *local* 'hidden variables" to it.

On the other hand, the negative heuristic of Bohm's theory led to Bell's[12] replacement of von Neumann's impossibility proof by yet another seemingly more physically plausible impossibility proof whose spell-binding effect appears now as potent as von Neumann's was. Nevertheless, it will be argued here that Bell's impossibility proof (like von Neumann's) not only falls short of its stated goal, but leaves the real problem untouched.

The negative heuristic of Bohm's theory, which apparently motivated Bell's impossibility proof, consists of certain anomalous features of «extraordinary character»[11] that are now being presumed to constitute a necessary part of any attempt to explain the quantum-statistical *correlations* exhibited in the EPR - Bohm[13] (EPRB) *ideal* experiment. However, this is not so. The local explanatory theory of the EPRB ideal experiment proposed here is free from such anomalous features. We shall briefly describe the anomalous features in Bohm's theory and how they lead to an impasse if interpreted in the usual way.

In Bohm's theory, whenever a pair of particles (s_1, s_2) is characterized by a *nonfactorizable* quantum state $|s_1, s_2>$, the differential equations determining the particle trajectories are *coupled* via the so-called "quantum potential" Q_g (the subscript g in Q_g stands for "Galilean"). Although in general Q_g is a function of the two positions $\underline{r}_1$ and $\underline{r}_2$ of s_1 and s_2, unlike a classical potential, the values of Q_g do *not* decrease as the distance $|\underline{r}_1—\underline{r}_2|$ increases. Thus, no matter how far away s_1 may be located from s_2 (they could be located a whole Universe apart!), their trajectories remain mutually coupled.

Naturally, a question arises. Is the coupling of the trajectories of s_1 and s_2, which is preserved at *any* distance $|\underline{r}_1—\underline{r}_2|$, due to some physical action and, if so, how is it related to Q_g?

Under the usual interpretation, Q_g is said to induce an instantaneous physical action at any distance $|\underline{r}_1—\underline{r}_2|$ which couples the trajectories of s_1 and s_2. This instantaneous physical action at any distance is now referred to as non-local action in discussions of the EPRB experiment. In the Galilean space-time underlying Bohm's (non-relativistic) theory, the non-local physical action attributed to Q_g may be tolerated as a causal influence acting instantaneously at any distance. However, in the more fundamental Minkowski space-time of special relativity, the corresponding variant of the "quantum potential" Q_m (where the subscript m in Q_m stands for "Minkowski") has no licence to induce a causal influence acting instantaneously at any distance in Minkowski space-time, even if one were to accept, provisionally, that Q_m could be constructed since admittedly[14] Bohm's theory has no consistent relativistic extension.

Nevertheless, a consistent construction of Q_m is usually taken for granted, and furthermore Q_m is being interpreted[15] as inducing a causal influence connecting *spacelike-separated* events, that is, events which lie outside each other's light cones. But if this interpretation of Q_m were true, it would be *inconsistent* with the *causality* of special relativity. For it is not difficult to show (we shall not do so here) that, if it exists, any such causal connection *clashes* with the *causal structure* (order) of individual events in Minkowski space-time.

In discussions of the EPRB-type experiment designed by Aspect et al.[16], where correlated photons γ_1 and γ_2 are being emitted in opposite directions by some suitable source and then separately have their polarizations measured in two spacelike-separated regions, it is often suggested that there is no real inconsistency with the theory of special relativity because the causal "influence" induced by Q_m could not manifest itself at the *statistical level* in the form of controllable information (signal) being exchanged faster than light between two spacelike-separated regions. Thus, the suggestion goes, no relativistic prohibitions are being violated in EPRB-type experiments. Although correct for a different reason, this suggestion does not address the real issue. For, as indicated, the conflict with relativistic causality lies *deeper* than the statistical level: It lies at the level of individual events and of their particular outcomes in Minkowski space-time, where (if it exists) the causal connection induced by Q_m is presumably at work by exerting instantaneous changes in the physical properties ("elements of reality") attributed to individual particles located in spacelike-separated regions.

With this we conclude our brief description of the anomalous features in Bohm's theory and how they lead to an impasse if the notion of the "quantum potential" is interpreted in the suggested way. Next we shall consider Bell's impossibility proof, which was apparently motivated by the negative heuristic of Bohm's theory.

2. BELL'S CONJECTURE OF NONLOCALITY

Bell's[12] impossibility proof, which we shall here call Bell's *conjecture* of nonlocality, purports to show that QF cannot be consistently extended by adjoining local "hidden variables" and "elements of reality" to it. According to a recent book review,[17] Bell's conjecture asserts:

> «The incompatibility of any local hidden variables theory with certain quantum mechanical predictions.» (UC)

This assertion we have elsewhere[18] called the Universality Claim (UC) of Bell et al.[19,20], that is, for EPRB-type experiments, ALL local theories give predictions *different* from those of QF. There, we explained the unrecognized[21] crucial significance of UC as follows: If UC is true, then QF itself *must* be an action at a distance theory irrespective of any possible interpretation of QF. On the other hand, if UC is false, then QF could be a local theory. And, by an *ad absurdum* disproof, we showed that a *weaker* UC is false. This logically implies the falsity of the *stronger* UC of Bell et al in the form stated above[22].

A more precise formulation of this conjecture of nonlocality can be found in Bell's[12] earlier paper. We shall slightly sharpen Bell's own formulation here. Take the QF function P_{12}^{QF} defined on D^2 by

$$p_{12}^{QF}: (\alpha, \beta) \to \frac{1}{2} \cos^2(\alpha - \beta), \ \forall(\alpha, \beta) \in D^2, \tag{1}$$

which describes the quantum-statistical correlations exhibited in the EPRB *ideal* experiment for all values assigned to the variables α and β in the range D, that is $(\forall\alpha,\beta \in D)$. Each value of the function p_{12}^{QF} is interpreted as the (conditional) joint probability for the *coincidence detection (count)* of both photons γ_1 and γ_2 emitted in opposite directions, and after passing their respective polarizers P_1 and P_2. The photons γ_1 and γ_2 are born by the spontaneous annihilation decay of the (nonfactorizable) *singlet state* $|\gamma_1,\gamma_2>$ prepared by a suitable source. Under different *value assignments* in D, the values (elements of D) $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ assigned to the variables α and β are interpreted as the directions of the settings of the polarizers P_1 and P_2 respectively. The QF marginal probability functions are $p_{12}^{QF} = \frac{1}{2}$ and $p_{12}^{QF} = \frac{1}{2}$.

Let (L1) $\wedge$ (L2) $\wedge$ (L3) denote the logical conjunction of the three formal postulates of locality enunciated by Bell et al[12,20] (cf. Section 3), where the symbol '$\wedge$' stands for the (truth-functional) *conjunction*. Let T denote a theory whose postulates consist of the quadruple $<\Lambda,\rho,\hat{p}_1,\hat{p}_2>$, where Λ is the range of the variable λ, ρ is a *specified* function defined on Λ, and $\hat{p}_1$, $\hat{p}_2$ are *specified* functions defined on $\Lambda \times D$. Then, Bell's conjecture asserts that :

There exists NO consistent theory T whose postulates $<\Lambda,\rho,\hat{p}_1,\hat{p}_2>$ satisfy (L1) $\wedge$ (L2) $\wedge$ (L3) and such that

$$\left(\forall\alpha,\beta \in D\right)\left[\frac{1}{2} \cos^2(\alpha - \beta) = \int_\Lambda \rho(\lambda)\hat{p}_1(\lambda,\alpha)\hat{p}_2(\lambda, \beta)d\lambda\right] \tag{2}$$

holds.

Or, in Bell's[12] own words, the QF probability function p_{12}^{QF} «cannot be represented, either accurately or arbitrarily closely, in the form (2).»

However, a consistent theory T has been constructed[23] whose postulates $<\Lambda,\rho,\hat{p}_1, \hat{p}_2>$ do satisfy (L1) $\wedge$ (L2) $\wedge$ (L3) and generate a family of functions $\{p_{12}^{\mu} | \mu \in M\}$ which converges uniformly to a unique *limit function* identical with the QF function p_{12}^{QF} for $\forall\alpha,\beta \in D$, as the *syntactical form* (2) precisely requires. Thus, T refutes Bell's conjecture.

The theory T will be further developed here into a local explanatory theory Th(G) of the quantum-statistical correlations exhibited in the EPRB experiment in terms of the initial ("hidden") directions of the planes of polarization of each and every photon pair (γ_1, γ_2) being born by the spontaneous annihilation decay of the singlet states prepared by the source.

The theory Th(G) gives a causal and local ("common cause") explanation of *the* characteristic trait of the EPRB ideal experiment, where the directions (given by values of the variable μ) of the planes of polarization of each and every photon pair (γ_1, γ_2) are being *chosen at random* by the spontaneous annihilation decay of the singlet states prepared by the source, and where the directions (given by values of the variables α and β) of the polarizer settings are being *chosen at random* by the switches whilst the photons are in full flight as in the experiment designed by Aspect et al. [16].

The theory Th(G) is based on postulates of a structural character as Einstein had in mind (cf. Section 4). The postulates provide a consistent local extension of QF, and thereby circumvent Bell's impossibility proof. Furthermore, the precise possessed values of the adjoined "hidden variables" λ and μ can be envisaged as Einstein's "elements of reality" existing independently of any measurement and to some extent missing from the specification of the quantum state $|\gamma_1,\gamma_2>$.

Postulates Π_1 and Π_2 of the theory Th(G) describe the probabilistic local interaction between individual photons and polarizers. The more important postulate Π_3 describes the breaking of the spherical symmetry of the singlet state by introducing a slightly *finer* description than that given by the singlet state. Postulate Π_3 stipulates a conditional probability distribution for the spherically symmetric singlet state $|\gamma_1, \gamma_2>$ to spontaneously disintegrate into two back-to-back photons plane-polarized in a specific but randomly chosen direction, given by a value of the variable μ, out of all the equally likely choices of directions given by the range Λ of values of the variable λ. And each value of μ is *sufficient* to completely specify the direction of the plane of polarization of the two emerging back-to-back photons at the instant the singlet state explodes.

Postulate Π_3 in Th(G) is the *local realistic* counterpart of the nonfactorizable singlet state $|\gamma_1, \gamma_2>$ in QF: The nonfactorizable (linear superposition) quantum state $|\gamma_1, \gamma_2>$ involving two mutually exclusive alternatives is the counterpart of the sum of two real-weighted probability distributions for the alternatives in question stipulated by Π_3. Furthermore, postulate Π_3 explains how the common phase of the two emerging back-to-back photons plays a rather important role in the local realistic extension of the quantum state $|\gamma_1, \gamma_2>$ proposed here.

By the postulates of the theory Th(G), each pair (γ_2, γ_1,) of back-to-back photons is characterized by a *value* of μ, and the *ensemble* of such pairs being emitted by the source is characterized by the whole *range* M of values of μ specifying the initial directions of the planes of polarization of each and every pair of back-to-back photons emerging at the instant the singlet states explode by the process of spontaneous annihilation.

Also, over many experiments, each experiment involving one pair of back-to-back photons, the ensemble (population) of such pairs of photons being emitted by the source is uniformly distributed — axially invariant — over the range M. But instead of assuming it, the axial invariance of the distribution can be *deduced* from the theory Th(G).

The formal part of our proposed local explanatory theory Th(G) of the EPRB ideal experiment is established by the proof of the *conditional sentence* Σ, which is displayed in Section 5. The sentence Σ expresses the formal definition of the uniform convergence of the family of functions $\{p_{12}^{\mu} \mid \mu \in M\}$. In fact, Σ is the formal definition of the "limiting case" itself. This is what the universal quantifier ($\forall \varepsilon > 0$) in the prefix of Σ means. The *consequent* in the conditional sentence Σ defines the unique *limit function* of $\{p_{12}^{\mu} \mid \mu \in M\}$ by

$$\lim_{\mu, N} \{p_{12}^{\mu}(\alpha,\beta)\} = p_{12}^{QF}(\alpha, \beta) = \frac{1}{2}\cos^2(\alpha - \beta), \ \forall(\alpha, \beta) \in D^2, \qquad (3)$$

which is identical with the QF probability function p_{12}^{QF} for *all* values in the range D assigned to the variables α and β.

