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ΜΑΘΗΜΑΤΙΚΑ.— **A Remark on the Space (H(X), h) of Fractals,** by Nicolas K. Artémiadis*, Regular member of the Academy of Athens.

ABSTRACT

Let (X,d) be a complete metric space. Let S(x,r) and S[x,r] be the open and the closed balls in X respectively, with center x and radius r. The metric d is said to be «round» if and only if $\overline{S}(x,r) = S[x,r]$. Let H(X) be the set whose elements are the nonempty compact subsets of X. It is well known that (H(X),h), called «The space of fractals», is a complete metric space if h is taken to be the Hausdorff distance between any two elements of H(X). In this paper we prove that the metric d is «round» in (X,d) if and only if the Hausdorff metric h is «round» in (H(X),h).

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1. PRELIMINARIES

Let (X,d) be a complete metric space. Then H(X) denotes the space whose points are the nonempty compact subsets of X. If $x \in X$ and $B \in H(X)$, the distance d(x,B) from x to the set B is $d(x,B) = \min\{d(x,y) : y \in B\}$. The existence of a minimum value in the set $\{d(x,y) : y \in B\}$ follows from the compactness and nonemptyness of the set B. In other words there is $\hat{y} \in B$ such

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that $d(x, B) = d(x, \hat{y})$. If $A,B \in H(X)$, the distance $d(A,B)$ from the set A to the set B is $d(A,B) = \max\{d(x,B) : x \in A\}$. The last definition is again meaningful because of the compactness of A and B. In particular there are points $\hat{x} \in A$ and $\hat{y} \in B$ such that $d(A,B) = d(\hat{x}, \hat{y})$. It can be shown that if $A,B \in H(X)$ then in general $d(A,B) \neq d(B,A)$, which implies that d does not provide a metric on H(X), [2].

The Hausdorff Distance.

Let (X, d) be a complete metric space. For $A \in H(X)$ and $\varepsilon > 0$ the ε - collar A_ε of A is defined by

$$A_\varepsilon = \{x \in X : d(x,y) \leqslant \varepsilon \quad \text{for some} \quad y \in A\}$$

For any two compact subsets A and B of X, the Hausdorff distance is

$$h(A,B) = \inf\{\varepsilon : A \subset B_\varepsilon \text{ and } \quad B \subset A_\varepsilon\} \tag{1}$$

According to Hausdorff the space of all compact subsets of X equipped with the Hausdorff distance, is another complete metric space. We prove that

$$h(A,B) = d(A,B) \vee d(B,A) \tag{2}$$

We use the notation $x \vee y$ to mean the maximum of the two real numbers x and y. Similarly $x \wedge y$ will mean the minimum of x and y. First, notice that if $\varepsilon < d(A,B)$ then $A \not\subset B_\varepsilon$. To see this it suffices to show that there is $a_0 \in A$ such that $d(a_0,y) > \varepsilon$ for all $y \in B$.

Indeed, we have $\varepsilon < d(A,B) = \max_{x \in A} d(x,B)$. Due to the compactness of A and B there is $a_0 \in A$ such that $\max_{x \in A} d(x, B) = d(a_0, B)$. Hence $\varepsilon < d(a_0, B) = \min_{y \in B} d(a_0, y)$ which implies $\varepsilon < d(a_0, y)$ for all $y \in B$. In a similar manner, one proves that if $\varepsilon < d(B, A)$ then $B \not\subset A_\varepsilon$. It follows that

$$h(A, B) \geqslant d(A, B) \vee d(B,A) \tag{3}$$

Next, we prove that $h(A, B) \leqslant d(A,B) \vee d(B,A)$. To this end we show that

$$B_{d(A,B)} \supset A, \quad A_{d(B,A)} \supset B \qquad (4$$

Put $d(A,B) = \max_{a \in A}\{\min_{y \in B} d(a, y)\} = \delta$. Let $a \in A$. Since B is compact there is $y_0 \in B$ such $d(a, y_0) = \min_{y \in B}\{d(a, y)\}$. It follows from the definition of d(A,B) that $\delta = d(A, B) \geqslant d(a, y_0)$. Hence $a \in B_\delta$ and $B_\delta \supset A$. In a similar manner one proves that $A_{d(B,A)} \supset B$, so that (4) holds. From (4) we get

$$h(A, B) \leqslant d(A, B) \vee d(B, A), \tag{5}$$

and (1) follows from (3) and (5).

