On remarque qu' aprés avoir effectué trois suspensions succesives l'échantillon ne présentant pas la constitution de la dolomie pure, s'enrichit progressivement jusqu' à une teneur 60 pour cent en oxyde de magnésium.

Quant au mineral demeuré en suspension en le traitant comme ci-dessus on obtient un 20 pour cent du produit en suspension ayant une teneur de 94 pour cent en CaO.

Le sédiment, soit le 20 pour cent de l'échantillon primitif présentant la constitution initiale, peut être remis dans le cycle des opérations.

ΒΙΒΛΙΟΓΡΑΦΙΑ

1. H. T. S. Britton, S. J. Gregg καὶ G. W. Winsor, J. Appl. Chem. 2 (1952) σ. 693.
2. M. καὶ A. Guedras, La chimie de la dolomie. Eyrolles Gauthier Villars, (1957) σ. 12.
3. Εμμ. Βογιατζακης, Δ. Γιαννακουδακης καὶ Γ. Βασιλικιωτης, Πρακτ. Ἀκαδημίας Ἀθηνῶν, τόμ. 33 (1958), σ. 292.
4. Roger's Industrial chemistry (1946) σ. 853.

ΓΕΩΜΕΤΡΙΑ.—The partition theorem of plane curves generalized with its geometrical interpretation *by Christos B. Glavas* *

The relations $f_1(a_1,b_1)=0$ and $f_2(a_2,b_2)=0$ are analytically equivalent if one can transform one to the other. The latter depends upon the existence of formulae of transformation between the coordinate systems (a_1,b_1) and (a_2,b_2). Thus the equations $x^2-y^2=a^2$ and $r^2\cos 2\theta=a^2$ are such relations.

The dual principle of geometrical equivalence has been already established in a paper communicated to the Academy of Athens [1].

Briefly, two curves $f(a_1,b_1)=0$ and $f(a_2,b_2)=0$ represented evidently by the same analytical relation are geometrically equivalent if one can transform geometrically one to the other. This is possible if $a_1=a_2$ and $b_1=b_2$, i. e. if the coordinates of the two systems are equal by pairs and if one can

---

* ΧΡΗΣΤ. Β. ΓΚΛΑΒΑΣ, Τὸ γενικευμένον θεώρημα κατανομῆς τῶν καμπυλῶν τοῦ ἐπιπέδου μετὰ τῆς γεωμετρικῆς ἑρμηνείας του.

[1] C. B. Glavas, The principle of geometrical equivalence and some of its consequences to the theory of curves, *Proceedings of the Academy of Athens*, 32 (1957), p. 122-131.

go geometrically from a point of the plane defined by one of the two systems to another point defined by the other system [1].

In the above mentioned paper it has been shown that the polar and the cathetic systems are both analytically and geometrically equivalent systems. In the cathetic system the coordinates of a point Q (Fig. 1) are the angle θ and $g = OA$ where A is the intersection of the perpendicular on OQ at Q with the axis OA. Really, if a circle is drawn with center at O and radius $OA = g$ the intersection of this circle with the extension of OQ determines the point P. The polar coordinates of P are θ and $r = OP = OA = g$. Hence the corresponding coordinates, representing similar magnitudes, of the two systems are equal by pairs and evidently one can go geometrically from the point P to the point Q and vice versa. On the other hand the two systems are analytically equivalent and the formula of transformation is $r = g \cos\theta$.

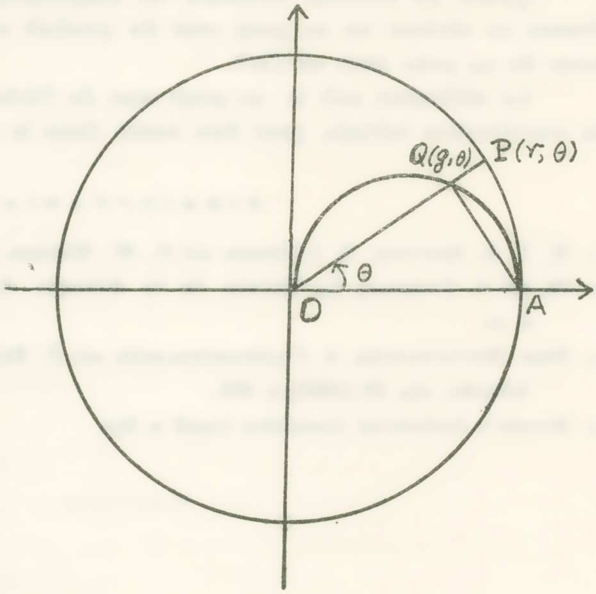

Fig. 1.