Let G denote a realization (structure) of the theory Th(G) in the model-theoretic sense (cf. Appendix B). By the *proof* of Σ, given in Appendix A, Σ is a *theorem* of Th(G), briefly expressed by Th(G) $\vdash \Sigma$. Thus, the sentence Σ is *valid* (*true*) *in* G, briefly $G \models \Sigma$, or G is a *model* of Σ. The proof of Σ and its interpretation in G demonstrates how the postulates of the theory Th(G) and the unique *limit function* (3) meticulously satisfy (L1) $\wedge$ (L2) $\wedge$ (L3).

The conditional sentence Σ and its physical interpretation, given in Section 5, underpins our proposed local explanatory theory Th(G) of the

EPRB ideal experiment. Also, the quantifiers occurring in the prefix of Σ hold the key to a proper understanding of the physics of the EPRB ideal experiment and of its characteristic trait (mentioned above). The role of these quantifiers will become more transparent in Section 5, where also some readily demonstrable formal features of the sentence Σ and their related physical interpretation will be discussed.

3. THE FORMAL POSTULATES OF LOCALITY

Bell et al [12,20] enunciated three conditions for locality, which intend to characterize physical locality. Here we shall collect together these conditions, and re-state them in the shape of three formal postulates of locality which any theory T, specified by some quadruple $<\Lambda,\rho,\hat{p}_1,\hat{p}_2>$, must satisfy if T is to qualify as a local theory in the sense of Bell et al.

(L1) Any joint probability function p_{12} must be defined as a specified instance of the syntactical form $p_{12}(\alpha, \beta) := \int_\Lambda \rho(\lambda)\hat{p}_1(\lambda,\alpha)\hat{p}_2(\lambda,\beta)d\lambda$, where any specified function $\hat{p}_1$ must not depend upon the variable β, and where any specified function $\hat{p}_2$ must not depend upon the variable α. [The form $\hat{p}_{12}(\lambda,\alpha,\beta) := \hat{p}_1(\lambda,\alpha)\hat{p}_2(\lambda,\beta)$ is known as the "factorizability condition".]

(L2) Any specified range Λ of the variable λ must not depend either upon the variable α or upon the variable β.

(L3) Any specified function ρ must not depend either upon the variable α or upon the variable β.

Note well that (L3) does not exclude the possibility that the function ρ may be chosen to depend upon some other variable, say μ, provided μ (like λ) is a variable *distinct* from both variables α and β.

Two formal reminders seem in order here. Firstly, two distinct variables may well have the same range. Otherwise, as Church[24] says, one would be faced with the absurdity that any two distinct variables x and y whose range is, say, some subset W of the set R of the real numbers must be identical. Secondly, in a formal language with equality, denoted by the predicate symbol '$\mathbf{=}$' (notice the difference between '$\mathbf{=}$' in boldface and '$=$' in what follows), the equation $x \mathbf{=} y$ can be satisfied even if the variables x and y are distinct. For by the Basic Semantic Definition[25] (BSD), the equation $x \mathbf{=} y$ is interpreted as follows. Let x_1 and y_1 denote respectively the values of the distinct variables x and y under the value assignment s_1 in some range W. Then, the equation $x \mathbf{=} y$ is true (satisfied) if $x_1 = y_1$, that is, if the values (elements of

W) assigned to the variables x and y under s_1 are the same. And the equation $x = y$ is false (not satisfied) if $x_1 \neq y_1$, that is, if the values assigned to the variables x and y under s_1 are different. Thus, the equation $x = y$ may be true for one value assignment s_1 in W and false for another value assignment s_2 in W. It is rather important to note that any value assignment s_1 in W is *independent* of any other value assignment s_2 in W even if s_1 and s_2 happen to agree on a given variable. On the other hand, two variables may be said to be identical if they are assigned the same values under *any* value assignment in *any* range.

These two formal reminders have the following physical significance. In the local theory Th (*G*), since each pair of back-to-back photons is characterized by a *value* of the variable μ, and since the initial directions (given by the range M of values of μ) of the planes of polarization of such pairs of photons born by the spontaneous annihilation process are random, *nothing* prevents the birth of a photon pair with a value μ_1 which *happens by pure chance* to be equal to a value α_1 or β_1 of the setting of a polarizer, that is, $\mu_1 = \alpha_1$ or $\mu_1 = \beta_1$ under some value assignment s_1 in *G*. It would be physically unreasonable to exclude *ad hoc* this perfectly local state of affairs. In fact, this local state of affairs would *refute* (L3), if (L3) were interpreted as forbidding the possibility $\mu_1 = \alpha_1$ or $\mu_1 = \beta_1$ under *some* value assignment s_1 in *G* (cf. paragraph G in Section 5). Furthermore, it would show that formal locality in the sense of Bell et al could *not* characterize physical locality in the very simple sense just described. In a parallel vein, nothing prevents the birth of another pair of back-to-back photons with a value μ_2 which happens by pure chance to be different (even light-years apart) from some values α_2 and β_2 of the settings of both polarizers so that $\mu_2 \neq \alpha_2$ and $\mu_2 \neq \beta_2$ under another value assignment s_2 in *G* (one would expect this situation to be true of almost all pairs of back-to-back photons emitted by the source). It will be shown that the conditional sentence Σ is *valid* (*true*) *in G* ($G \models \Sigma$) for *all* value assignments s in *G*, and therefore Σ is valid (true) for all such random choices of values *irrespective* of whether the *distance* $|\mu - \alpha|$ and $|\mu - \beta|$ between *any* values assigned to the variables μ,α,β is arbitrarily small or arbitrarily large (cf. paragraph B in Section 5).

4. THE POSTULATES OF THE THEORY Th(*G*)

The theory Th(*G*) is based on postulates of a structural character, which provide a consistent local extension of QF. The first two postulates Π_1 and Π_2

describe the standard probabilistic local interaction between individual photons and polarizers. Postulates Π_1 and Π_2 stipulate respectively two specified probability functions $\hat{p}_1$ and $\hat{p}_2$ defined on Λ x D by

$$(\Pi_1) \qquad \hat{p}_1\colon (\lambda,\alpha) \to \cos^2(\lambda-\alpha), \quad \forall(\lambda,\beta) \in \Lambda \times D$$

$$(\Pi_2) \qquad \hat{p}_2\colon (\lambda,\beta) \to \cos^2(\lambda-\beta), \quad \forall(\lambda,\beta) \in \Lambda \times D.$$

Since the symbol β does not occur in the definition of $\hat{p}_1$, and since the symbol α does not occur in the definition of $\hat{p}_2$, the functions $\hat{p}_1$ and $\hat{p}_2$ do manifestly satisfy the "factorizability condition" (see (L1) above) at the syntactical level. Also, $\hat{p}_1$ and $\hat{p}_2$ are *symmetrical* in the sense of being the same functions of their respective arguments. Furthermore, the values of $\hat{p}_1$ and $\hat{p}_2$ are bounded by 0 and 1, as probabilities should be. This answers what Feynman[26] has called the «fundamental problem».

The more important third postulate Π_3 describes how the spontaneous annihilation process itself breaks the spherical symmetry of the singlet state. It does so by introducing a slightly *finer* description than that given by the singlet state. In this sense, postulate Π_3 could be said to describe how the spontaneous annihilation process itself "collapses" or "disentangles" the singlet state, a description missing from QF.

Consider a photon pair (γ_1, γ_2) — or rather an ensemble of photon pairs — characterized by the quantum state

$$|\gamma_1, \gamma_2\rangle = \frac{1}{\sqrt{2}}\,[|x(\gamma_1)\rangle\,|x(\gamma_2)\rangle + |y(\gamma_1)\rangle\,|y(\gamma_2)\rangle], \tag{4}$$

known as the singlet state. Under the usual interpretation, $|x(\gamma_1)\rangle$ denotes the plane-polarized quantum state of photon γ_1 in the x direction and $|y(\gamma_1)\rangle$ denotes the plane-polarized quantum state of photon γ_1 in the y direction. Similarly, $|x(\gamma_2)\rangle$ and $|y(\gamma_2)\rangle$ denote the corresponding quantum states of photon γ_2.

We propose here the following realistic interpretation of the singlet state $|\gamma_1, \gamma_2\rangle$ as describing the *exclusive disjunction* (ED):

"both photons are plane-polarized in the x direction OR
both photons are plane-polarized in the y direction". $\qquad (ED_1)$

Next we shall describe the two characteristic features of our local realistic extension of the singlet state $|\gamma_1, \gamma_2>$, which are incorporated into the definition of ρ_p stipulated by postulate Π_3. This definition extends the linear superposition (nonfactorizable) quantum state $|\gamma_1, \gamma_2>$ of two mutually exclusive alternatives into the sum of two real-weighted probability distributions for the alternatives in question.

The first characteristic feature is this: There is nothing special about the orthogonal directions x and y in ordinary space (here "orthogonal" in ordinary space does correspond to "orthogonal" in the Hilbert space sense). One could equally well choose any other pair of orthogonal directions in ordinary space, say, λ and $\lambda + \frac{1}{2}\pi$ $\left(\text{or, say, } \lambda \text{ and } \lambda - \frac{1}{2}\pi\right)$, *all the more so since the singlet state is spherically symmetric.* Thus, the exclusive disjunction (ED_1) now reads:

"both photons are plane-polarized in the λ direction OR
both photons are plane-polarized in the $\lambda + \frac{1}{2}\pi$ direction", (ED_2)

where the values of λ specify any arbitrary direction in ordinary space.

The second characteristic feature is this: *Before* the spontaneous annihilation decay of the spherically symmetric singlet state $|\gamma_1,\gamma_2>$ occurs, *all* choices of directions λ are equally likely. *After* the spontaneous annihilation decay of $|\gamma_1,\gamma_2>$ has taken place, a *specific* direction in ordinary space, given by a value of the variable μ, has been randomly chosen by the spontaneous annihilation process itself.

The following postulate Π_3 incorporates these two characteristic features, partly shared by and partly missing from the specification of the singlet state $|\gamma_1,\gamma_2>$, into the definition of ρ_p (the subscript p in ρ_p stands for "photon pair"). Thus, we postulate the following probability distribution ρ_p and range Λ :

$$(\Pi_3) \qquad \rho_p(\lambda - \mu) := \frac{1}{2}\left[\delta(\lambda - \mu) + \delta\left(\lambda - \mu + \frac{1}{2}\pi\right)\right]$$

$$(\Pi_4) \qquad \Lambda := \{\lambda \mid -\infty < \lambda < +\infty\},$$

where δ is the Dirac distribution (*functional*; see below) and the range Λ includes all possible directions λ. The norm of ρ_p is one, that is, $\int_\Lambda \rho_p(\lambda - \mu) d\lambda = 1$. Since the symbols α and β do not occur in the definitions of ρ_p and Λ, postulates Π_3 and Π_4 do manifestly satisfy (L3) and (L2) respectively at the syntactical level.

Postulate Π_3 stipulates the conditional probability distribution ρ_P for the spherically symmetric singlet state $|\gamma_1,\gamma_2>$ to spontaneously disintegrate into two back-to-back photons plane-polarized in a *specific* but randomly chosen direction, given by a *value* of μ, out of *all* the equally likely choices of directions given by the range Λ of values of the variable λ (since $\int_\Lambda \rho_P(\lambda-\mu)d\lambda = 1$). The postulated distribution ρ_P gives each emerging pair of back-to-back photons the mark of its birth by the spontaneous annihilation decay of the singlet state. Also, the variables λ and μ, whose precise possessed values can be envisaged as Einstein's 'elements of reality', can be identified, as Bell[27] suggests, with the common "causal factors", where each value of μ can be regarded as the new local "element of reality" created by the spontaneous annihilation process itself. Furthemore, each value of the variable μ is *sufficient* to completely specify the initial direction of the *common* plane of polarization of the two emerging back-to-back photons at the instant the singlet state explodes: For by the definition of ρ_P, the exclusive disjunction (ED_2) becomes

"both photons are plane-polarized in the μ direction OR

both photons are plane-polarized in the $\mu - \frac{1}{2}\pi$ direction", (ED_3)

since integration over λ using a Dirac distribution almost amounts to substituting μ for λ (see below), and where each of the two mutually exclusive alternatives has probability of occurrence equal to $\frac{1}{2}$.