The roundness of a metric.

Let (X,d) be a metric space and let S(x,r) and S[x,r] be the open and the closed balls in X respectively, with center x and radius r. In «Teoria functsiy veshchestvennoy peremennoy» (Theory of functions of a real variable), by I. P. Natanson, Moscow 1957, 2nd edition, p. 508, one finds the following statement: «The closure $\bar{S}(x,r)$ of the open ball is the closed ball S[x,r]». The falsness of this statement had raised the question of finding all the metric spaces for which the statement is true. The following, Theorem A, of this author, communicated to the Academy of Holland [1], by H. Freudental, answers this question.

Definition. The metric d is said to be «round» if and only if $\bar{S}(x, r) = S[x, r]$, for $x \in X$ and $r > 0$.

Given any two distinct points a and b in (X,d) put:

$$E(a, b) = \{z \in X : d(z,a) < d(a, b), \quad d(z, b) < (a,b)\} \cup \{a\} \cup \{b\}$$
$$E'(a,b) = \text{the derived set of } E(a, b). \tag{6}$$

Theorem A.

Let (X,d) be a metric space. Then d is round if and only if, for every a and b in X, $a \neq b$, the set E(a,b) is dense in itself; i.e. $E(a,b) \subset E'(a,b)$.

Several consequences of Theorem A are given in [1].

The purpose of this paper is to show that the Hausdorff metric h is round if and only if the underlying metric d is round.

2. THE MAIN RESULT

Theorem B.

The Hausdorff metric h in (H(X),h) is round if and only if d is round in (X,d).

Proof. Suppose h is round in (H(X),h). Then it follows from Th. A that

$$E_h(A,B) \subset E'_h(A,B)$$

where $A,B \in H(X), A \neq B$ and

$$E_h(A,B) = \{A\} \cup \{B\} \cup \{Z \in H(X) : h(Z,A) < h(A,B), h(Z,B) < h(A,B).\}$$

To prove that d is round in (X,d) we show that:

For every a,b in X, $a \neq b$, we have $E_d(a,b) \subset E'_d(a,b)$

where $E_d(a,b)$ equals E(a,b) as in (6).

Let $\zeta \in E_d(a,b)$. It is easily seen that $\{\zeta\} \in E_h(\{a\},\{b\})$.

For if $\zeta = a$ or $\zeta = b$ then $\{\zeta\} \in E_h(\{a\},\{b\})$. If $\zeta \neq a$ and $\zeta \neq b$ then $d(a,z) < d(a,b)$, $d(b,\zeta) < d(a,b)$,
or equivalently

$$h(\{a\},\{\zeta\}) < h(\{a\},\{b\}), \; h(\{b\},\{\zeta\}) < h(\{a\},\{b\}),$$

which proves that $\{\zeta\} \in E_h(\{a\},\{b\})$, since $\{\zeta\} \in H(X)$.

By assumption $\{\zeta\} \in E'_h(\{a\},\{b\})$. Hence: for every $\delta < 0$ there is $Z \in H(X)$, $Z \neq \{\zeta\}$, such that $h(\{\zeta\},Z) < \delta$.

Let $\zeta_0 \in Z$ such that $\zeta_0 \neq \zeta$. Such a ζ_0 exists because $Z \neq \{\zeta\}$. We have

$$d(\zeta,\zeta_0) = h(\{\zeta\},\{\zeta_0\}) \leq h(\{\zeta\},Z) < \delta.$$

Take $\delta < \{d(a,b) - d(\zeta,a)\} \wedge \{d(a,b) - d(\zeta,b)\}$. If $\zeta = a$ or $\zeta = b$ the last double inequality provides

$$h(\{a\},\{\zeta_0\}) = d(a,\zeta_0) < \delta < d(a,b)$$

$$h(\{b\},\{\zeta_0\}) = d(\zeta_0,b) < \delta < d(a,b)$$

If $\zeta \neq a$ and $\zeta \neq b$ we have

$$d(\zeta_0,a) \leqslant d(\zeta_0,\zeta) + d(\zeta,\alpha) < \delta + d(\zeta,a) < d(a,b) - d(\zeta,a) + d(\zeta,\alpha) = d(a,b)$$

Similarly we get $d(\zeta_0,b) < d(a,b)$. Hence $\zeta_0 \in E_d(a,b)$, which implies that d is round in (X,d).

