στημιακῶν συναδέλφων τοῦ ἐκλιπόντος σοφοῦ, Ἑλλήνων τε καὶ ξένων, ὡς καὶ ἄλλων ἐπιστημόνων. Τὸ περιεχόμενόν του, οὗτινος προτάσσεται μακρὸν βιογραφικὸν σημείωμα καὶ ἀναγραφὴ τῶν δημοσιευμάτων τοῦ Κωνστ. Ἀμάντου, ἀναφέρεται εἰς θέματα ἁπτόμενα τῶν ἐνδιαφερόντων τοῦ ἀειμνήστου διδασκάλου καὶ ἱστορικοῦ τοῦ ἔθνους, ἤτοι εἰς τὴν βυζαντινὴν ἱστορίαν καὶ φιλολογίαν τῶν χρόνων τῆς τουρκοκρατίας, εἰς τὴν τοπωνυμιολογίαν, εἰς τὴν ἀρχαιολογίαν καὶ εἰς τὴν λαογραφίαν.

Αἱ 43 ἐν ὅλῳ μελέται τὰς ὁποίας ὁ τόμος περιλαμβάνει ἀποτελοῦσι συμβολὰς ἀνταξίας τοῦ τιμωμένου ἀνακαινιστοῦ τῆς ἐρεύνης τῆς ἱστορίας τῶν μέσων καὶ τῶν νεωτέρων χρόνων τῆς Ἑλλάδος, οἷος ὑπῆρξεν ὁ Κωνσταντῖνος Ἄμαντος.

ΑΝΑΚΟΙΝΩΣΕΙΣ ΜΗ ΜΕΛΟΥΣ

ΓΕΩΜΕΤΡΙΑ.—Some Remarks on Groups of Plane Curves in Relation to Coordinate Systems and to their Maxima or Minima, *by Chr. B. Glavas**. Ἀνεκοινώθη ὑπὸ τοῦ κ. Ἰωάνν. Ξανθάκη.

Ὁ Ἀκαδημαϊκὸς κ. **Ἰωάννης Ξανθάκης** ἀνακοινῶν τὴν μελέτην ταύτην εἶπε τὰ ἑξῆς.

Ὁ κ. Χ. Γκλαβᾶς συνεχίζει προηγουμένας ἐρεύνας του ἐπὶ τῶν ἀριθμητικῶν τιμῶν τῶν συντεταγμένων τῶν μεγίστων ἢ ἐλαχίστων σημείων μιᾶς ὁμάδος καμπυλῶν, παραγομένων βάσει τῆς ἀρχῆς τῆς γεωμετρικῆς ἰσοδυναμίας.

Εἰς τὴν ἀνακοίνωσιν ταύτην ἐξετάζονται δύο προβλήματα. Ἡ δυνατότης πρῶτον τοῦ προσδιορισμοῦ μιᾶς ἀρχικῆς καμπύλης, ὅταν δίδωνται δύο συστήματα συντεταγμένων, ἀναλυτικῶς καὶ γεωμετρικῶς ἰσοδύναμα, καὶ ἀριθμός τις μεγίστων ἢ ἐλαχίστων σημείων διὰ τῶν συντεταγμένων των. Καὶ δεύτερον ἡ δυνατότης προσδιορισμοῦ δύο συστημάτων συντεταγμένων, ἱκανῶν νὰ παραγάγουν μίαν ὁμάδα μεγίστων ἢ ἐλαχίστων σημείων, ὅταν δίδεται μία ἀρχικὴ καμπύλη καὶ ἀριθμός τις μεγίστων ἢ ἐλαχίστων σημείων.

Ὑπὸ ὡρισμένας συνθήκας ἀλγεβρικῶν κυρίως δυνατοτήτων ἡ ἀπάντησις εἶναι καταφατικὴ δι' ἀμφότερα τὰ προβλήματα.