This also explains, in physical terms, why the *ensemble* of photon pairs must be characterized by the *whole* range M of values of μ specifying the initial directions of the common planes of polarization of *each* and *every* pair of back-to-back photons born by the spontaneous annihilation decay of the singlet states prepared by the source. Thus, by the rules of substitution the ranges Λ and M of the variables λ and μ must be the same so that

$$M := \{\mu \mid -\infty < \mu < +\infty\}, \tag{5}$$

where again the symbols α and β do not occur in the definition of the range M which (like Λ) manifestly satisfies (L2).

One could equally well choose for the singlet state the equivalent representation

$$|\gamma_1,\gamma_2> = \frac{1}{\sqrt{2}}[|R(\gamma_1)>|R(\gamma_2)> + |L(\gamma_1)>|L(\gamma_2)>]. \tag{6}$$

Under the usual interpretation, $|R(\gamma_1)>$ denotes the right-handed circularly (RHC) polarized quantum state of photon γ_1 and $|L(\gamma_1)>$ denotes the left-handed circularly (LHC) polarized quantum state of photon γ_1. Similarly, $|R(\gamma_2)>$ and $|L(\gamma_2)>$ denote the corresponding quantum states for photon γ_2. Now the proposed realistic interpretation of this representation of $|\gamma_1,\gamma_2>$ takes the shape of the following exclusive disjunction:

"both photons are RHC polarized OR
both photons are LHC polarized", (ED_4)

where again each of the two mutually exclusive alternatives has probability of occurrence equal to $\frac{1}{2}$.

The representation (6) of $|\gamma_1,\gamma_2>$ may suggest that it does not require any pair of orthogonal directions to define it. For it could be suggested that if a photon is RHC or LHC, it should not have anything to do with the x and y directions (or the λ and $\lambda + \frac{1}{2}\pi$ directions). But, as Feynman[28] stresses, it is not true that a RHC or a LHC photon looks the same for any pair of orthogonal directions. Its *phase* keeps track of the x (or y) direction. Similarly, the common phase of the two emerging back-to-back photons keeps track of the μ direction chosen by the spontaneous annihilation process itself. Thus, the common phase of the two emerging back-to-back photons plays a rather important role in the local realistic extension of $|\gamma_1,\gamma_2>$ proposed here. Yet, as far as we know, in discussions of the EPRB experiment the role of the common phase has remained virtually unrecognized.

By way of a heuristic illustration, one may depict the two photons as spinning rifle bullets, whose spin is either right-handed or left-handed with respect to their momentum directions, and think of the values of μ as specifying the common orientation of two back-to-back bullets just before the instant they are fired off in opposite directions. And given the pitch of each spinning bullet and its orientation at any other instant, one can always find which was the value of the initial common orientation.

To sum up: The postulates of the theory Th(G) have a structural character as Einstein had in mind. In particular, postulate Π_3 holds the key to a causal and local ("common cause") explanation of the quantum-statistical correlations $\frac{1}{2}\cos^2(\alpha-\beta)$ exhibited in the EPRB ideal experiment in terms of the initial ("hidden") directions μ of the common planes of polarization of each and every pair of back-to-back photons being born by the spontaneous annihilation decay of the singlet states prepared by the source.

Another reminder seems in order here concerning the Dirac distribution δ in terms of which the postulated distribution ρ_p is defined. By the term "distribution F" one means a functional defined on some *function space* Φ whose elements are called *test functions* φ. Thus, the "arguments" on which a functional F operates are the test functions $\varphi \in \Phi$ so that $F : \varphi \rightarrow F(\varphi)$. Yet the usual notation '$\delta(x)$' sometimes gives the rather unfortunate impression that the arguments of the distribution δ are real numbers rather than functions belonging to a suitable function space. This impression essentially treats the object '$\delta(x)$' as if it were a function (as opposed to a functional), and then incorrectly identifies the set $\{0\}$ of values of the variable x at which the "function" $\delta(x)$ is non-zero with, say, the set X on which the test functions $\varphi \in \Phi$ are defined. One may be inclined to dismiss this reminder as splitting hairs. But if the usual impression were true, then all integrals involving the "function" $\delta(x)$ would be zero since the set $\{0\}$ has measure zero. In fact, the integral of any function is zero when integrating over a set of measure zero. Thus, the usual notation "$\delta(x)$" must be read as $< \delta_X, \varphi > = \varphi(x)$, which means pick the value $\varphi(x)$ of the test function φ at $x \in X$ rather than of the distribution δ which has no defined value at any $x \varepsilon X$ at all (for there is no such thing as «the value of a distribution F at a point»[29]). In the case considered here, the test functions are $\hat{p}_1$ and $\hat{p}_2$, given by Π_1 and Π_2, and integration over λ using Dirac's "$\delta(\lambda - \mu)$" etc essentially amounts to substituting the variable μ for the variable λ occurring in $\hat{p}_1(\lambda,\alpha)$ and $\hat{p}_2(\lambda,\beta)$.

5. THE SENTENCE Σ AND ITS PHYSICAL INTERPRETATION

The local explanatory theory Th(G) of the EPRB ideal experiment proposed here is based on the (first-order) conditional sentence Σ

$$(\forall \varepsilon > 0)\, (\exists \eta > 0)\, (\forall \mu \in M)\, (\forall \alpha, \beta \in D)\, [(|\mu - \alpha| < \eta) \vee (|\mu - \beta| < \eta) \supset |p^{\mu}_{12}(\alpha,\beta) - p^{QF}_{12}(\alpha,\beta)| < \varepsilon], \qquad (\Sigma)$$

and on a structure G in which Σ is satisfied. Since a sentence (like Σ) *is valid iff it is satisfiable*, G is a *model* of Σ ($G \models \Sigma$). The *domain* (universe) of G is the set $R^{+} \times M \times D$ (cf. Appendix B).

Before we discuss the physical interpretation of the sentence Σ, it might be helpful if we were to begin by explaining the notation, terminology and certain salient points concerning the formal features of Σ. Later we shall explain the role played by the quantifiers occurring in the prefix of Σ, which hold the key to a proper understanding of the physics of the proposed local explanatory theory Th(*G*) of the EPRB ideal experiment.

The proof of the sentence Σ is given in Appendix A. The symbol '$\vdash$' is used to express the fact that the sentence Σ is a *theorem* of Th(*G*), briefly Th(*G*) $\vdash$ Σ. The variables ε and $\eta = 2\varepsilon$ (so that the choice of η depends only on ε) are assigned values in the set R^+ of positive real numbers. The symbols ($\forall$x) and ($\exists$x) stand for the *universal* and *existential* quantifiers respectively (when the operator variable is x). The symbol "v" stands for the (truth-functional) *inclusive disjunction*, and may be read as "or". The symbol "$\supset$" stands for the (truth-functional) *conditional* which, with some caution, may be read as "If..., then...". More details of the notation and terminology used here can be found elsewhere.[24,25]

The *antecedent* $(|\mu - \alpha| < 2\varepsilon) \vee (|\mu - \beta < 2\varepsilon)$ in the conditional sentence Σ is a *propositional form* [24] and as such it may be assigned the truth-value *truth* under one value assignment s_1 is *G* to the variables $\mu, \alpha, \beta, \varepsilon$, and the truth-value *falsehood* under another value assignment s_2 in *G*. With the sentence Σ *proved* (Th(*G*) $\vdash$ Σ), the sentence Σ is *valid (true) in G* (*G* $\models$ Σ) for *all* value assignments s in *G irrespective* of whether the antecedent in Σ is true of false by virtue of the Basic Semantic Definition[25] (BSD) of the (truth-functional) conditional connective "$\supset$" occurring in Σ. Furthermore, *whenever* the antecedent in Σ is satisfied (true), the *consequent* $|p^{\mu}_{12}(\alpha,\beta) - p^{QF}_{12}(\alpha,\beta)| < \varepsilon$ in Σ can be deduced from Σ by *modus ponens*. The so deduced consequent defines the unique *limit function* (3) of $\{p^{\mu}_{12} | \mu \in M\}$, which is identical with the QF joint probability function p^{QF}_{12} for *all* values in D assigned to the variables α and β.

The conditional sentence Σ expresses the *formal definition* of the uniform convergence of the family of functions $\{p^{\mu}_{12} \mid \mu \in M\}$. Since a family[30] (like a sequence) is itself a *function*, what is considered here is the function with domain the *index set* M and codomain the *indexed set* $\{p^{\mu}_{12} \mid \mu \in M\}$ of functions. Thus, each value of μ corresponds to a *member* p^{μ}_{12} of the set $\{p^{\mu}_{12} \mid \mu \in M\}$, where each function p^{μ}_{12} defined on $D^2 \times M$ is deduced as a *specified instance* of the syntactical form stipulated by (L1) using the conjunction $\Pi_1 \wedge \Pi_2 \wedge \Pi_3 \wedge \Pi_4$ of the postulates of Th(*G*) so that

$$p^{\mu}_{12}(\alpha,\beta) := \int_{\Lambda}\rho_p(\lambda-\mu)\hat{p}_1(\lambda,\alpha)p_2(\lambda,\beta)d\lambda$$
$$= \frac{1}{4}[1+\cos 2(\mu-\alpha)\cos 2(\mu-\beta)] \tag{7}$$

$$p^{\mu}_1 := \int_{\Lambda}\rho_p(\lambda-\mu)\,\hat{p}_1(\lambda,\alpha)\,d\lambda = \frac{1}{2} \tag{8}$$

$$p^{\mu}_2 := \int_{\Lambda}\rho_p(\lambda-\mu)\,p_2(\lambda,\beta)\,d\lambda = \frac{1}{2} \tag{9}$$

The (constant) marginal probability functions p^{μ}_1 and p^{μ}_2 are identical with the (constant) QF marginal probability functions $p^{\mu}_1 = p^{QF}_1 = \frac{1}{2}$ and $p^{\mu}_2 = p^{QF}_2 = \frac{1}{2}$.

Given that each value of μ determines the common plane of polarization of each pair of back-to-back photons born at the instant the singlet state explodes, each joint probability function p^{μ}_{12} describes a purely *local* and *probabilistic* interaction between each correlated photon γ_1 (γ_2) and its corresponding polarizer P_1 (P_2), where each polarizer acts independently of the other polarizer precisely as stipulated by the "factorizability condition" (L1). Using the properties of the postulated distribution ρ_p, this can be seen from

$$p^{\mu}_{12}(\alpha,\beta) = \frac{1}{2}\left[\hat{p}_1(\mu,\alpha)\,\hat{p}_2(\mu,\beta) + \hat{p}_1(\mu-\frac{1}{2}\pi,\alpha)\,\hat{p}_2(\mu-\frac{1}{2}\pi,\beta)\right], \tag{10}$$

which shows that each p^{μ}_{12} can be written as the *sum* of two real-weighted products $\hat{p}_1\hat{p}_2$ of probabilities corresponding to the two mutually exclusive alternatives described by (ED_3). Recall that each function $\hat{p}_1$ ($\hat{p}_2$), given by Π_1 (Π_2), determines the probability that each photon γ_1 (γ_2) *will* get through its corresponding polarizer P_1 (P_2), *given* that photon γ_1 (γ_2) is plane-polarized in the μ direction OR in the $\mu-\frac{1}{2}\pi$ direction. Thus, given any value of μ, each polarizer interacts with its own photon since, in Bell's[27] own words, (10) already incorporates his «hypothesis of "local causality" or "no action at a distance"».