This proves the first half of the theorem.

Next we assume that d is round in (X,d) and prove that h is round in (H(X),h). Again by Th. A it suffices to show that:

For every $A \in H(X)$ and $r > 0$ we have $S[A,r] = \overline{S}(A,r)$. We first show that $\overline{S}(A,r) \subset S[A,r]$. Let $Z \in \overline{S}(A,r)$. If $Z \in S(A,r)$ then $Z \in S[A,r]$.
Suppose $Z \in S'(A,r)$. Then $h(Z,A) \leqslant r$, because if we had $h(Z,A) > r$ then the neighborhood $S\left(Z, \frac{h(Z,A) - r)}{2}\right)$ of Z would contain no point of S(A,r). For if

$$B \in S(A,r) \cap S\left(Z, \frac{h(Z,A) - r}{2}\right)$$

then $h(A,Z) \leqslant h(A,B) + h(B,Z) < r + \frac{h(A,Z) - r}{2}$

which implies $h(A,Z) < r$, contradicting $h(Z,A) > r$. Hence the ball $S\left(Z, \frac{h(Z,A) - r}{2}\right)$ contains no point of S(A,r) which contradicts the assumption $Z \in S'(A,r)$. This proves that $\overline{S}(A,r) \subset S[A,r]$. Next we show

$$S[A,r] \subset \overline{S}(A,r) \tag{7}$$

Let $Z \in S[A,r]$. If $h(Z,A) < r$ then $Z \in S(A,r) \subset \overline{S}(A,r)$ so that (7) holds. Suppose $h(Z,A) = d(A,Z) \vee d(Z,A) = r$. We have

(a) $d(Z,A) = \max_{z \in Z}\{\min_{a \in A} d(z,a)\} \leqslant r$

(b) $d(A,Z) = \max_{a \in A}\{\min_{z \in Z} d(z,a)\} \leqslant r$

We want to prove that:

(*) For every $\varepsilon > 0$ there is $\Psi \in (H(X),h)$ such that $h(A,\Psi) < r$, $h(\Psi,Z) < \varepsilon$.

Let $a \in A$. Since Z is nonempty and compact there is $z_a \in Z$ such that $d(a,z_a) = \min_{z \in Z} d(z,a)$.

From (b) we get $d(a,z_a) \leqslant r$.

Since d is round in (X,d) there is $z'_a \in S(z_a,\varepsilon)$ such that $d(a,z'_a) < d(a,z_a) \leqslant r$.

Observe that since $d(a,z_a)$ is the minimum of the distances of a to z, and $d(a,z'_a) < d(a,z_a)$, we have $z'_a \notin Z$.

Let $0 < \lambda < 1$ and put $\rho_a = \lambda[d(a,z_a) - d(a,z'_a)] > 0$.

Let $x \in S(a,\rho_a)$. We have $d(z'_a,x) \leqslant d(z'_a,a) + d(a,x) < d(z'_a,a) + \rho_a =$ $= (1-\lambda)d(a,z'_a) + \lambda d(a,z_a) < (1-\lambda)d(a,z_a) + \lambda d(a,z_a) = d(a,z_a)$. Hence

$$d(z'_a,x) < r \text{ if } x \in S(a,\rho_a) \tag{8}$$

Consider the open covering $\{S(a,\rho_a)\}_{a \in A}$ of A. There is a finite subcovering of A say $\{S(\alpha_1,\rho_a),\ldots,S(\alpha_n,\rho_{a_n})\}$. Put $\Psi = \{z'_{a_1},z'_{a_2},\ldots, z'_{a_n}\}$.