The use of analytical and geometrical equivalence in relation to these two systems led to the establishment of the Partition Theorem of the total set of plane curves [2]. Now this theorem is going to be generalized for any two plane systems provided that they are analytically and geometrically equivalent.

Let (x, y) and (x', y') be two analytically and geometrically equivalent plane coordinate systems. Let the formulae of transformation between these systems be:

$$(1) \qquad x' = \varphi_1(x, y), \quad y' = \varphi_2(x, y)$$
$$x = \sigma_1(x', y'), \quad y = \sigma_2(x', y')$$

[1] Loc. cit.

[2] GLAVAS, op. cit., p. 126-28.

Since the systems are geometrically equivalent, given two points $P(x, y)$ and $Q(x', y')$ we shall have $x = x'$ and $y = y'$. One can also go geometrically from P to Q and vice versa. This definition of geometric equivalence leads to an equivalence relation. Really:

1. $P(x, y) \sim P(x, y)$ because $x = x$ and $y = y$ and evidently one can go from P to the same P.

2. If $P(x, y) \sim Q(x', y')$, then $Q(x', y') \sim P(x, y)$. The first relation gives $x = x'$ and $y = y'$. Then $x' = x$ and $y' = y$ and of course one can go geometrically from Q to P by reversing the process.

3. Finally, if $P(x, y) \sim Q(x', y')$ and $Q(x', y') \sim R(x'', y'')$ then $P(x, y) \sim R(x'', y'')$. The two first equivalence relations give $x = x'$, $y = y'$ and $x' = x''$, $y = y''$. Hence $x = x''$, $y = y''$ and since one can go geometrically from P to Q and from Q to R one can go from P to R. The three properties of an equivalence relation, i.e. the reflexive, the symmetric and the transitive, exist here and geometric equivalence constitutes thus an equivalence relation.

Since geometric equivalence between $P(x, y)$ and $Q(x', y')$ is an equivalence relation then the set of geometrically equivalent curves $f(x, y) = 0$, $f(x', y') = 0$ described by P and Q respectively may be partitioned into mutually exclusive subsets. This comes out from a well known general theorem on equivalence relation [1]. It now remains to show how each such subset can be derived.

Take an initial curve $f(x, y) = 0$. The geometrical equivalent curve of the latter is $f(x', y') = 0$ which is the same with its analytically equivalent $f(\varphi_1(x, y), \varphi_2(x, y)) = 0$ resulting from the application of the formulae of transformation from the system (x', y') to (x, y).

Again the geometrically equivalent of the latter is $f(\varphi_1(x', y'), \varphi_2(x', y')) = 0$ which is the same with:

$$f(\varphi_1(\varphi_1(x, y), \varphi_2(x, y)), \quad \varphi_2(\varphi_1(x, y), \varphi_2(x, y)) = 0$$

Continuing this process we produce the geometrically equivalent curves of the left column of Table I expressed in the (x, y) system.

Reversing the above process and starting from $f(x', y') = 0$ we get as its geometrically equivalent the curve $f(x, y) = 0$. The latter is the same

[1] E. R. Lorch, *Theory of Functions*. New York, Columbia University, 1951, p. 1.

with $f(\sigma_1(x',y'), \sigma_2(x',y'))=0$ whose geometrically equivalent curve is $f(\sigma_1(x,y), \sigma_2(x,y))=0$.

Continuing in this way we produce the geometrieally equivalent curves of the right column of Table I.