Before we explain how the conditional sentence Σ describes the *local* overall response of the apparatus (polarizers & detectors) to each and every pair of back-to-back photons emitted by the source, we should discuss some preliminary formal features of Σ and their physical interpretation.

(A) Since *any* two value assignments s_1 and s_2 in G are *independent* of each other (even if s_1 and s_2 happen to agree on a given variable), for *all* value assignments s in G, the value in M assigned to the variable μ is *independent* of the values in D assigned to the variables a and β. In other words, the *choice* of any value from M assigned to the variable μ is *independent* of the *choice* of any values from D assigned to the variables α and β for all $\mu \in M$ and all $\alpha, \beta \in D$.

On this formal feature of Σ is based the physical explanation of *the* characteristic trait of the EPRB ideal experiment, where the directions (given by values of the variable μ) of the common planes of polarization of each and every pair of back-to-back photons are being *chosen at random* by the spontaneous annihilation decay of the singlet states prepared by the source, and where the directions (given be values of the variables α and β) of the polarizer settings are being *chosen at random* by the switches whilst the photons are in full flight as in the experiment designed by Aspect et al[16].

(B) Since Σ is a *theorem* of Th(G) (Th(G) $\vdash \Sigma$), the sentence Σ is *valid (true) in* G ($G \models \Sigma$) for *all* value assignments s in G *irrespective* of whether the antecedent $(|\mu - \alpha| < 2\varepsilon) \vee (|\mu - \beta| < 2\varepsilon)$ in Σ is true or false. Note well that the antecedent in Σ does not impose any restriction on the value assignments s in G. All value assignments s in G are equally free. Thus, the sentence Σ is valid in G for all value assignments s in G *irrespective* of whether the *distance* $|\mu - \alpha|$ and $|\mu - \beta|$ between *any* values assigned to the variables μ, α, β is arbitrarily small or arbitrarily large (cf. last paragraph of Section 3).

On this formal feature of Σ is based the physical explanation that the source may choose to emit a pair of back-to-back photons with a value $\mu_1 \in M$ being arbitrarily close to, OR another pair of back-to-back photons with a value $\mu_2 \in M$ being light-years apart from any chosen values from D assigned to the variables α and β, *this choice being made by the spontaneous annihilation process itself or, say, by the outcome of the toss of a coin. The sentence* Σ *is valid (true) for all such random choices.*

(C) Each *instantiation* of the conditional sentence Σ describes as ONE experiment the *local* overall response of the apparatus (polarizer & detectors) to a *single* pair of back-to-back photons emitted by the source. To see this, write the antecedent in Σ in the form $S_\alpha \cup S_\beta$, where $\{S_\alpha \mid \alpha \in D\}$ and $\{S_\alpha \mid \beta \in D\}$ are indexed subsets of M defined by

$$S_\alpha := \{\mu \mid -2\varepsilon + \alpha < \mu < \alpha + 2\varepsilon) \tag{11}$$

$$S_\beta := \{\mu \mid -2\varepsilon + \alpha < \mu < \alpha + 2\varepsilon), \tag{12}$$

and such that

$$M = U_\alpha S_\alpha = U_\beta S_\beta. \tag{13}$$

That is, the range (set) M is the set-theoretic union of the subsets S_α or S_β, which can be interpreted as angular sectors. Note that the index variable α (β) occurring in $U_\alpha S_\alpha$ ($U_\beta S_\beta$) is *bound* ("dummy") so that any other index variable, say, ξ could replace α (β).

What each instantiation of Σ says is the following. For *any* chosen values of α and β, *whenever* a value of μ, characterizing the *random* direction of the common plane of polarization of a single pair of back-to-back photons, *happens by pure chance* to belong to subset S_α or S_β, this single pair of back-to-back photons gets through polarizers P_1 and P_2 *and causes* a coincidence count with probability given by a value of the QF probability function p_{12}^{QF}.

More precisely, let s_1 be a value assignment in *G* of some instantiation of the sentence Σ, and let μ_1, α_1, β_1 be the values in M and D assigned to the variables μ,α,β under s_1. *Whenever* the antecedent in the conditional sentence Σ is satisfied (true), that is,

$$\mu_1 \in S_{\alpha_1} \subset M \quad \text{OR} \quad \mu_1 \in S_{\beta_1} \subset M, \tag{14}$$

the consequent in Σ can be deduced from Σ by *modus ponens*. And the so deduced consequent determines the probability of the single pair of back-to-back photons characterized by the (random) value $\mu_1 \in M$ to get through the polarizers *and cause* a coincidence count, this propability being equal to the value $\frac{1}{2}\cos^2(\alpha_1 - \beta_1)$ under s_1 of the QF probability function p_{12}^{QF} defined by (1).

Naturally, a question arises. Do the detectors *only* register those pairs of back-to-back photons with values of μ belonging to the subsets S_{α_1} or S_{β_1}? The answer is: No. To see this, consider the following question which the local theory Th(*G*) readily answers.

What happens to the single pair of back-to-back photons if its value $\mu_1 \in M$ is such that the antecedent in Σ is *not* satisfied, that is, if

$$\mu_1 \notin S_{\alpha_1} \quad \text{AND} \quad \mu_1 \notin S_{\beta_1}? \tag{15}$$

If so, then the consequent in Σ *cannot* be deduced from Σ (we should warn that it is *not* sound to deduce the negation of the consequent in Σ from the conditional Σ and the negation of the antecedent in Σ; to presume that it is sound is to commit the common fallacy known in logic as "denying the antecedent"). Thus, it cannot be asserted that the single pair of back-to-back photons with $\mu_1 \in M$ causes a coincidence count with probability $\frac{1}{2}\cos^2(\alpha_1 - \beta_1)$. But the single pair of back-to-back photons with $\mu_1 \in M$ may fall inside another subset, say, $S_{\alpha 4}$ or $S_{\beta 4}$ of the set M, that is,

$$\mu_1 \in S_{\alpha 4} \subset M \quad \text{OR} \quad \mu_1 \in S_{\beta 4} \subset M, \tag{16}$$

so that it causes a coincidence count with a *different* probability $\frac{1}{2}\cos^2(\alpha_4 - \beta_4)$, determined by the consequent in Σ deduced from Σ (by *modus ponens*) under another value assignment s_4 in G which agrees with s_1 on the variable μ ($\mu_1 = \mu_4$).

This argument, based upon instantiating Σ, also shows how the apparatus responds, as it should, to the *whole* range M characterizing the *ensemble* of pairs of back-to-back photons emitted by the source and not only to those pairs of back-to-back photons with values of μ belonging, say, to the "small' subset $S_{\alpha 1}$ or $S_{\beta 1}$. This is what the universal quantifiers ($\forall \mu \in M$) and ($\forall \alpha, \beta \in D$) in the prefix of Σ do: They take into account ALL the "small" subsets S_α or S_β of M.

(D) The universal quantifiers ($\forall \mu \in M$) and ($\forall \alpha, \beta \in D$) occurring in the prefix of the sentence Σ take into account the *whole* array of such possibilities (by instantiation) so that the detectors accordingly register coincidence (and single) counts with the same probabilities as those given by QF for each and every pair of back-to-back photons being emitted by the source. Thus, in the light of the conditional sentense Σ, each value of the QF probability function p_{12}^{QF} can now be interpreted, in purely *local* terms, as the measure of the *chance* of each pair of back-to-back photons, upon being born by the spontaneous annihilation process, to get through the corresponding polarizers P_1 and P_2 and cause a coincidence count.

Furthermore, since the QF probability function p_{12}^{QF} is *deducible* from the postulates of Th(G) via the consequent in the conditional sentence Σ, and since in all experiments p_{12}^{QF} has been found to correctly describe the response of the apparatus (coincidence & single counts), it follows that the postulates of Th(G) not only correctly describe the *local* overall response of the apparatus to the whole ensemble of photon pairs emitted by the source (characterized by M), but more importantly the postulates of Th(G) also give a detailed realistic description of this response in terms of the precise possessed values of the *local* variable μ, which specify the initial ("hidden") directions of the common planes of polarization of each and every pair of back-to-back photons born at the instant the singlet states explode. This realistic "common cause" explanation of the quantum-statistical correlations exhibited in the EPRB ideal experiment is missing from QF.

(E) Over many experiments, each experiment involving one pair of back-to-back photons, the ensemble of such pairs of photons being emitted by the source is uniformly distributed — axially invariant — over the range M. But instead of assuming it, the axial invariance of the distribution can be deduced from Th (G) as follows.

Let the source be fixed. Rotate both polarizers P_1 and P_2 about their common z axis so that their *relative* setting $\theta = |\alpha - \beta|$ is fixed to some arbitrary value (we may fix both polarizers and rotate the source about the z axis; the situation is completely symmetrical). Since the *choice* of any value from M assigned to μ is *independent* of the *choice* of any values from D assigned to α and β for all $\mu \in$ M and all $\alpha, \beta \in$ D (cf. paragraph A above), as the rotating polarizers *sweep different* directions, such that $|\alpha_1 - \beta_1| = |\alpha_2 - \beta_2| = \ldots = \theta =$ constant, different subsets S_{α_1} (S_{β_1}), S_{α_2} (S_{β_2}),... of photon pairs are being selected from the ensemble emitted by the source. But, according to the *limit function* (3) and the constant marginal probability functions (8) and (9), *nothing* changes as the polarizers sweep around: *The number of coincidence and single counts remains the same (invariant).* Given that the source generates a constant flux of photon pairs, this implies that the number of photon pairs in each and every subset S_{α_1} (S_{β_1}), S_{α_2} (S_{β_2}),... of M is the *same*, and furthermore that the directions (given by values of μ) of the common planes of polarization remain the *same (invariant)* as the polarizers sweep around selecting different subsets of pairs of back-to-back photons emitted by the source. Thus, the number and directions of the common planes of polarization of the pairs of back-to-back photons are uniformly distributed — axially invariant — over the range M[31].

But actually a little more has been shown than said above. Since the source generates a constant flux of photon pairs, the total number of photon pairs impinging upon the polarizers is the same for all values of θ; and so it is for each and every subset $S_{\alpha 1}$ ($S_{\beta 1}$), $S_{\alpha 2}$ ($S_{\beta 2}$),... of M. Different values of θ do not influence the uniform distribution of the ensemble of photon pairs impinging upon the polarizers. Rather what happens with different choices of values of θ is that a *different proportion* of photon pairs gets through the polarizers to cause a *different* number of coincidence counts, whilst the same total number of photon pairs *still* impinges upon the polarizers as before. For example, whenever the polarizers are parallel ($\theta = 0$), the number of coincidence counts is equal to the number of single counts (every coincidence count corresponds to a single count); and whenever the polarizers are orthogonal ($\theta = \frac{1}{2}\pi$), the number of coincidence counts is zero, but the number of single counts remains the same as before because each and every correlated photon, contained in each pair (γ_1,γ_2) of back-to-back photons, still impinges upon its corresponding polarizer causing the same number of single counts for all values of θ according to (8) and (9).

The same argument allows us to add a little to the local interpretation of the QF probability function p_{12}^{QF} given in paragraph D above:

(F) Since the choice of any value from M assigned to μ is independent of the choice of any values from D assigned to α and β for *all* $\mu \in M$ and *all* $\alpha, \beta \in D$, choosing different settings (values of α and β) of the polarizers has no influence upon the directions (values of μ) of the common planes of polarization of the pairs of back-to-back photons being emitted by the source. Thus, choosing a *different* setting, say, α_4 ($\neq \alpha_1$) of polarizer P_1, whilst the setting β_1 of polarizer P_2 is held fixed ($\beta_4 = \beta_1$), simply means that a *different* subset $S_{\alpha 4}$ ($\neq S_{\alpha 1}$) of photon pairs is being selected by the setting a_4 from the ensemble emitted by the source causing a *different* number of coincidence counts with a *different* probability value $\frac{1}{2}\cos^2(a_4 - \beta_4)$ of p_{12}^{QF} (cf. text just before (16) in paragraph C above). Note again, however, that the number of single counts remains the same for any choice of settings.