Let $\{z_1,z_2,\ldots,z_m\}$ be an $\frac{\varepsilon}{2}$ — net of Z. From (a) we get for each $z \in Z$, $\min_{a \in A} d(z,a) \leqslant r$. Hence $\min_{a \in A} d(z_k,a) \leqslant r$, $1 \leqslant k \leqslant m$. Since A is compact there are elements, $a'_1,a'_2,\ldots a'_m$ of A such that $\min_{a \in A} d(z_k,a) = d(z_k a'_k) \leqslant r$, $k = 1,2,\ldots,$ m. In other words to the elements $z_1,z_2,\ldots,z_m$ of the $\frac{\varepsilon}{2}$ — net of Z correspond some elements $a'_1,a'_2,\ldots,a'_m$ of A such that $d(z_k,a'_k) \leqslant r$, $k = 1,2,\ldots,m$.

Since d is round in (X,d), in each $S\left(z_k, \frac{\varepsilon}{2}\right)$ there is z'_k such that

$$d(a'_k,z'_k) < r, \; k = 1,2,\ldots,m. \tag{9}$$

Put $\Psi_2 = \{z'_1,z'_2,\ldots,z'_m\}$ and $\Psi = \Psi_1 \cup \Psi_2$. We claim that the set Ψ is the one we are looking for in (*). Clearly Ψ as a finite set belongs to H(X).

Proof of $h(A,\Psi) < r$.

We have $h(A,\Psi) = d(A,\Psi) \vee d(\Psi,A)$ and $d(A,\Psi) = \max_{a \in A}\{\min_{y \in \Psi} d(a,y)\}$.

Since A is compact there is $\hat{a} \in A$ such that $d(A,\Psi) = \min_{y \in \Psi} d(\hat{a},y)$.

Suppose $\hat{a} \in S(a_k,\rho_{a_k})$. It follows from (8) that for $y = z'_{a_k} \in \Psi$ we have $d(z'_{a_k}, \hat{a}) < r$, so that $\min_{y\in\Psi} d(\hat{a},y) < r$. Hence $d(A,\Psi) < r$.

We also have $d(\Psi,A) = \max_{y\in\Psi}\{\min_{a\in A} d(a,y)\}$. For some $\hat{y} \in \Psi$ we get $d(\Psi,A) = \min_{a\in A} d(a,\hat{y})$. Suppose $\hat{y} \in \Psi_1$ and $\hat{y} = z'_{a_k}$, $1 \leqslant k \leqslant n$. For $a = a_k$ we have from (8) $d(z'_{a_k},a_k) < r$. Hence $d(\Psi,A) = \min_{a\in A} d(a,\hat{y}) \leqslant d(z'_{a_k},a_k) \leqslant r$.

Suppose $\hat{y} \in \Psi_2$ and $\hat{y} = z'_k$. From (9) we get $d(a'_k,z'_k) < r$, $1 \leqslant k \leqslant m$.

It follows that $d(\Psi,A) = \min_{a\in A} d(a,\hat{y}) \leqslant d(a'_k,z'_k) < r$. Hence $d(\Psi,A) \leqslant r$.

The inequalities $d(A,\Psi) < r$, $d(\Psi,A) < r$, imply $h(A,\Psi) < r$.

Proof of $h(\Psi,Z) < \varepsilon$.

We have $h(\Psi,Z) = d(\Psi,Z) \vee d(Z,\Psi)$, $d(\Psi,Z) = \max_{y\in\Psi}\{\min_{z\in Z} d(y,z)\}$. For some $\hat{y} \in \Psi$ we have $d(\Psi,Z) = \min_{z\in Z} d(\hat{y},z)$. Suppose $\hat{y} \in \Psi_1$ and $\hat{y} = z'_{a_k}$.

For $z = z_{a_k} \in Z$ we have $d(\hat{y},z) = d(z'_{a_k},z_{a_k}) < \varepsilon$, so that $d(\Psi,Z) < \varepsilon$.

Suppose $\hat{y} \in \Psi_2$ and $\hat{y} = z'_k$, $1 \leqslant k \leqslant m$. For $z = z_k$ we get $d(z'_k,z_k) < \dfrac{\varepsilon}{2} < \varepsilon$, so that $d(\Psi,Z) = \min_{z\in Z} d(\hat{y},z) \leqslant d(z'_k,z_k) < \varepsilon$. Hence $d(\Psi,Z) < \varepsilon$.