Looking at the equations of the curves of Table I we see that the first equation of the left column is produced from the initial equation $f(x,y)=0$ if x and y are substituted there by $\varphi_1(x,y)$ and $\varphi_2(x,y)$ respectively. The third equation of the same column is produced from $f(x,y)=0$ after three successive substitutions of x and y by $\varphi_1(x,y)$ and $\varphi_2(x,y)$ respectively.

$f(\varphi_1(x,y), \varphi_2(x,y))=0$	$f(\sigma_1(x,y), \sigma_2(x,y))=0$
$f(\varphi_1(\varphi_1(x,y), \varphi_2(x,y)), \varphi_2(\varphi_1(x,y), \varphi_2(x,y))=0$	$f(\sigma_1(\sigma_1(x,y), \sigma_2(x,y)), \sigma_2(\sigma_1(x,y), \sigma_2(x,y))=0$
$f(\varphi_1(\varphi_1(\varphi_1,\varphi_2), \varphi_2(\varphi_1,\varphi_2)), \varphi_2(\varphi_1(\varphi_1,\varphi_2), \varphi_2(\varphi_1,\varphi_2))=0$	$f(\sigma_1(\sigma_1(\sigma_1,\sigma_2), \sigma_2(\sigma_1,\sigma_2)), \sigma_2(\sigma_1(\sigma_1,\sigma_2), \sigma_2(\sigma_1,\sigma_2))=0$
—	—
—	—

Table I.

And generally the nth equation of either column is produced from $f(x,y)=0$ by n successive such substitutions of x and y by $\varphi_1(x,y)$ and $\varphi_2(x,y)$ for the left column and $\sigma_1(x,y)$ and $\sigma_2(x,y)$ for the right one.

It is now necessary to find a new symbolism for the equations of Table I because they become longer and longer and it is difficult to cope with them. Table II shows the new representation of the equations of Table I. $\Delta^n f(\varphi_1,\varphi_2)=0$ or $E^n f(\varphi_1, \varphi_2)=0$ mean the equation which is found after n successive substitutions as defined above of x and y in $f(x,y)=0$ by $\varphi_1(x,y)$ and $\varphi_2(x,y)$ or $\sigma_1(x,y)$ and $\sigma_2(x,y)$.

In other words the symbols Δ^n and E^n mean substitutions whose number is determined by n.

$$\Delta^0 f(\varphi_1,\varphi_2)=E^0 f(\sigma_1,\sigma_2)=0$$

$\Delta^1 f(\varphi_1,\varphi_2)=0$	$E^1 f(\sigma_1,\sigma_2)=0$
$\Delta^2 f(\varphi_1,\varphi_2)=0$	$E^2 f(\sigma_1,\sigma_2)=0$
$\Delta^3 f(\varphi_1,\varphi_2)=0$	$E^3 f(\sigma_1,\sigma_2)=0$
—	—
—	—
$\Delta^n f(\varphi_1,\varphi_2)=0$	$E^n f(\sigma_1,\sigma_3)=0$

Table II.

We define the following law of composition symbolized by a small circle o for any two equations which we shall call henceforth elements of the set of curves of Table II. The combination of any two elements $[f(A,B)=0] \circ [f(C,D)=0]$ where A, B, C and D are functions of x and y gives a new element which is found if x and y in A(x,y) and B(x,y) are substituted by C and D respectively. Before showing that the combination of any two elements of Table II under this law of composition gives a new element belonging to the same set it is necessary to determine the relationship between the symbols Δ^n and E^n. We have for $n=1$:

$$[\Delta^1 f(\varphi_1, \varphi_2)=0] \circ [E^1 f(\sigma_1 \sigma_2)=0] = [f(\varphi_1(x,y), \varphi_2(x,y))=0] \circ f(\sigma_1(x,y), \sigma_2(x,y))] = [f(\varphi_1(\sigma_1(x,y), \sigma_2(x,y)), \varphi_2(\sigma_1(x,y), \sigma_2(x,y))=0].$$