1. It has been already shown that given an initial curve and two geometrically and analytically equivalent coordinate systems one may determine a group of curves with the initial one as their identity element. The numerical values of the maxima or minima of these curves, if they

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* ΧΡΗΣΤ. Β. ΓΚΛΑΒΑ, Μερικαὶ παρατηρήσεις ἐπὶ τῶν ὁμάδων τῶν ἐπιπέδων καμπυλῶν ἐν σχέσει πρὸς συστήματα συντεταγμένων καὶ μέγιστα ἢ ἐλάχιστα σημεῖα.

exist, form a group under a certain law of composition [1]. Thus to a given curve and coordinate systems there corresponds a definite group of plane curves and a definite group of maxima or minima.

Now there are two more problems for examination. First to determine an initial curve after the coordinate systems are given. And second to find the systems if the initial curve is known.

2. Let two coordinate systems (x,y), (x',y') be given. The systems are supposed to be geometrically and analytically equivalent [2]. Let the formulae of transformation from one system to the other be:

(1) $$x' = f_1(x, y), \quad y' = f_2(x, y)$$

Suppose that one takes a certain form for the initial curve, say $f(x, y) = 0$. To find this curve it is necessary to determine its unknown coefficients. The maxima or minima of $f(x, y) = 0$ may be found by simultaneous solution of $f(x, y) = 0$ and $f_x(x, y) = 0$. The geometrically equivalent of $f(x, y) = 0$ is $f(x', y') = 0$. By the substitution of x', y' from (1) into the latter equation:

(2) $$f(f_1(x, y), f_2(x, y)) = 0$$

To find the maxima or minima of this curve one should take $f_x(f_1, f_2) = 0$ and solve simultaneously the latter equation with (2). If the coefficients of $f(x, y) = 0$ are for example four, then it is clear that it is necessary to know two number pairs of maxima or minima (x_0, y_0) and (x_1, y_1), the first being the identity element of the whole group of maxima or minima. Substituting (x_0, y_0) and (x_1, y_1) in the four equations $f(x, y) = 0$, $f_x(x, y) = 0$, $f(f_1(x, y), f_2(x, y)) = 0$ and $f_x(f_1, f_2) = 0$ and solving for the four unknown coefficients one may be able to determine them and therefore the initial curve $f(x, y) = 0$.

It is well understood that if $f(x, y) = 0$ is finally found then (x_0, y_0) and (x_1, y_1) will be elements of the group of maxima or minima with the (x_0, y_0) as the identity element. If $f(x, y) = 0$ has n coefficients then an adequate number of additional pairs of maxima or minima should be given in order to secure n equations for solution.

Example 2. 1. Let two analytically and geometrically equivalent sys-

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1. C. B. Glavas, «On Maxima or Minima of a Group of Plane Curves», *Proceedings of the Academy of Athens*, 32 (1957), 507 - 517.

2. C. B. Glavas, «The Principle of Geometrical Equivalence and Some of its Consequences to the Theory of Curves», *Proceedings of the Academy of Athens*, 32 (1957), 122 - 124.

tems (x, y) and (x′, y′) be given with formulae of transformation $x'=x, y'=x+y$. Let the ordered pairs of numbers (1, 1) and $\left(\frac{1}{2}, \frac{1}{4}\right)$ represent the coordinates of the two given maxima or minima points. If the form of the required initial curve is $ax^3+bx^2+cx+dy=0$, then the maxima or minima points of this curve are determined by the solution of the two equations, the second of which represents the partial derivative of the first with respect to x:

$$ax^3+bx^2+cx+dy=0, \quad 3ax^2+2bx+c=0$$

Putting in these two equations $x=1$ and $y=1$, i.e. the given coordinates of the first max. or min., one gets:

(3) $$a+b+c+d=0, \quad 3a+2b+c=0$$

The geometrically equivalent of the given curve is $ax'^3+bx'^2+cx'+dy'=0$. Substituting x', y' by their equals from the formulae of transformation one finds the equation $ax^3+bx^2+cx+d(x+y)=0$ or $ax^3+bx^2+(c+d)x+dy=0$. Taking the partial derivative of this equation with respect to x, which is $3ax^2+2bx+c+d=0$, and substituting $x=\frac{1}{2}$, $y=\frac{1}{4}$ in the two latter equations, one gets:

(4) $$a+2b+4c+6d=0, \quad 3a+4b+4c+4d=0$$

The simultaneous solution of the four equations (3) and (4) for a, b, c, d gives $a=0$, $b=1$, $c=-2$, $d=1$. The initial therefore curve has the form $x^2-2x+y=0$.