We also have $d(Z,\Psi) = \max_{z\in Z}\{\min_{y\in\Psi} d(z,y)\}$.

For some $\hat{z} \in Z$ we have $d(Z,\Psi) = \min_{y\in\Psi} d(\hat{z},y)$. Suppose $\hat{z} \in S\left(z_k, \dfrac{\varepsilon}{2}\right)$.

For $y = z'_k \in Z$ we get $d(\hat{z},z_k) \leqslant d(\hat{z},z'_k) + d(z_k,z'_k) < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$.

Hence $d(Z,\Psi) = \min_{y\in\Psi} d(\hat{z},y) \leqslant d(,\hat{z}'_k) < \varepsilon$.

The inequalities $d(\Psi,Z) < \varepsilon$, $d(Z,\Psi) < \varepsilon$, imply $h(\Psi,Z) < \varepsilon$. It follows that the set Ψ is the one sought in (*), so that (7) holds. The theorem is proved.

REMARKS.

(a) Observe that the completeness of the metric space (X, d) has not been used in the proof of Th. B. The hypothesis on «completeness» was made to put emphasis on a result which applies on the well known space (H (X), h) where the fractals live.

Clearly a similar proposition to Th. B holds if (X, d) is not supposed to be complete.

(b) The importance of the result provided by Th. B is quite obvious. It applies, practically, to all fractals we are dealing with. The euclidean metric on R^n being round the corresponding Hausdorff metric is also round. We often work in some complete metric space as (R^2, Euclidean) which we denote by (X, d). But then when we wish to discuss pictures, drawings, «black-on-white» subsets of the space, it becomes natural to introduce the space (H (X), h) where we know, by Th. B, that h is round.

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ΠΕΡΙΛΗΨΗ

Παρατηρήσεις ἐπὶ τοῦ Χώρου (H(X), h) τῶν Fractals

Ἔστω (X,d) ἕνας πλήρης μετρικὸς χῶρος, καὶ ἔστωσαν S(x,r), S[x,r] οἱ ἀνοικτὲς καὶ οἱ κλειστὲς σφαῖρες στὸν (X,d) ἀντιστοίχως, μὲ κέντρο x καὶ ἀκτίνα r.

Ἡ μετρικὴ d θὰ λέγεται «στρογγύλη» στὸν (X,d) τότε, καὶ μόνο τότε, ὅταν $\overline{S}(x,r) = S[x,r]$.

Ἔστω H(X) τὸ σύνολο, τὰ στοιχεῖα τοῦ ὁποίου εἶναι τὰ μὴ κενὰ καὶ συμπαγῆ ὑποσύνολα τοῦ μετρικοῦ χώρου (X,d). Εἶναι γνωστὸ [2] ὅτι, ἂν ἐπὶ τοῦ H(X) θεωρήσομε τὴ μετρικὴ h τοῦ Hausdorff, τότε ὁ χῶρος (H(X),h), καλούμενος καὶ χῶρος τῶν Fractals, εἶναι ἕνας πλήρης μετρικὸς χῶρος.

Στὴν παροῦσα ἐργασία ἀποδεικνύεται (Theorem B) ὅτι ἡ μετρικὴ h τοῦ Hausdorff στὸ χῶρο (H(X),h) εἶναι «στρογγύλη», τότε καὶ μόνο τότε ὅταν ἡ μετρικὴ d εἶναι «στρογγύλη» στὸ χῶρο (X,d).

Ἡ ὑπόθεση ὅτι ὁ χῶρος (X, d) εἶναι πλήρης δὲν χρησιμοποιήθηκε κατὰ τὴν ἀπόδειξη τοῦ Th. B. Οἱ παρατηρήσεις ποὺ ἀκολουθοῦν τὴν ἀπόδειξη τοῦ Th. B ἀναφέρονται στὴν ἀναγκαιότητα τῆς ὑποθέσεως αὐτῆς καθὼς ἐπίσης καὶ στὴν σπουδαιότητα τοῦ παρεχομένου ἀποτελέσματος ὑπὸ τοῦ Th. B.