Next we turn to consider two other readily demonstrable formal features of the sentence Σ and their physical interpretation, which is of some importance. As already mentioned,

(G) The sentence Σ expresses the *formal definition* of the uniform convergence of the family of functions $\{p^{\mu}_{12} \mid \mu \in M\}$ to the function p^{QF}_{12}. In fact, Σ is the formal definition of the "limiting case" *itself*; this is what the universal quantifier $(\forall \varepsilon > 0)$ in the prefix of Σ means. Naturally, a question may arise: Does the formal definition Σ of the "limiting case" license the identification of the variable μ either with the variable α or with the variable β, denoted respectively by the equations $\mu = \alpha$ or $\mu = \beta$? The answer is: No.

This can be demonstrated as follows. Since Σ is *valid (true) in* G ($G \models \Sigma$) for *all* value assignments s in G, and since there are value assignments s in G such that both equations $\mu = \alpha$ and $\mu = \beta$ are *false* (notice again the "$=$" in boldface), it follows that the formal definition Σ of the "limiting case" does *not* logically imply the identification $\mu = \alpha$ or $\mu = \beta$ in G, *nor* licenses this identification anywhere in our construction (cf. penultimate paragraph of Section 3).

The same result can be demonstrated for other models G' *elementarily equivalent* to G (G and G' are said to be *elementarily equivalent*, denoted by $G \equiv G'$, if for any sentence σ we have $G \models \sigma \Longleftrightarrow G' \models \sigma$)[25], and such both equations $\mu = \alpha$ and $\mu = \beta$ are always *false* in G'.

Before we consider the physical significance of any such model G' for the formal postulate of locality (L3), we should explain how to obtain such a model G' for the sentence Σ ($G' \models \Sigma$).

Let the sets M and D be *disjoint*. By the Downward Löwenheim-Skolem Theorem[25], the sentence Σ has an elementarily equivalent model G' ($G' \models \Sigma \Longleftrightarrow G \models \Sigma$) such that the sets M′ and D′ are also disjoint and such that M′ can be chosen to be the set Z_0 of odd integers and D′ can be chosen to be the set Z_e of even integers. Then, any values in M′ assigned to the variable μ are *always different* from any values in D′ assigned to the variables α and β, and therefore the equations $\mu = \alpha$ and $\mu = \beta$ are *false* in G' for all value assignments s′ in G'. Thus, the identification $\mu = \alpha$ or $\mu = \beta$ is impossible in G'.

The physical significance of this result for (L3), as envisaged in the light of the postulated distribution ρ_ρ, is this: If (L3) were interpreted as *demanding* that $\mu \neq \alpha$ and $\mu \neq \beta$ should be *always true*, then (L3) can be meticulously satisfied in G' since both equations $\mu = \alpha$ and $\mu = \beta$ are false in G' for *all* value assignments s′ in G' (note that $\mu = \alpha$ is the *negation* of $\mu \neq \alpha$ etc.). But, although everything said so far about the physical interpretation of the sentence Σ goes through, we would submit rather cogently that such an interpretation of (L3) would not only be physically unreasonable, but must be actually *false* since *nothing* prevents the birth of a pair of back-to-back photons with a value μ_1

which *happens by pure chance* to be equal to a value α_1 or β_1 of the setting of a polarizer so that $\mu_1 = \alpha_1$ or $\mu_1 = \beta_1$ is actually true (cf. last paragraph of Section 3). Thus, a physically tenable interpretation of (L3) must be based on a model (like G) which allows for the possibility that $\mu_1 = \alpha_1$ or $\mu_1 = \beta_1$ is true under *some* value assignment.

On the other hand, any demand that the equations $\mu = \alpha$ or $\mu = \beta$ should be always true must be equally *false* on the physical grounds that nothing prevents the birth of another pair of back-to-back photons with a value μ_2 which happens by pure chance to be *different* (even light-years apart) from some values α_2 and β_2 of the settings of both polarizers so that $\mu_2 \neq \alpha_2$ and $\mu_2 \neq \beta_2$ is actually true. One would expect this situation to be true of almost all photon pairs being emitted by the source.

In the same vein, the *ad hoc* demand that $\mu = \alpha$ or $\mu = \beta$ should be always true would also have the following bizarre conspiracy as a consequence, which may even leave Laplace's demon gaping: The random directions (given by values of μ) of the common planes of polarization of each and every pair of back-to-back photons being born by the spontaneous annihilation process would always "fit" the random directions (given by values of α and β) of the polarizer settings chosen whilst the photons are in full flight (as in the experiment designed by Aspect et al). This would indeed be "spooky action at a distance" with a vengeance! But this bizarre conspiracy, based on the spurious demand $\mu = \alpha$ or $\mu = \beta$, is ruled out *ab initio* by the models G and G' of the conditional sentence Σ (cf. also paragraph B above).

Nevertheless, "hanging on" to the spurious demand $\mu = \alpha$ or $\mu = \beta$ would seemingly wipe out our proposed realistic local ("common cause") explanation of the quantum-statistical correlation exhibited in the EPRB ideal experiment, an explanation missing from QF itself (cf. paragraphs C and D above). But no surprise if the price of "hanging on" would be the unreal spooky stuff advocated by Bell et al.

To sum up : The *ad hoc* demand that $\mu = \alpha$ or $\mu = \beta$ should always be true would not only be a spurious additional assumption unwarranted by the models G and G' of the conditional sentence Σ, but must be actually *false* on physical grounds. Similarly, if (L3) were interpreted as demanding that $\mu \neq \alpha$ and $\mu \neq \beta$ should be always true, then such an interpretation of (L3) must be equally *false* on physical grounds. This leaves G as the more physically reasonable model of (L3) and of the conditional sentence Σ.

Next we turn to consider the physical significance of the *uniform convergence* of the familly of functions $\{p^{\mu}_{12} \mid \mu \in M\}$. to the *limit function* (3).

(H) Since in the formal definition Σ the existential quantifier $(\exists \eta > 0)$ *precedes* the universal quantifier $(\forall \alpha, \beta \in D)$, what we have here is *uniform convergence* and the choice of the variable η depends only upon the variable ε. Thus, as chosen $(\eta = 2\varepsilon)$ in the *proof* of Σ (cf. Appendix A), the same $\eta = 2\varepsilon$ serves at every point of D^2 since uniform convergence has been proved with respect to the *whole* domain of the *limit function* (3) (cf. Definition A2). The physical significance of this result is rather important: The uniform convergence on D^2 of the familly of functions $\{ p^{\mu}_{12} \mid \mu \in M\}$ to the *limit function* (3) does *not* depend upon any particular values in D assigned to the variables α and β, and thus this mode of (uniform) convergence is independent of the settings of the polarizers.

This point essentially concerns the (familiar) distinction between uniform convergence and pointwise convergence[32,33] based upon the *order* in which the quantifiers are being applied. If the sentence Σ were valid (true) in G only for the case where the universal quantifier $(\forall \alpha, \beta \in D)$ preceded the existential quantifier $(\exists \eta > 0)$, then convergence would be pointwise only. Then, the choice of the variable η would not only depend upon ε but also upon values in D, say, $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2 \ldots$ assigned to the variables α and β, and thus this mode of (pointwise) convergence would depend upon the settings of the polarizers. But we must not have this implicit dependence. The stronger condition of uniform convergence obtained here excludes any such implicit dependence by ensuring that the variable η is *independent* of *any* values in D assigned to the variables α and β.

Next we would like to add here a brief note concerning the "no-enhancement" hypothesis[20].

(I) The "no-enhancement" hypothesis asserts that for each and every photon pair emitted by the source the probability of a count with polarizers in place is less than or equal to the probability of a count with polarizers removed. In other words, this physically reasonable hypothesis asserts that the presence of the polarizers does not produce an enhanced detection of photon "downstream" of the polarizers.

It has been shown elsewhere[22] that the postulates of Th(G) do *satisfy* the "no-enhancement" hypothesis. Th(G) is the only local theory that satisfies this hypothesis and agrees with QF on the EPRB *ideal* experiment. Furthermore, by introducing measures of the inefficiency of the apparatus (polarizers & detectors), it was predicted that there is no enhancement in the non-ideal EPRB photon-cascade type experiments. The recent Stirling experiment[34] confirmed this additional prediction.

Before we conclude this Section, we wish to add that the question of how the consistent local theory Th(G) (cf. paragraph B4 of Appendix B) circumvents Bell's "impossibility proof" will be discussed in another paper.

6. CONCLUSION

EPR left open the question whether or not a finer description exists than that stipulated by the quantum state $|\psi>$, and concluded with the belief that such a theory is possible. The proposed theory Th(G) seems to answer the question and conclusion posed by EPR in the following sense. Postulate Π_3 of Th(G) stipulates a *finer* state specification than that given by the quantum state $|\gamma_1,\gamma_2>$. Postulate Π_3 introduces the variable μ whose values can be interpreted as the new *local* "elements of reality" created at the instant the spontaneous annihilation process breaks the spherical symmetry of the singlet state $|\gamma_1,\gamma_2>$ into a pair of back-to-back photons having only axial symmetry about the direction of their motion. In particular, the values of the variable μ— missing from the specification of the singlet state $|\gamma_1,\gamma_2>$ — specify the initial ("hidden") directions of the common planes of polarizations of each and every pair of back-to-back photons being born at the instant the singlet states explode, and thereby provide via the postulates of Th(G) a realistic and local ("common cause") explanation of the quantum-statistical correlations exhibited in the EPRB ideal experiment. Thus, in this sense, Th(G) affirms Einstein's deep commitment to realism and locality.

APPENDIX A. THE PROOF OF THE SENTENCE Σ

Here two definitions are formulated and the theorem Σ of Th(G) (Th(G) $\vdash \Sigma$) is proved. Some remarks and comments are also made.

Definition A.1: Let X be a topological space and $x \in X$. Let N_x be the set of *basic neighbourhoods* U of x ordered by the relation $\leqq$ on X such that $U_1 \leqq U_2 \Longleftrightarrow U_2 \subset U_1$. Then N_x is said to be a *directed* (by downward inclusion) set of basic neighbourhoods of x, and $\leqq$ is said to be a *direction on* X. The collection $\mathcal{N} := \{ N_x \mid x \in X \}$ is said to be a *direction in* X[32].

The notion of *uniform convergence* is not limited to sequences and series of functions, but can be validly extended to a family of functions. We now define it more formally, keeping the notation as close as possible to that used in the text.

Definition A.2: Let D^2 be a subset of R^2. Let M be a subset of R, and let N be a direction in M. Let $f : D^2 x\ M \rightarrow R$ be a function. To each value of μ in M, let there correspond a function $f^{\mu}: D^2 x\ M \rightarrow R$ defined by $f^{\mu}(x) := f(\mu, x)$. Let $g: D^2 \rightarrow R$ be a function. The family of functions $\{f^{\mu} \mid \mu \in M\}$ is said to *converge uniformly* to g on D^2 if for every $\varepsilon > 0$ there exists an $\eta > 0$ (with η depending only on ε) corresponding to a basic neighbourhood N_{η} in N such that for $\forall \mu \in M$ and for $\forall x \in D^2$ *whenever* the values of μ are in N_{η}, then $| f^{\mu}(x) - g(x) | < \varepsilon$ holds. The function g is said to be the limit function of the family of functions $\{f^{\mu} \mid \mu \in M\}$, and it is uniquely determined. In symbols, $\lim\limits_{\mu, N} \{f^{\mu}(x)\} = g(x)$ *uniformly on* D^2[32].

This definition requires that for each ε one single $\eta(\varepsilon)$, depending *only* on ε, can be found which serves at every point x of D^2. In other words, the corresponding $N_{\eta(\varepsilon)}$ can freely move about the *whole* domain D^2 since uniformity is required with respect to the whole domain D^2 of the *limit function* g. Thus, $N_{\eta(\varepsilon)}$ is independent of any values in D^2 assigned to the variable x.