If one continues the above process one finds as max. or min. the pairs (0, 0), $\left(-\frac{1}{2}, \frac{1}{4}\right)$, ...By reversing the process the max. or min. are (3/2, 9/4), (2, 16/4), ...The two sets of the coordinates of the max. or min. points are:

(x) $$\ldots, \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, \ldots$$

(y) $$\ldots, \frac{25}{4}, \frac{16}{4}, \frac{9}{4}, 1, \frac{1}{4}, 0, \frac{1}{4}, \ldots$$

These two sets constitute groups under a certain law of composition which has been already described[1]. This example shows that given two geometrically and analytically equivalent coordinate systems and any two

1. C. B. Glavas, «On Maxima or Minima of a Group of Plane Curves», *Proceedings of the Academy of Athens*, 32 (1957), 516.

(or more) ordered pairs of numbers representing the max. or min. of two (or more) points one may be able to determine an initial curve such that the latter has the first point as max. or min. and the derivative curves have the other points as such in their order of succession. Thus there are generated a group of curves with the first one as their identity element and two groups of the numerical values of the coordinates of their max. or min. It is clear now that through this arrangement the given max. or min. form a group with the rest of the max. or min. points.

The next example constitutes an attempt to determine an initial curve under the conditions of the previous example 2.1 but with a different form of the initial curve.

Example 2.2. Let the initial curve have the form $ax^2+by^2+cx+dy=0$ under the conditions of example 2.1. Again taking the partial derivative of this equation with respect to x and substituting in the two equations $x=1$ and $y=1$, one gets:

$$a+b+c+d=0, \qquad 2a+c=0$$

The geometrically equivalent of the initial curve is $ax'^2+by'^2+cx'+dy'=0$. Substituting x', y' for x and $x+y$ one gets $ax^2+b(x+y)^2+cx+d(x+y)=0$. The partial derivative of this equation with respect to x is $(2a+2b)x+2by+c+d=0$. Substituting in the latter two equations $x=\frac{1}{2}$ and $y=\frac{1}{4}$, one finally gets:

$$4a+9b+8c+12d=0, \qquad 2a+3b+2c+2d=0$$

Solving the four equations for a,b,c,d one finds $a=1$, $b=0$, $c=-2$, $d=1$, i. e. the same initial curve $x^2-2x+y=0$ is again determined. This leads to the suggestion that the initial curve is independent of the form taken for its determination and therefore it is uniquely determined.

Now we are going to prove the latter assertion. Let a form of the initial curve be $f(x,y)=0$. Let the formulae of transformation from the coordinate system (x', y') to the (x, y) be $x'=\varphi_1(x,y)$, $y'=\varphi_2(x,y)$. If the coordinates of the two given max. or min. points are (x_0, y_0) and (x_1, y_1), then the two equations $f(x,y)=0$ and $f_x(x,y)=0$ must be verified by $x=x_0$ and $y=y_0$. Suppose that $f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = f_1(x,y)$ and $f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = f_2(x,y)$. The geometrically equivalent of $f(x,y)=0$ is $f(x',y')=0$. Substituting x', y' by their equals from the formulae of transformation one gets $f(\varphi_1(x,y), \varphi_2(x,y))=0$.

The values $x=x_1$, $y=y_1$ must satisfy the two equations $f(\varphi_1, \varphi_2)=0$ and $f_x(\varphi_1, \varphi_2)=0$. The latter can be written as $f_1(\varphi_1, \varphi_2)\varphi'_{1x}+f_2(\varphi_1, \varphi_2)\varphi'_{2x}=0$. If the undetermined coefficients of $f(x, y)=0$ are four, then the solution of the four equations $f(x, y)=0$, $f_1(x, y)=0$, $f(\varphi_1, \varphi_2)=0$ and $f_1(\varphi_1, \varphi_2)\varphi'_{1x}+ f_2(\varphi_1, \varphi_2)\varphi'_{2x}=0$ suffice to give the value of those coefficients.