Putting x′ and y′ instead of x and y respectively in (1) we get $x' = \sigma_1(x,y)$ and $y' = \sigma_2(x,y)$. Substituting x′ and y′ for $\sigma_1(x,y)$ and $\sigma_2(x,y)$ respectively in the above last expression we take:

$$[f(\varphi_1(x',y'), \varphi_2(x',y'))=0]$$

But from (1) $\varphi_1(x',y')=x$ and $\varphi_2(x',y')=y$. Therefore we shall finally get:

$$[\Delta^1 f(\varphi_1,\varphi_2)=0] \circ [E^1 f(\sigma_1,\sigma_2)=0] = [f(x,y)=0].$$

The last result means that the combination of the first corresponding elements of Table I gives the initial element. It is very easy also to prove that the combination of $[f(x,y)=0]$ with any element of the set of Table I gives again the same element which means that $[f(x,y)=0]$ is a neutral (or identity) element for the set in question. Hence it follows that Δ^1 and E^1 are symbols of inverse meaning and it is justifiable to put $[E^1 f(\sigma_1,\sigma_2)= =0] = [\Delta^{-1} f(\varphi_1,\varphi_2)=0]$. It remains to show that this holds for any two symbols E^n and Δ^n.

Suppose that

$$[\Delta^{n-1} f(\varphi_1,\varphi_2)=0] \circ [E^{n-1} f(\sigma_1,\sigma_2)=0] = [f(\varphi_1,\varphi_2)=0] \circ [f(\sigma_1,\sigma_2)=0] = [f(x,y)=0]$$

is true. Then using the last result:

$$[\Delta^n f(\varphi_1,\varphi_2)=0] \circ [E^n f(\sigma_1,\sigma_2)=0] = [\Delta^{n-1} f(\varphi_1(\varphi_1,\varphi_2), \varphi_2(\varphi_1,\varphi_2))=0] \circ \circ [E^{n-1} f(\sigma_1(\sigma_1,\sigma_2), \sigma_2(\sigma_1,\sigma_2))=0] = [f(\varphi_1(\varphi_1,\varphi_2), \varphi_2(\varphi_1,\varphi_2)=0] \circ \circ [f(\sigma_1(\sigma_1,\sigma_2), \sigma_2(\sigma_1,\sigma_2))=0] = [f(x,y)=0].$$

It should be clarified that in the last computations $\Delta^n f$ and $E^n f$ are found from their preceding expressions $\Delta^{n-1} f$ and $E^{n-1} f$ if x and y in the latter are substituted by φ_1, φ_2, and σ_1, σ_2 respectively as this has been observed previously. The proof of

$$[f(\varphi_1(\varphi_1,\varphi_2), \varphi_2(\varphi_1,\varphi_2))=0] \circ [f(\sigma_1(\sigma_1,\sigma_2), \sigma_2(\sigma_1,\sigma_2))=0]=[f(x,y)=0]$$

is similar to the one previously made for the first two corresponding relations of Table II.

From the above discussion we conclude that Δ^n and E^n are symbols of inverse meaning and since $\Delta^n = E^n$ for $n=1$ and since it has been shown that if the latter equality is true for $n-1$ it is true for n it follows by mathematical induction that it holds for any value of n. Table II can therefore be replaced by Table III.

It is necessary now to show that the combination of any two elements of the set in question gives a new element belonging to the same set. Suppose first that the two elements belong to the same column, say the left column. Then

$$[\Delta^n f(\varphi_1,\varphi_2)=0] \circ [\Delta^m f(\varphi_1,\varphi_2)=0]$$

means the substitution of x and y in $\Delta^n f(\varphi_1,\varphi_2)=0$ by the first and second expressions within the parenthesis of $\Delta^m f(\varphi_1,\varphi_2)=0$.

$\Delta^0 f(\varphi_1,\varphi_2)=0$

$\Delta^1 f(\varphi_1,\varphi_2)=0$	$\Delta^{-1} f(\varphi_1,\varphi_2)=0$
$\Delta^2 f(\varphi_1,\varphi_2)=0$	$\Delta^{-2} f(\varphi_1,\varphi_2)=0$
$\Delta^3 f(\varphi_1,\varphi_2)=0$	$\Delta^{-3} f(\varphi_1,\varphi_2)=0$
—	—
—	—
$\Delta^n f(\varphi_1,\varphi_2)=0$	$\Delta^{-n} f(\varphi_1,\varphi_2)=0$

Table III.