THEOREM A.3: The family of functions $\{p^{\mu}_{12} \mid \mu \in M\}$ *converges uniformly on* D^2 *to the function* p^{QF}_{12} *and its unique limit function is*

$$\lim_{\mu, N} \{p^{\mu}_{12}(\alpha,\beta)\} = p^{QF}_{12}(\alpha,\beta) = \frac{1}{2}\cos^2(\alpha-\beta), \ \forall (\alpha,\beta) \in D^2. \tag{A1}$$

Proof: Using the identity

$$\cos 2(\alpha-\beta) = \cos 2(\mu-\alpha)\cos 2(\mu-\beta) + \sin 2(\mu-\alpha)\sin 2(\mu-\beta), \tag{A2}$$

valid for $\forall \mu \in M$ and for $\forall \alpha,\beta \in D$, and the identity $\cos^2(\alpha-\beta) = \frac{1}{2}[1 + \cos 2(\alpha-\beta)]$ together with the definition of $p^{\mu}_{12}(\alpha,\beta) := \frac{1}{4}[1+\cos 2(\mu-\alpha)\cos 2(\mu-\beta)]$, given by (7), we obtain $| p^{\mu}_{12}(\alpha,\beta) - p^{QF}_{12}(\alpha,\beta) | = \frac{1}{4} | \sin 2(\mu-\alpha)$

$\sin 2(\mu-\beta)|$. Using the inequality $|\sin 2(\mu-\alpha)\sin 2(\mu-\beta)|\leq|\sin 2(\mu-\alpha)|$ together with the inequality $|\sin z|\leq|z|$, we deduce $|p^{\mu}_{12}(\alpha,\beta)-p^{QF}_{12}(\alpha,\beta)|=\frac{1}{4}|\sin 2(\mu-\alpha)\sin 2(\mu-\beta)|\leqslant\frac{1}{4}|\sin 2(\mu-\alpha)|\leqslant\frac{1}{2}|\mu-\alpha|$ for, $\forall\mu\in M$ and $\forall\alpha,\beta\in D$. It suffices to choose $\eta=2\varepsilon$ (η depending only on ε) to establish that for $\forall\mu\in M$ and for $\forall\alpha,\beta\in D$ whenever the values of μ are also in $N_{2\varepsilon}$: $|\mu-\alpha|<\eta=2\varepsilon$, then $|p^{\mu}_{12}(\alpha,\beta)-p^{QF}_{12}(\alpha,\beta)|<\varepsilon$ follows. Thus, $\lim\limits_{\mu,N}\{p^{\mu}_{12}(\alpha,\beta)\}=p^{QF}_{12}(\alpha,\beta)=\frac{1}{2}\cos^2(\alpha-\beta)$ uniformly on D^2.

Comment 1: Using the inequality $|\sin 2(\mu-\alpha)\sin 2(\mu-\beta)|\leqslant|\sin 2(\mu-\beta)|$ the same result can be deduced. Thus, the antecedent of this deduction is $(|\mu-\alpha|<\eta)\vee(|\mu-\beta|<\eta)$ with $\eta=2\varepsilon$. Whence, by the Deduction Theorem [25], the conditional sentence Σ is proved ($\mathrm{Th}(G)\vdash\Sigma$). Σ is displayed in Section 5.

Comment 2: The conditional sentence Σ is *symmetric* under the interchange of the variables α and β with one another (this is as it should be on physical grounds).

Comment 3: Another direction N_d in X (cf. Definition A.1) can also be defined using the collection of *deleted* basic neighbourhoods of X. This collection N_d consists of all directed sets of the form $U\setminus\{x\}$, where U is a basic neighbourhood of $x\in X$. Then, using the *strict* inequality $|\sin z|<|z|$, valid for $z\neq 0$, the above proof goes through as before, but the antecedent $(|\mu-\alpha|<\eta)\vee(|\mu-\beta|<\eta)$ in Σ must be replaced by $(0<|\mu-\alpha|<\eta)\vee(0<|\mu-\beta|<\eta)$ with $\eta=2\varepsilon$ as before. The resulting conditional sentence is here denoted by Σ_d, and its proof has been given elsewhere[23], The structure G', discussed in paragraph G of Section 5, is a *model* of Σ_d ($G'\models\Sigma_d$). Recall that $\mu\neq\alpha$ and $\mu\neq\beta$ are true in G' for all value assignments s′ in G'.

APPENDIX B. SOME MODEL-THEORETIC NOTIONS

Model theory studies the relationship between sets of (first-order) sentences and the structures in which they are satisfied, that is, their models. Here only some model-theoretic notions will be outlined. They are mainly intended as a brief introduction to the terminology used in the text. An advanced treatment can be found elsewhere[25].

The notions of *satisfiability* and *truth* may be intuitively clear, but a rigorous definition can be given following Tarski (cf. Chapters 2 and 5 of Ref. 25). Such a definition is necessary for the formulation and understanding of precise proofs.

(B1) A *well-formed formula* (wff), like the quantified formula (sentence) Σ, is a *syntactic* object. A wff acquires *semantic* significance (meaning) only when an interpretation is given to the symbols occurring in it. In order to interpret a wff it is necessary to specify a *structure* $\mathcal{B}$. A structure $\mathcal{B}$ consists of a non-empty set B, called the *domain* (or universe) of $\mathcal{B}$, together with mappings which assign to each predicate symbol, function symbol and constant symbol occurring in a wff a specific relation, function and individual object respectively.

Given a structure $\mathcal{B}$, it is further necessary to specify a denumerable sequence $s = < b_0, b_1, \ldots >$ of elements of the domain B as an *assignment* of values to the variables $v_0, v_1, \ldots$ so that the variable v_i is assigned the element $b_i \in B$ *under* s. The elements b_i of s need not be distinct and indeed each b_i in the sequence s may be the same element of B. Such a sequence s is called a *value assignment in* $\mathcal{B}$. In other words, a value assignment $s: V \rightarrow B$ is a mapping from the set V of *all* variables into the domain B of $\mathcal{B}$. Let S be the set of all value assignments s in $\mathcal{B}$ (that is, the set of all denumerable sequences s of elements of B). Note well that all choices of s from S are equally free.

(B2) Let $\mathcal{B}$ be a structure with domain B. Let S be the set of all value assignments s in $\mathcal{B}$. Let φ be a wff. Then,

(a) A value assignment s *satisfies* φ *in* $\mathcal{B}$, denoted by $\mathcal{B} \models_s \varphi$, iff when all predicate, function and constant symbols occurring in φ are interpreted in $\mathcal{B}$, and when all free occurrences of the variables v_i in φ are replaced by their values $b_i \in B$ under s (for each i), the resulting proposition is true in $\mathcal{B}$.

(b) A wff φ is *valid (true)* in $\mathcal{B}$, denoted by $\mathcal{B} \models \varphi$, iff $\mathcal{B} \models_s \varphi$ for *all* value assignments s in $\mathcal{B}$.

The formal definition of satisfaction of φ in $\mathcal{B}$ proceeds by induction on the *degree* of φ (degφ). The technical details are not needed here, but can be found elsewhere (cf. Chapter 5 of Ref. 25).

(B3) A wff which has no free variables — so that all occurrences of variables in it, if any, are *bound* (e.g. by quantifiers) — is called a *sentence* (like Σ). For a structure B and a sentence σ, either

(a) B satisfies σ with every value assignment s in B, or

(b) B does not satisfy σ with any value assignment s in B.

If alternative (a) holds, then σ is said to be *valid (true)* in B ($B \models \sigma$), or that B is a *model of* σ. And if alternative (b) holds, then of course σ is *false in* B. Alternatives (a) and (b) are mutually exclusive. Note also that a sentence σ is *valid iff* σ *is satisfiable.*

(B4) Let Th(B) be the set of all sentences σ valid in B, that is,

$$\mathrm{Th}(B) := \{\sigma \mid \sigma \text{ is a sentence and } B \models \sigma\}.$$

The set Th(B) is called a *theory of* B and is closed under logical implication, that is, if $B \models \sigma$, then $\sigma \in$ Th(B). Furthermore, for any one structure B, Th(B) is always a *complete* theory (and consistent) in the sense that, for any well-formed sentence σ of the formal language of Th(B), either $\sigma \in$ Th(B) or $\neg\, \sigma \in$ Th(B), but not both since no structure B can be a model of both a sentence σ and its negation $\neg\, \sigma$. (In this sense, Th(G) is a complete and consistent theory of G.)

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ΠΕΡΙΛΗΨΙΣ

Ἐπὶ τοῦ προβλήματος τῆς τοπικῆς ἐπεκτάσεως τοῦ Κβαντικοῦ Φορμαλισμοῦ

Εἰς τὴν παροῦσαν ἐργασίαν ἐπανεξετάζεται καὶ ἀποδεικνύεται τὸ μὴ εὐσταθὲς τῶν διαφόρων ἐπιχειρημάτων ὅτι εἶναι ἀδύνατος ἡ συνεπὴς τοπικὴ ἐπέκτασις τοῦ Κβαντικοῦ Φορμαλισμοῦ ἐντὸς τοῦ πλαισίου τοῦ ἰδανικοῦ πειράματος τῶν Einstein, Podolsky, Rosen, Bohm (EPRB). Ἐν συνεχείᾳ προτείνεται νέα, συνεπὴς καὶ τοπικὴ θεωρία, ἡ θεωρία Th (G) ἡ ὁποία ἐπεκτείνει τὸν Κβαντικὸν Φορμαλισμόν. Μὲ βάσιν τὴν θεωρίαν Th (G) ἀποδεικνύεται ὅτι ἡ τοπικὴ δρᾶσις ἐπαρκεῖ νὰ ἐξηγήσῃ ὅλα ὅσα ὁ Κβαντικὸς Φορμαλισμὸς προβλέπει διὰ τὸ ἰδανικὸν EPRB πείραμα, καθὼς ἐπίσης δίδει ἐπὶ πλέον ἄλλα σημαντικὰ ἀποτελέσματα.

Ἡ θεωρία Th (G) βασίζεται εἰς τέσσαρας ἀξιωματικὰς παραδοχὰς (Π_1, Π_2, Π_3, Π_4), τὸ δὲ γράμμα "G" δηλοῖ ἕνα ἀπὸ τὰ πρότυπα (models) τῆς προτεινομένης θεωρίας Th (G). Ἡ ὕπαρξις τουλάχιστον ἑνὸς προτύπου G ἀποδεικνύει αὐστηρῶς ὅτι ἡ θεωρία Th (G) εἶναι συνεπής (consistent).

Ἡ παραδοχὴ Π_3 συνομολογεῖ ἕνα πλέον λεπτομερῆ καθορισμὸν καταστάσεως ἀπ' ἐκεῖνον τῆς Κβαντικῆς καταστάσεως ἐπαλληλίας $|\gamma_1, \gamma_2>$, γνωστῆς ὡς singlet state. Αὐτὸς ὁ πλέον λεπτομερὴς καθορισμὸς καταστάσεως περιγράφει τὴν διάσπασιν τῆς σφαιρικῆς συμμετρίας τῆς Κβαντικῆς καταστάσεως $|\gamma_1, \gamma_2>$ εἰς ἕνα ζεῦγος φωτονίων μὲ ἀξονικὴν συμμετρίαν περὶ τὴν κίνησιν τῶν φωτονίων πρὸς ἀντιθέτους κατευθύνσεις. Αὐτὴ ἡ διάσπασις συμμετρίας (breaking of symmetry) εἶναι ἀποτέλεσμα τῆς αὐθορμήτου διαδικασίας ἀποσυνθέσεως (spontaneous annihilation process) εἰς τὴν ὁποίαν ὑπόκειται ἡ Κβαντικὴ κατάστασις $|\gamma_1, \gamma_2>$.