Now suppose that another initial curve $g(x, y)=0$ has to be determined. Then its final form depends upon the solution of the equations $g(x, y)=0$, $g_1(x, y)=0$, $g(\varphi_1, \varphi_2)=0$ and $g_1(\varphi_1, \varphi_2)\varphi'_{1x}+g_2(\varphi_1, \varphi_2)\varphi'_{2x}=0$, where $g_1(x, y) \ \frac{\partial}{\partial x} g(x, y) \ \ g_x(x,y)$ and $g_2(x, y) \ \frac{\partial}{\partial y} g(x, y) \ \ g_y(x, y)$. Eliminating φ'_{1x} and φ'_{2x} between the latter equation and the similar one of the previous paragraph one gets;

$$-\frac{f_1(\varphi_1, \varphi_2)}{f_2(\varphi_1, \varphi_2)} = -\frac{g_1(x, y)}{g_2(x, y)}$$

The left side of this equation represents the derivative of φ_2 with respect to φ_1. Putting $\varphi_2=k(\varphi_1)$ which is another way of writing $f(\varphi_1, \varphi_2)=0$, one can represent the left side by $\frac{d\varphi_2}{d\varphi_1} \quad k'(\varphi_1)$. Similarly for the right side $\frac{d\varphi_2}{d\varphi_1}=l'(\varphi_1)$, where $\varphi_2=l(\varphi_1)$ is another form of $g(\varphi_1, \varphi_2)=0$.

Therefore:

$$k'(\varphi_1)=l'(\varphi_1)$$

Hence:

$$k(\varphi_1)=l(\varphi_1)+C$$

But $\varphi_2(x_1, y_1)=k(\varphi_1(x_1, y_1))$ and $\varphi_2(x_1, y_1)=l(\varphi_1(x_1, y_1))$ by hypothesis. It follows that $C=0$ and $k(\varphi_1)=l(\varphi_1)$ which is the same as $f(\varphi_1, \varphi_2)=g(\varphi_1 \varphi_2)$. Replacing the variable φ_1 by x in the formula $k(\varphi_1)=l(\varphi_1)$ we find:

$$k(x)=l(x)+C_1$$

For $x=x_o$, we get $k(x_o)=y_o=l(x_o)$ by hypothesis, since $y=k(x)$ and $y=l(x)$ is another expression of the equations $f(x, y)=0$ and $g(x, y)=0$ respectively. Hence $C_1=0$ and $f(x, y)=g(x, y)$, which means that the initial curve is unique.

3. It remains to examine the third problem of determining two geometrically and analytically equivalent coordinate systems if the initial curve is given and an adequate number of maxima or minima points.

Let the equation of the initial curve be $f(x, y)=0$ and (x_1, y_1), (x_2, y_2) two max. or min. points. Suppose that the formulae of transformation from one coordinate system (x', y') to another one (x, y) have the form:

$$x'=\varphi_1(x, y), \quad y'=\varphi_2(x, y)$$

The functions $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are supposed to have four undetermined coefficients. The max. or min. of $f(x, y) = 0$ may be found by the solution simultaneously of $f(x, y) = 0$ and $f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = f_1(x, y) = 0$. Since $f(x, y) = 0$ is known the values of the max. or min. of this curve may be found.