But it is known that $\Delta^n f(\varphi_1,\varphi_2)=0$ results from $f(x,y)=0$ after n successive substitutions of x and y by φ_1 and φ_2 respectively, while $\Delta^m f(\varphi_1,\varphi_2)=0$ is the outcome of m such substitutions. Therefore it is clear that the substitution of x and y in $\Delta^n f(\varphi_1 \varphi_2)=0$ by the first and the second expressions within the parenthesis of $\Delta^m f(\varphi_1,\varphi_2)=0$ is equivalent to $n+m$ successive substitutions of x and y in $f(x,y)=0$ by φ_1 and φ_2 respectively. Hence:

$$[\Delta^n f(\varphi_1,\varphi_2)=0] \circ [\Delta^m f(\varphi_1,\varphi_2)=0]=[\Delta^{m+n} f(\varphi_1,\varphi_2)=0].$$

The proof is the same for any two elements of the right column of Table II. It remains to show that the combination of any two elements each belonging to each column gives a new element belonging to our set. Take $[\Delta^n f(\varphi_1\varphi_2)=0] \circ [\Delta^{-m} f(\varphi_1,\varphi_2)=0]$. Suppose $n>m$. Then $n=m+k$, where k is a positive integer. Substituting n for $k+m$ in the last expression we get:

$$[\Delta^{k+m} f(\varphi_1,\varphi_2)=0] \circ [\Delta^{-m} f(\varphi_1,\varphi_2)=0].$$

Following the lines of the proof for the combination of two elements of the same column we see that the first element of the last expression means k successive substitutions of x and y in $f(x,y)=0$ by φ_1 and φ_2 respectively followed by m similar substitutions. On the other hand the combination of the two elements means m further substitutions of the inverse form. But if has been already shown that the combination of the elements $[\Delta^m f(\varphi_1,\varphi_2)=0]$ and $[\Delta^{-m} f(\varphi_1,\varphi_2)=0]$ gives the neutral element $[f(x,y)=0]$ which leaves unchanged the first k substitutions. Therefore we shall have:
$[\Delta^{k+m} f(\varphi_1,\varphi_2)=0] \circ [\Delta^{-m} f(\varphi_1,\varphi_2)=0]=[\Delta^k f(\varphi_1,\varphi_2)=0]=[\Delta^{n-m} f(\varphi_1,\varphi_2)=0]$.

The above result shows that if $n>m$ we obtain the $(n-m>$th) element of the left column of Table III. It is similarly shown that if $m>n$ we get the $(m-n)$ th element of the right column.

In conclusion it has been shown that the combination of any two elements of the set of Table III gives a new element belonging to the same set and that therefore the set in question is closed under the defined law of composition. It is not difficult now to show that this set constitutes a group. Really:

1. The set is closed under the defined law of composition.

2. The set is associative under the same law because:

$$[\Delta^m f(\varphi_1,\varphi_2)=0] \circ [\![\Delta^n f(\varphi_1,\varphi_2)=0] \circ [\Delta^p f(\varphi_1,\varphi_2)=0]\!]=[\Delta^{m+(n+p)} f(\varphi_1,\varphi_2)=0]$$

And:

$$[\![\Delta^m f(\varphi_1,\varphi_2)=0] \circ [\Delta^n f(\varphi_1,\varphi_2)=0]\!] \circ [\Delta^p f(\varphi_1,\varphi_2)=0]=[\Delta^{(m+n)+p} f(\varphi_1,\varphi_2)=0]$$

But the two above results are the same since $m+(n+p)=(m+n)+p$ is true for the integers

3. It has been shown already that to any element there corresponds an inverse one.

4. It has been shown also that there is a neutral (or identity) element, namely $[f(x,y)=0]$.

Besides it is very easy to show that the above group is commutative. The set in question is a subset of the total set of plane curves. At the beginning of this paper it is shown that geometrical equivalence constitutes an equivalence relation which forces a partition of the set of plane curves into mutually exclusive subsets of equivalent curves. These findings lead to the establishment of the following generalized Partition Theorem.