Ἡ παραδοχὴ Π_3 εἰσάγει δύο ἀνεξαρτήτους μεταβλητὰς λ καὶ μ, αἱ ὁποῖαι συνάπτουν ἐπὶ τοῦ Κβαντικοῦ Φορμαλισμοῦ ὡρισμένα τοπικὰ «στοιχεῖα πραγματικότητος» τὰ ὁποῖα ἐλλείπουν ἀπὸ τὸν καθορισμὸν τῆς Κβαντικῆς καταστάσεως ἐπαλληλίας $|\gamma_1, \gamma_2>$.

Ἰδιαιτέρως, ἑκάστη τιμὴ τῆς μεταβλητῆς μ ἀρκεῖ διὰ νὰ προσδιορίσῃ τελείως τὴν ἀρχικὴν (τυχαίαν) κατεύθυνσιν τοῦ κοινοῦ ἐπιπέδου πολώσεως τῶν δύο φωτονίων τὰ ὁποῖα ἀναδύονται τὴν στιγμὴν κατὰ τὴν ὁποίαν ἡ κατάστασις ἐπαλληλίας $|\gamma_1, \gamma_2>$ ἀποσυντίθεται, καὶ δι' αὐτὸ ἑκάστη τιμὴ τῆς μεταβλητῆς μ μπορεῖ νὰ ἑρμηνευθῇ ὡς τὸ νέον τοπικὸν «στοιχεῖον πραγματικότητος», τὸ ὁποῖον δημιουργεῖται ἀπὸ αὐτὴν καθ' ἑαυτὴν τὴν αὐθόρμητον διαδικασίαν ἀποσυνθέσεως. Ἐπίσης, ἡ παραδοχὴ Π_3 ἐξηγεῖ πῶς ἡ κοινὴ φάσις τῶν δύο ἀναδυομένων φωτονίων παίζει ἕνα ἀρκετὰ

σπουδαῖον ρόλον εἰς τὴν προτεινομένην τοπικὴν ἐπέκτασιν τῆς Κβαντικῆς καταστάσεως ἐπαλληλίας $| \gamma_1, \gamma_2 >$.

Ἡ παραδοχὴ Π_3 εἰς τὴν θεωρίαν Th (G) εἶναι τὸ τοπικὸν καὶ ρεαλιστικὸν ἀντίστοιχον τῆς μὴ παραγοντοποιησίμου καταστάσεως ἐπαλληλίας $| \gamma_1, \gamma_2 >$ εἰς τὸν Κβαντικὸν Φορμαλισμόν: Τὸ *ἄθροισμα* τῶν γινομένων τῶν καταστατικῶν διανυσμάτων τοῦ Κβαντοφορμαλισμοῦ, τὰ ὁποῖα χαρακτηρίζουν τὰς δύο ἀμοιβαίας ἀποκλειστικὰς περιπτώσεις τὰς ὁποίας συνομολογεῖ ἡ μὴ παραγοντοποιήσιμος κατάστασις ἐπαλληλίας $| \gamma_1, \gamma_2 >$, μεταγράφεται εἰς τὸ *ἄθροισμα* τῶν γινομένων τῶν ὑπὸ ὅρους πιθανοτήτων (conditional probabilities) εἰς τὴν θεωρίαν Th (G), αἱ ὁποῖαι χαρακτηρίζουν τὰς δύο ὁμολόγους ἀμοιβαίως ἀποκλειστικὰς περιπτώσεις. Τοῦτο ἀποδεικνύει πῶς αἱ γνωσταὶ μαθηματικαὶ συνθῆκαι «τοπικῆς αἰτιότητος» ἱκανοποιοῦνται εἰς τὴν θεωρίαν Th (G). Ἄλλα πρότυπα G' στοιχειωδῶς ἰσοδύναμα πρὸς τὸ πρότυπον G ἐπίσης κατασκευάζονται καὶ ἡ φυσικὴ ἑρμηνεία των ἡ ὁποία ἀφορᾶ τὰς μαθηματικὰς συνθήκας «τοπικῆς αἰτιότητος» συζητεῖται.

Ἡ θεωρία Th (G) δίδει μίαν αἰτιατὴν καὶ τοπικήν (κοινῆς αἰτίας) ἐξήγησιν τοῦ κυριωτέρου χαρακτηριστικοῦ του ἰδανικοῦ πειράματος EPRB, ὅπου αἱ ἀρχικαὶ κατευθύνσεις (αἱ ὁποῖαι δίδονται μὲ τὰς τιμὰς τῆς μεταβλητῆς μ) τῶν κοινῶν ἐπιπέδων πολώσεως ἑκάστου ζεύγους φωτονίων *ἐπιλέγονται τυχαίως* ἀπ' αὐτὴν καθ' ἑαυτὴν τὴν αὐθόρμητον διαδικασίαν ἀποσυνθέσεως τῶν καταστάσεων ἐπαλληλίας $| \gamma_1, \gamma_2 >$ αἱ ὁποῖαι παράγονται ἀπὸ κατάλληλον πηγήν, καὶ ὅπου αἱ κατευθύνσεις τῶν πολωτῶν *ἐπιλέγονται τυχαίως* ἀπὸ τοὺς μεταλλάκτας ἐνῶ τὰ φωτόνια εὑρίσκονται εἰς πλήρη πτῆσιν ὅπως ἀκριβῶς πραγματοποιεῖται εἰς τὸ πείραμα τῶν Aspect καὶ ἄλλων.

Ὁ Ἀκαδημαϊκὸς κ. **Περικλῆς Θεοχάρης** προσθέτει τὰ ἑξῆς σχετικὰ πρὸς τὴν ἀνωτέρω ἐργασίαν:

Τὸ θέμα τῆς παρούσης ἐργασίας ἀνάγεται στὸ πρόβλημα τῆς τοπικῆς ἐπεκτάσεως τοῦ Κβαντικοῦ Φορμαλισμοῦ καὶ ἀφορᾶ τὶς δύο βασικὲς θεωρίες τῆς φυσικῆς τοῦ 20οῦ αἰῶνος. Ἀφ' ἑνὸς τῆς Κβαντοθεωρίας καὶ ἀφ' ἑτέρου τῆς θεωρίας τῆς σχετικότητος. Οἱ δύο αὐτὲς θεωρίες θεωροῦνται ἀσυμβίβαστοι διότι, ἡ μὲν θεωρία τῆς σχετικότητος λέγει ὅτι αἱ φυσικαὶ ἐπιδράσεις δὲν μποροῦν νὰ μεταδίδωνται μὲ ταχύτητα μεγαλυτέραν τοῦ φωτός, δηλαδὴ εἶναι τοπικαί, ἐνῶ ἡ Κβαντοθεωρία μὲ βάσιν τὸ ἰδανικὸν πείραμα τῶν Ἀϊνστάιν, Ποντόλσκυ, Ρόζεν, Μπόμ (EPRB) λέγει ὅτι φυσικαὶ ἐπιδράσεις μὴ τοπικαὶ ὑπάρχουν, βασιζομένη καὶ ἐπὶ τοῦ λεγομένου θεωρήματος Bell.

Ἡ ἐργασία ἀποτελεῖ θέμα μεγάλης ἐπιστημονικῆς σημασίας διότι ἀνοίγει νέους ορίζοντας εἰς τὴν ἐπέκτασιν τοῦ φορμαλισμοῦ τῆς Κβαντοθεωρίας, οὕτως ὥστε νὰ γίνη ἐφικτὸς ὁ συμβιβασμός της μὲ τὴν θεωρίαν τῆς σχετικότητος, πρᾶγμα τὸ ὁποῖον ἐθεωρεῖτο μέχρι σήμερον ἀκατόρθωτον.

Ἂν καὶ πλέον ἔχουν λησμονηθῆ, ἐν τούτοις ὅλοι οἱ διάσημοι φυσικοὶ τῆς γενεᾶς τοῦ von Neumann ἦσαν γοητευμένοι ἀπὸ τὴν ἀπόδειξίν του, ἡ ὁποία ἔγινε τὸ ἔτος 1932 καὶ διὰ τῆς ὁποίας ἀπεδείκνυε τὴν ἀδυναμίαν συστηματικῆς ἐπεκτάσεως τοῦ Κβαντικοῦ Φορμαλισμοῦ διὰ προσθήκης εἰς αὐτὸν τῶν λεγομένων κρυφῶν μεταβλητῶν. Μὲ τὸν ὅρον κρυφὴ μεταβλητὴ ὡρίζετο ὁ,τιδήποτε ἄλλο τὸ ὁποῖον δὲν ἐλαμβάνετο ὑπ' ὄψιν κατὰ τὸν καθορισμὸν τῆς Κβαντικῆς καταστάσεως ποὺ ἐχαρακτήριζε τὸ φυσικὸν σύστημα. Ἡ ἀπόδειξις τοῦ von Neumann κατέδειξεν ὅτι ἡ ὁρισθεῖσα ὕπαρξις τοιούτων μεταβλητῶν ἀντιβαίνει πρὸς τὸν κβαντικὸν φορμαλισμὸν ὁ ὁποῖος πρέπει νὰ εἶναι ἀντικειμενικῶς λανθασμένος γιὰ νὰ εἶναι δυνατὸς ὁ λεπτομερέστερος καθορισμὸς τῆς καταστάσεως τοῦ φυσικοῦ συστήματος, τῆς ὁριζομένης ἀπὸ τὴν Κβαντικὴν κατάστασιν.

Τὸ θέμα ἦτο ἂν ὑπῆρχον βαθύτεραι διαστρώσεις τῆς φυσικῆς πραγματικότητος ὅπως αὐταὶ ποὺ ἀντιμετωπίσθησαν ἀπὸ τὸν de Broglie, τὸν Einstein, καὶ αἱ ὁποῖαι δὲν ἐγένοντο ἀντιληπταὶ ἀπὸ τὸν κανονικὸν Κβαντικὸν Φορμαλισμόν. Δεδομένου ὅτι ὁ Κβαντικὸς Φορμαλισμός, αὐτὸς καθ' ἑαυτόν, δὲν ἐθεωρεῖτο λανθασμένος, ἡ ἀπόδειξις τοῦ von Neumann ἐθεωρήθη ὡς ἀποκλείουσα ταυτοχρόνως τὴν ἀξιωματικὴν ὕπαρξιν τῶν κρυφῶν μεταβλητῶν καθὼς καὶ τὶς ἰδιόρρυθμες ἔννοιες τῶν de Broglie καὶ Einstein τὶς σχετιζόμενες μὲ τὴν φυσικὴν κατάστασιν τὴν ἐπεκτεινομένην πέραν τοῦ ὁρίζοντος τοῦ Κβαντικοῦ Φορμαλισμοῦ καὶ τὴν βασικὴν ἑρμηνείαν τὴν εἰσαχθεῖσαν ἀπὸ τὸν Bohr καὶ Heisenberg. Ἐν τούτοις ὅμως, ἐπὶ τῇ βάσει τῆς ἀρχῆς τῆς αἰτιοκρατίας, ὁ von Neumann φαίνεται ὅτι δὲν ἀπέδιδε στὸ θεώρημά του τὶς ἐξαιρετικὲς ἀπαιτήσεις ποὺ ἔδωσαν ἄλλοι εἰς αὐτό.