The geometrically equivalent of $f(x, y) = 0$ is $f(x', y') = 0$. Substituting x', y' by their equals from the formulae of transformation we get $f(\varphi_1(x, y), \varphi_2(x, y)) = 0$. The partial derivative of this equation with respect to x is $f_{\varphi_1} \varphi'_{1x} + f_{\varphi_2} \varphi'_{2x} = 0$. Representing $f_{\varphi_1} = f_1(\varphi_1, \varphi_2) = \frac{\partial}{\partial \varphi_1} f(\varphi_1, \varphi_2)$ and $f_{\varphi_2} = \frac{\partial}{\partial \varphi_2} f(\varphi_1, \varphi_2) = f_2(\varphi_1, \varphi_2)$ we shall finally have:

$$f(\varphi_1, \varphi_2) = 0, \quad f_1(\varphi_1, \varphi_2) \varphi'_{1x} + f_2(\varphi_1, \varphi_2) \varphi'_{2x} = 0$$

The above two equations are verified by hypothesis by $x = x_1$ and $y = y_1$.

Continuing in this way, i.e. taking the geometrically equivalent of $f(\varphi_1, \varphi_2) = 0$, substituting there x', y' by their equals from the formulae of transformation and taking the partial derivative with respect to x, we find two more equations verified by $x = x_2$, $y = y_2$. Therefore we have four equations with four unknown coefficients. If these equations can be solved for the four coefficients then the two systems are determined and the points (x_1, y_1), (x_2, y_2) together with the max. or min. point (x_0, y_0) of $f(x, y) = 0$ and the rest of the points definitely constitute a group under a certain law of composition. The difficulty is of an algebraic nature and depends upon the ability to solve a certain number of equations to determine the coefficients of the formulae of transformation of the coordinate systems.

Example 3.1. Let $f(x, y) = x^2 - 2x + y = 0$. Let $(x_1, y_1) = \left(\frac{1}{2}, \frac{1}{4}\right)$ and the formulae of transformation be $x' = ax$, $y' = by$. The max. or min. of $x^2 - 2x + y = 0$ is the solution of the latter equation and $2x - 2 = 0$. Therefore $x = 1$ and $y = 1$ or $x_0 = 1$, $y_0 = 1$, where (x_0, y_0) represents the coordinates of the max. or min. of the given initial curve. The geometrically equivalent of $x^2 - 2x + y = 0$ is $x'^2 - 2x' + y' = 0$. Replacing x', y' by ax and by from the formulae we get $a^2x^2 - 2ax + by = 0$. Taking the partial derivative with respect to x, we get $2a^2x - 2a = 0$ or $2a(ax - 1) = 0$. It follows $a = 0$ or $ax - 1 = 0$. But $a = 0$ gives $b = 0$, i.e. no solution to our problem. Therefore we consider the equations $a^2x^2 - 2ax + by = 0$, $ax - 1 = 0$.

The values of the coordinates of this second curve are $x_1 = \frac{1}{2}$ and

$y_1 = \frac{1}{4}$. Replacing x and y by $\frac{1}{2}$ and $\frac{1}{4}$ respectively in the latter two equations we get finally $a^2 - 4a + b = 0$, $a = 2$. Therefore $a = 2$, $b = 4$ and the formulae become $x' = 2x$, $y' = 4y$. Substituting in the equation $a^2x^2 - 2ax + by = 0$ a and b by their equals we get $4x^2 - 4x + 4y = 0$ or $x^2 - x + y = 0$. Continuing the process we find that the max. or min. of the next curve $(2x)^2 - 2x + 4y = 0$ or $2x^2 - x + 2y = 0$ is $\left(\frac{1}{4}, \frac{1}{6}\right)$. Finally we can write the two sets of the «abscissae» (x) and «ordinates» (y) of the whole group of curves:

(x) $\quad \ldots, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

(y) $\quad \ldots, 64, 16, 4, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

It is not difficult to see that the two sets (x) and (y) constitute groups with (1, 1), which is the max. or min. of the initial curve, as their identity element and with the usual multiplicative law of composition. Thus the law of composition for (x) is $2^m \circ 2^n = 2^{m+n}$ where m and n are integers. For $m = -n$, we get $2^0 = 1$, i.e. the identity element. For the set (y) the law is $4^m * 4^n = 4^{m+n}$ (n and m integers).

Example 3.2. Let again the initial equation be $x^2 - 2x + y = 0$ and the given points $\left(\frac{1}{2}, \frac{1}{4}\right)$ and (0, 0). If the form of the formulae of transformation is $x' = ax + y$, $y' = cy + d$, then four coefficients must be determined and the number of equations for their determination is therefore four.