Theorem I. The total set of plane curves may theoretically be partitioned into an infinite number of subsets of geometrically equivalent curves. Each such subset constitutes a commutative or Abelian group under a defined operation.

The above theorem can find its interpretation in the Euclidean plane. Suppose that to go from a point $P(x,y)$ determined by the (x,y) system to the point $Q(x',y')$ determined by the system (x',y') equivalent to the first one is necessary to do two successive geometrical constructions. If the first of these constructions is denoted by a and the second by b let their «product» be denoted by $T^{(1)} = (a \times b)^{(1)}$.

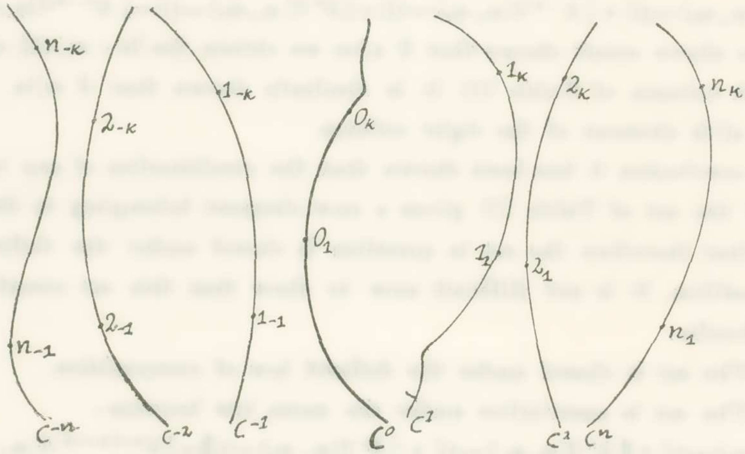

Fig. 2.

Let the initial curve $f(x,y)=0$ be represented by C^0 (Fig. 2). From the point $0_1(x,y)$ of the latter one can go to the point $1_1(x',y')$ of its geometrically equivalent curve C^1 with equation $f(x',y')=0$ by the product of constructions $T^{(1)}=(a\times b)^{(1)}$. But $f(x',y')=0$ is exactly the same with the curve $f(\varphi_1(x,y), \varphi_2(x,y))=0$ or $\Delta^1 f(\varphi_1,\varphi_2)=0$. Considering now the point 1_1 in the (x,y) system one can go to the next point 2_1 of C^2 whose equation is $\Delta^2 f(\varphi_1,\varphi_2)=0$ by another product $T^{(1)}=(a\times b)^{(1)}$. Therefore one can go from 0_1 to 2_1 by two successive products $T^{(1)}$, which may be symbolized by $T^{(2)}=(a\times b)^{(2)}$. And generally speaking one can go from 0_1 to n_1 by the products $T^{(n)}=(a\times b)^{(n)}$.

If we seek to go from $1_1(x',y')$ to $0_1(x,y)$ it is clear that we must reverse the order of constructions and therefore we shall have the product $T^{(-1)}=(b\times a)^{(-1)}$. Similarly the same product of constructions $T^{(-1)}$ is required to go from 0_1 to 1_{-1}. And generally to go from 0_1 to n_{-1} we must use the products $T^{(-n)}=(b\times a)^{(-n)}$.

The same process obviously can be applied if we start from any other point O_k of C^0. It still remains to make the geometrical interpretation of the defined law of composition for the subset of Table III. By definition

$$[\Delta^m f(\varphi_1,\varphi_2)=0] \circ [\Delta^n f(\varphi_1,\varphi_2)=0]=[\Delta^{m+n} f(\varphi_1,\varphi_2)=0]$$

where m and n are any integers, means to take the points of the curve C^m, whose equation is $[\Delta^m f(\varphi_1,\varphi_2)=0]$, n positions to the right of the latter if $n>0$, to the left if $n<0$ and to leave them where they are if $n=0$. Therefore if $n>0$ we shall transform the curve C^m to C^{m+n}, if $n<0$ to C^{m-n} and to itself if $n=0$. The same definition is applied leading exactly to the same result in case the order of the two elements in the last expression is reversed.