Τὸ 1935 ἡ Grete Hermann ἐδημοσίευσεν ἐμπεριστατωμένην κριτικὴν τοῦ θεωρήματος τοῦ von Neumann καὶ ἰδιαιτέρως τοῦ ἰσχυρισμοῦ τοῦ von Neumann ὅτι ἕνα ἀπὸ τὰ ἀξιώματα τῆς ἀποδείξεώς του καὶ δὴ τὸ ἀξίωμα προσθετικότητος ἴσχυεν εἰς ὅλας τὰς περιπτώσεις. Τὸ ἀξίωμα αὐτό, τὸ ὁποῖον ἀναφέρεται καὶ ὡς ἀπαίτησις παγκοσμιότητος, εἰσήγαγε τὴν ἔννοιαν ὅτι τὸ ἀξίωμα προσθετικότητος ἦτο ἰσχυρὸν διὰ τὴν τάξιν ὅλων τῶν ἀνεξαρτήτων καταστάσεων αἱ ὁποῖαι περιελάμβανον τὴν τάξιν ὅλων τῶν κβαντικῶν καταστάσεων καθὼς καὶ τὴν τάξιν ὅλων τῶν καταστάσεων τῶν κρυφῶν μεταβλητῶν. Ἐξ ἄλλου ἡ Hermann διηρωτᾶτο ἂν τὸ ἀξίωμα τῆς προσθετικότητος δὲν θὰ ἠδύνατο νὰ θεωρηθῇ ὡς ἰσχῦον διὰ τὴν τάξιν τῶν καταστάσεων ὅλων τῶν κρυφῶν μεταβλητῶν ὅπως ἐθεώρει ὁ von Neumann. Κατ' αὐτὸν τὸν τρόπον ἡ συλλογιστικὴ τῆς Hermann ἐδημιούργησε τὴν ἔννοιαν τοῦ λανθασμένου τῆς ἐννοίας τῆς παγκοσμιότητος τοῦ von Neumann.

Ἐπίσης τὸ 1935, οἱ Ἀϊνστάιν, Ποντόλσκυ, Ρόζεν (EPR) προέτειναν τὸ περίφημον ἐπιχείρημα τὸ ὁποῖον, χωρὶς νὰ ἀντιφάσκῃ πρὸς τὸν Κβαντικὸν Φορμαλισμόν, ἀπεδείκνυε τὴν ὕπαρξιν «στοιχείων πραγματικότητος» τὰ ὁποῖα εἶχαν διαφύγει ἀπὸ τὸ πλέγμα τοῦ κανονικοῦ καθορισμοῦ τῶν καταστάσεων τῶν κβάντων, δηλαδή, ὅτι ἕνα σωματίδιον δύναται ταυτοχρόνως νὰ κατέχῃ σαφῆ θέσιν καὶ ροπὴν ἀνεξαρτήτως τοῦ τύπου τῆς μετρήσεως. Αἱ προτάσεις τῶν Ἀϊνστάιν, Ποντόλσκυ καὶ Ρόζεν, (EPR) ἐβασίζοντο εἰς τὴν σιωπηρὰν ἀλλὰ βασικὴν παραδοχὴν τῆς ἀνυπαρξίας δράσεως ἐξ ἀποστάσεως. Ἡ βασικὴ αὐτὴ παραδοχὴ ἦτο σαφὴς ἐνόψει τῆς θεωρίας τῆς εἰδικῆς σχετικότητος τοῦ Einstein ἡ ὁποία ἀπαγορεύει κάθε δρᾶσιν ἢ ἐπίδρασιν ἡ ὁποία διαδίδεται ταχύτερον ἀπὸ τὴν ταχύτητα τοῦ φωτὸς καὶ ἡ ὁποία ἀργότερον διετυπώθη σαφῶς ἀπὸ τὸν Einstein ὡς ἡ ἀρχὴ τῆς τοπικῆς δράσεως. Οὕτω ἐδείχθη κατὰ τὸν συλλογισμὸν τῶν Ἀϊνστάιν, Ποντόλσκυ καὶ Ρόζεν (EPR), ὅτι ἡ κβαντικὴ κατάστασις δὲν παρέχει πλήρη περιγραφὴν τῆς φυσικῆς πραγματικότητος ἀλλὰ ἀφήνει ἀνοικτὴν τὴν ἐρώτησιν ἂν ναὶ ἢ ὄχι μιὰ εὐαισθητοτέρα περιγραφὴ δύναται νὰ ὑπάρξῃ, καὶ κατέληγε μὲ τὴν πίστιν ὅτι μία τοιαύτη θεωρία εἶναι δυνατή.

Τὸ 1952 ὁ Bohm προέτεινεν ἔξυπνη ἐπέκτασιν τῆς θεωρίας τοῦ de Broglie, τῶν ὁδηγῶν κυμάτων, ἀποδεικνύοντας σαφῶς ὅτι αἱ κρυφαὶ μεταβληταὶ δύνανται νὰ προσαρτηθοῦν συστηματικῶς πρὸς τὸν Κβαντικὸν Φορμαλισμὸν καὶ κατ' αὐτὸν τὸν τρόπον παρέκαμψε τὴν ἀπόδειξιν ἀδυναμίας τοῦ von Neumann. Ἐπὶ πλέον ἔδειξε πῶς δύνανται νὰ ἑρμηνευθοῦν ὡς καθωρισμέναι, τροχιαὶ τῶν σωματιδίων εἰς τὸν Γαλιλαῖον χωροχρόνον ὅπου βασίζεται ἡ θεωρία τοῦ Bohm.

Ἡ θεωρία τοῦ Bohm ἔδειξε μέχρις ἑνὸς σημείου τί δρόμους πρέπει νὰ ἀκολουθῇ κανεὶς καὶ τί νὰ ἀποφεύγῃ. Οὕτω εἰσήχθη ἡ θετικὴ καὶ ἡ ἀρνητικὴ εὑριστικὴ διαδικασία. Ἡ θετικὴ εὑριστικὴ τῆς θεωρίας τοῦ Bohm ὡδήγησεν εἰς τὴν ἐπαινετικὴν κριτικὴν τοῦ Bell ἡ ὁποία προσέθεσεν οὐσιαστικῶς εἰς τὴν κριτικὴν τῆς Hermann τὴν διαμόρφωσιν ἀντιπαραδείγματος. Οὕτω ὁ Bell ἀπέδειξεν ἐκ νέου τὸ λανθασμένον τῆς προτάσεως παγκοσμιότητος τοῦ von Neumann, παρουσιάζοντας ἀντιπαράδειγμα ἀποδεικνῦον ὅτι τὸ ἀξίωμα προσθετικότητος δὲν ἱκανοποιεῖτο δι' ὡρισμένας καταστάσεις κρυφῶν μεταβλητῶν αἱ ὁποῖαι ὁλοκληρούμεναι ἔδιδον ἀποτέλεσμα σύμφωνα μὲ τὸν κβαντικὸν φορμαλισμόν.

Οὕτω, διὰ τῆς ὑπάρξεως τοῦ ἀντιπαραδείγματος αὐτοῦ καὶ τῆς θεωρίας τοῦ Bohm, τὸ θεώρημα τοῦ von Neumann προοδευτικῶς ἐγκατελείφθη. Ἐξ ἄλλου ἡ ἐργασία τῶν Ἀϊνστάιν, Ποντόλσκυ καὶ Ρόζεν (EPR), μετὰ τῆς ἐργασίας τοῦ Bell ἄφησαν ἀνοικτὴν τὴν ἐρώτησιν ἐὰν ὁ κβαντικὸς φορμαλισμὸς δύναται ἢ ὄχι νὰ ἐπεκταθῇ μὲ συνέπειαν καὶ νὰ συνδεθῇ τοπικῶς μὲ τὰς κρυφὰς μεταβλητάς.

Ἐξ ἄλλου ἡ ἀρνητικὴ εὑριστικὴ τῆς θεωρίας τοῦ Bohm ὡδήγησεν εἰς τὴν ἀντικατάστασιν τῆς ἀποδείξεως ἀδυναμίας τοῦ von Neumann (ὡς αὕτη συνεπληρώθη ἀπὸ τὴν προσθήκην τοῦ Bell), δι' ἑτέρας φαινομενικῶς περισσότερον πειστικῆς ἀποδείξεως ἀδυναμίας, ἐξ ἴσου δυναμικῆς ὡς καὶ ἡ ἀπόδειξις von Neumann. Εἰς τὴν ὑπὸ ἀνακοίνωσιν ἐργασίαν ἐρευνᾶται καὶ ἀποδεικνύεται ὅτι ἡ ἀπόδειξις ἀδυναμίας τοῦ Bell, ὅπως καὶ τοῦ von Neumann, ὄχι μόνον εἶναι ἀνεπαρκὴς διὰ τὸν ἐπιζητούμενον σκοπὸν ἀλλὰ καὶ ἀφήνει τὸ πραγματικὸν πρόβλημα ἄθικτον.

Ἡ ἀρνητικὴ εὑριστικὴ διαδικασία τῆς θεωρίας Bohm συνίσταται ἀπὸ ὡρισμένα ἀνώμαλα χαρακτηριστικὰ ἐκτάκτου χαρακτῆρος, τὰ ὁποῖα ὑποτίθεται ὅτι ἀποτελοῦν ἀναγκαῖον τμῆμα κάθε προσπαθείας ἑρμηνείας τῶν κβαντικῶν στατιστικῶν συσχετίσεων ἐκτιθεμένων εἰς τὸ EPRB ἰδανικὸν πείραμα. Ἀποδεικνύεται εἰς τὴν ἐργασίαν ὅτι τὰ ἐν λόγῳ δὲν ἰσχύουν. Ἡ τοπικὴ ἐπεξηγηματικὴ θεωρία τοῦ ἰδανικοῦ EPRB πειράματος, ἡ ὁποία προτείνεται εἰς τὴν ἐργασίαν, εἶναι ἀπελευθερωμένη ἀπὸ τοιαῦτα ἀνώμαλα χαρακτηριστικά. Εἰς τὴν ἐργασίαν αὐτὴν περιγράφονται καὶ ἀποδεικνύονται τὰ ἀνώμαλα αὐτὰ χαρακτηριστικὰ τῆς θεωρίας Bohm καὶ ἐξηγεῖται πῶς αὐτὰ ὁδηγοῦν εἰς ἀδιέξοδον ἐὰν ἑρμηνευθοῦν κατὰ τὸν κανονικὸν τρόπον.

Εἰς τὴν ἀρχὴν τῆς ἐργασίας γίνεται ἱστορικὴ ἀναφορά, ἀπὸ τὸ 1935 καὶ ἐντεῦθεν τῶν ἐργασιῶν πολλῶν διασήμων ἐπιστημόνων ποὺ ἠσχολήθησαν μὲ τὸ πρόβλημα, ὅπως τῶν von Neumann, de Broglie, Hermann, Einstein, Podolsky, Rosen, Bohm Bohr, Heisenberg, Bell, ὡς ἀνεφέρθησαν προηγουμένως, ὡς ἐπίσης καὶ εἰς τὴν ὅλην ἐξέλιξιν τῶν ὑπὲρ καὶ τῶν κατὰ ἀπόψεων ποὺ ἀντηλλάγησαν καὶ συνεζητήθησαν χωρὶς νὰ δοθῇ καμμία λύσις. Ἐν συνεχείᾳ, ἐπανεξετάζεται τὸ θεώρημα τοῦ Bell καὶ ἀποδεικνύεται ὅτι δὲν ἐπιτυγχάνει τοῦ σκοποῦ του. Τελικῶς προτείνεται νέα συνεπὴς τοπικὴ θεωρία, μὲ τὴν ὁποίαν ἀποδεικνύεται ὅτι ἡ τοπικὴ δρᾶσις ἐπαρκεῖ νὰ ἐξηγήσῃ ὅλα ὅσα ὁ Κβαντικὸς Φορμαλισμὸς προβλέπει διὰ τὸ ἰδανικὸν πείραμα τῶν Ἀϊνστάιν, Ποντόλσκυ, Ρόζεν, Μπόμ (EPRB), καθὼς ἐπίσης δίδει ἐπὶ πλέον ἄλλα σημαντικὰ ἀποτελέσματα. Ἡ ἐργασία αὐτή, κατὰ τὴν γνώμην τοῦ παρουσιάζοντός την, ἀποτελεῖ σημαντικὴν συμβολὴν εἰς τὰ βασικὰ προβλήματα τῆς συγχρόνου θεωρητικῆς φυσικῆς καὶ διὰ τὸν λόγον αὐτὸν ἐκφράζονται καὶ ἀπὸ τῆς θέσεως ταύτης τὰ θερμὰ συγχαρητήρια.