The max. or min. of the initial curve is as before the point (1, 1). The geometrically equivalent to the initial curve is $x'^2 - 2x' + y' = 0$. Substituting here x' and y' by $ax + b$ and $cy + d$ respectively from the formulae of transformation we get:

(5) $$(ax + b)^2 - 2(ax + b) + cy + d = 0$$

The partial derivative with respect to x of this equation is $2a(ax + b) - 2a = 0$ or $ax + b = 1$. Substituting 1 for $ax + b$ in equation (5), we find $cy + d = 1$. But $ax + b = 1$ and $cy + d = 1$ must be verified by the coordinates $x = \frac{1}{2}$ and $y = \frac{1}{4}$ of the first given pair of numbers. Therefore we shall get after the substitution:

(6) $$a + 2b = 0, \quad c + 4d = 4$$

Continuing the process we see that the geometrically equivalent of (5) is found if x and y are substituted by x' and y' respectively. Then putting there $ax + b$ and $cy + d$ for x' and y' respectively we find:

$$[a(ax + b) + b]^2 - 2[a(ax + b) + b] + c(cy + d) + d = 0$$

Or:

$$(a^2x + ab + b)^2 - 2(a^2x + ab + b) + c^2y + cd + d = 0 \tag{7}$$

The partial derivative of this equation with respect to x is:

$$2a^2(a^2x + ab + b) - 2a^2 = 0$$

Or:

$$a^2x + ab + b = 1 \tag{8}$$

Substituting 1 in (7) for $a^2x + ab + b$ we finally get:

$$c^2y + cd + d = 1 \tag{9}$$

Equations (8) and (9) are verified according to our hypothesis by $x=0$, $y=0$.

Then:

$$ab + b = 1, \quad cd + d = 1 \tag{10}$$

Equations (6) and (10) solved simultaneously for a, b, c, d give $a=0$, $b=1$ and $a=1$, $b=\frac{1}{2}$. Also $c=0$, $d=1$ and $c=3$, $d=\frac{1}{4}$ Taking first $a=0$, $b=1$, $c=0$, $d=1$ we find $x'=1$, $y'=1$ which does not represent a transformation between the two coordinate systems. Putting now $a=1$, $b=\frac{1}{2}$, $c=3$, $d=\frac{1}{4}$, we get:

$$x' = x + \frac{1}{2}, \quad y' = 3y + \frac{1}{4} \tag{11}$$

The application of these formulae will surely give the two max. or min. $\left(\frac{1}{2}, \frac{1}{4}\right)$ and (0, 0). Substituting in (7) the above determined values of a, b, c, d we shall finally obtain:

$$(x + 1)^2 - 2(x + 1) + 9y + 1 = 0$$

The max. or min. of this curve is $x=0$, $y=0$ and this is well known in advance. The geometrically equivalent of the latter curve can be found if x', y' are replaced for x, y respectively. Then applying formulae (11) we get:

$$\left(x + 1 + \frac{1}{2}\right)^2 - 2\left(x + 1 + \frac{1}{2}\right) + 9\left(3y + \frac{1}{4}\right) + 1 = 0$$

Or:

$$\left(x + \frac{3}{2}\right)^2 - 2\left(x + \frac{3}{2}\right) + 27y + \frac{13}{4} = 0 \tag{12}$$

The partial derivative with respect to x of this curve is:

$$2\left(x+\frac{3}{2}\right)-2=0$$

Or:

$$x=-\frac{1}{2}$$

Replacing the above value in (12) we finally get $y=-1/12$

Continuing in the same way we find as numerical values of the next max. or min. the ordered pairs $(-1, -1/9)$, $(-3/2, -13/108)$....