It is worthwhile to observe that if $m=-n$ then:

$$[\Delta^n f(\varphi_1,\varphi_2)=0] \circ [\Delta^{-n} f(\varphi_1,\varphi_2)=0]=[\Delta^0 f(\varphi_1,\varphi_2)=0]=[f(x,y)=0].$$

The first of the above curves is C^n while the other is C^{-n}. According to the previously given geometric interpretation of the law of composition of the group of curves of Fig. 2 the points of the curve C^n should be taken n places to the left of it and therefore to produce the initial curve which is the identity element of the group in question. This shows that the combination of any two inverse curves C^n ant C^{-n} under the geometrically interpreted law of composition gives the initial curve C^0.

ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν ἐργασίαν ταύτην γίνεται γενίκευσις τῆς ἀρχῆς τῆς γεωμετρικῆς ἰσοδυναμίας καὶ τοῦ θεωρήματος κατανομῆς τῶν καμπυλῶν τοῦ ἐπιπέδου. Τὰ θέματα ταῦτα ἐτέθησαν τὸ πρῶτον ὑπὸ εἰδικὴν μορφὴν εἰς ἀνακοίνωσίν μου κατὰ τὴν συνεδρίαν τῆς Ἀκαδημίας Ἀθηνῶν τῆς 28ης Φεβρουαρίου 1957 [*Πρακτικὰ τῆς Ἀκαδημίας Ἀθηνῶν*, τόμ. 32 (1957), σ. 122-131].

Δίδεται ἐνταῦθα κατὰ πρῶτον ὁ ὁρισμὸς τῶν γεωμετρικῶς ἰσοδυνάμων καμπυλῶν. Ὡς τοιαῦται ὁρίζονται αἱ καμπύλαι, αἱ ὁποῖαι παρίστανται ὑπὸ τὴν ἰδίαν ἀναλυτικὴν σχέσιν εἰς δύο διάφορα συστήματα συντεταγμένων, δυνατῆς οὔσης τῆς μεταβάσεως ἐξ ἑνὸς σημείου τῆς μιᾶς εἰς ἓν ἄλλο (ἀντίστοιχον) τῆς ἄλλης διὰ πεπερασμένου ἀριθμοῦ γεωμετρικῶν κατασκευῶν. Τοῦτο εἶναι δυνατόν, ἂν τὰ χρησιμοποιούμενα συστήματα εἶναι «γεωμετρικῶς καὶ ἀναλυτικῶς ἰσοδύναμα», ποὺ σημαίνει, ὅτι ὑπάρχουν ἀναλυτικαὶ σχέσεις μετασχηματισμοῦ τοῦ ἑνὸς εἰς τὸ ἄλλο καὶ αἱ συντεταγμέναι αὐτῶν εἶναι ἴσαι κατὰ ζεύγη, ὥστε νὰ καθίσταται δυνατὴ ἡ μετάβασις ἐξ ἑνὸς σημείου, ὁριζομένου ὑπὸ τοῦ ἑνὸς συστήματος, εἰς ἓν ἄλλο, ὁριζόμενον ὑπὸ τοῦ ἄλλου διὰ πεπερασμένου ἀριθμοῦ γεωμετρικῶν κατασκευῶν.

Ὁ ὁρισμὸς τῆς γεωμετρικῆς ἰσοδυναμίας ἐφαρμοζόμενος ἐπὶ δύο σημείων τοῦ

ἐπιπέδου, ὁριζομένων ὑφ' ἑκάστου τῶν συστημάτων τούτων καὶ περιγραφόντων δύο καμπύλας τοῦ ἐπιπέδου, συνιστᾷ μίαν σχέσιν ἰσοδυναμίας. Ἡ τελευταία προκαλεῖ κατανομὴν τῶν καμπυλῶν τοῦ ἐπιπέδου εἰς διακεκριμένα ἀλλήλων σύνολα. Ἀποδεικνύεται ἐν συνεχείᾳ κατὰ τρόπον γενικόν, ὅτι τὰ σύνολα ταῦτα συνιστοῦν ὁμάδας ὑπὸ ὡρισμένον νόμον συνθέσεως.

Ἡ διαφορὰ καὶ ἡ δυσκολία τῆς ἐργασίας ταύτης ἐν σχέσει πρὸς τὴν ἀνακοινωθεῖσαν, ὡς ἀνωτέρω, κατὰ τὴν συνεδρίαν τῆς 28ης Φεβρ. 1957 ἔγκειται εἰς τὴν λῆψιν τυχούσης καμπύλης ὡς ἀρχικῆς καὶ εἰς τὴν χρησιμοποίησιν συστημάτων συντεταγμένων, τῶν ὁποίων οἱ τύποι μετασχηματισμοῦ εἶναι ἐπίσης γενικευμένοι. Κατὰ τὴν πορείαν τῆς ἀποδείξεως τοῦ θεωρήματος κατανομῆς ἐν τῇ γενικῇ του μορφῇ προέκυψαν θέματα συμβολικῆς παραστάσεως ὡρισμένων τύπων, τῶν ὁποίων τὸ μέγεθος καθίστα ἄκρως δύσκολον τὴν ἀνάπτυξιν τῆς ἀποδείξεως. Διὰ τῆς χρησιμοποιήσεως καταλλήλου συμβολισμοῦ παρεκάμφθησαν αἱ δυσκολίαι καὶ ἐπετεύχθη ἡ ἀπόδειξις τοῦ θεωρήματος ὑπὸ τὴν γενικήν του μορφήν.

Τέλος γίνεται γεωμετρικὴ ἑρμηνεία τοῦ γενικευμένου τούτου θεωρήματος ἐπὶ τοῦ Εὐκλειδείου ἐπιπέδου. Αἱ παριστώμεναι καμπύλαι κατατάσσονται εἰς δύο ὑποσύνολα. Ἑκάστη καμπύλη τοῦ ἑνὸς ὑποσυνόλου ἔχει τὴν γεωμετρικῶς «ἀντίστροφόν» της εἰς τὸ ἄλλο. Ἀξιοσημείωτον εἶναι, ὅτι ἡ γεωμετρικὴ πρᾶξις αὕτη ἐφαρμοζομένη ἐπὶ δύο οἱωνδήποτε ἀντιστρόφων καμπυλῶν ὁδηγεῖ εἰς τὴν κατασκευὴν τῆς ἀρχικῆς καμπύλης, ἡ ὁποία ἀποτελεῖ ἓν εἶδος «στοιχείου ταυτότητος» τῆς ὅλης ὁμάδος καμπυλῶν.

ΜΑΘΗΜΑΤΙΚΗ ΑΝΑΛΥΣΙΣ.— **On the evaluation of double integrals containing a large parameter,** *by* ***N. Chako*** *. Ἀνεκοινώθη ὑπὸ τοῦ κ. Ἰωάνν. Ξανθάκη.

I. INTRODUCTION

In this paper we shall be concerned with the evaluation of double integrals of the type

$$(A) \qquad U(k) = \iint_D g(x,y)\, e^{ik\Phi(x,y)}\, dxdy,$$

for large values of the real parameter k. The amplitude and phase functions, g(x,y) and Φ(x,y) are real in the real variables (x,y) subject to certain restrictions which will be specified later, and D is a finite domain of integration. Integrals of this type occur often in mathematical physics, especially in diffraction and scattering problems (1 - 3).

The method which will be developed here for evaluating integrals of this kind will follow closely the method developed by Poincaré (4) and

* N. CHAKO, Ἐπὶ τοῦ προσδιορισμοῦ διπλῶν ὁλοκληρωμάτων μὲ μεγάλας παραμέτρους.