Reversing the process we go from $x^2-2x+y=0$ to the equation:

$$\left(x'-\frac{1}{2}\right)^2-2\left(x'-\frac{1}{2}\right)+\frac{y'}{3}-\frac{1}{12}=0$$

This equation is obtained from the initial one by the substitution of x and y by $x'-\frac{1}{2}$ and $\frac{y'}{3}-\frac{1}{12}$ from (11). Taking the partial derivative with respect to x′ we obtain as max. or min. the point $x'=\frac{3}{2}$, $y'=\frac{13}{4}$. Continuing in the same way we find next max. or min. the pairs (2, 10), $\left(\frac{5}{2}, \frac{121}{4}\right)$, (3, 91)....

Now we can write the two sets of the «abscissae» (x) and «ordinates (y) of the max. or min. points of our group of plane curves:

(x) $\quad \ldots, 3, \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$

(y) $\quad \ldots, 91, \frac{121}{4}, 10, \frac{13}{4}, 1, \frac{1}{4}, 0, -\frac{1}{12}, -\frac{1}{9}, -\frac{1}{108}, \ldots$

The set (x) constitutes a group as it is well known from example 2.1. To show that the set (y) does the same let us take 1 and the left to 1 numbers and apply the Calculus of Finite Differences:

n	U_n		ΔU_n
0	U_0	1	
			$\frac{3^2}{2^2}$
1	U_1	$\frac{13}{4}$	
			$\frac{3^3}{2^2}$
2	U_2	10	
			$\frac{3^4}{2^2}$
3	U_3	$\frac{121}{4}$	
			$\frac{3^5}{2^2}$
4	U_4	91	
•	•	•	•
•	•	•	•
•	•	•	•

If we continue we shall see that the next differences $\Delta^2 U_n$, $\Delta^3 U_n, \ldots$ do not become constant. But we observe that in our case we have $U_n - pU_{n-1} = qr^n$, where $p=1$, $q=\frac{3}{2^2}$ and $r=3$. The solution of the above difference equation is given by the formula [1]:

$$U_n = \frac{Ap^n + qr^{n+1}}{r-p}$$

Substituting p, q, r by their equals:

$$U_n = \frac{4A + 3^{n+2}}{2^3}$$

To determine the constant A we put $n=0$ and $U_0=1$. We then finally find $A = -\frac{1}{4}$. Hence the general term becomes:

$$U_n = \frac{3^{n+2}-1}{2^3}$$

Substituting in this formula n for all its integral values positive or negative we obtain all numbers of set (y). For $n=0$ we get the identity element 1 and the law of composition of group (y) is evidently:

$$\frac{3^{n+2}-1}{2^3} * \frac{3^{m+2}-1}{2^3} = \frac{3^{m+n+2}-1}{2^3}$$

Here m and n are supposed to be any integers.

Example 3.3. Let again the initial curve be the same $x^2-2x+y=0$ and the given max. or min. points $\left(\frac{1}{2}, \frac{1}{4}\right)$, (0, 0), i.e. the same as in example 3.2. Let the formulae of transformation have the form $x' = a_1 x + b_1 y$, $y' = a_2 x + b_2 y$. The problem is to determine a_1, b_1, a_2, b_2 and show that the resulting sets of (x) and (y) constitute groups with the given points $x_1 = \frac{1}{2}$, $y_1 = \frac{1}{4}$ and $x_2 = 0$, $y_2 = 0$. Following exactly the method of the previous examples we find $a_1 = 1$, $b_1 = 0$, $a_2 = 1$, $b_2 = 1$. Therefore the formulae of transformation are $x' = x$ and $y' = x + y$. The two sets of the «abscissae» (x) and of the «ordinates» (y) are the same with those of example 2.1 and both are known to constitute groups.

In examples 3.1, 3.2, 3.3 the same initial curve $x^2 - 2x + y = 0$ was taken. In 3.1 the given pair was $\left(\frac{1}{2}, \frac{1}{4}\right)$ while in 3.2,3.3. the given pairs were $\left(\frac{1}{2}, \frac{1}{4}\right)$ and (0, 0). In the three examples three different coordinate

[1] S. Barnard and J. M. Child, *Higher Algebra*, London, Macmillan Co., 1947, p. 368-69.

systems were determined and this led to the determination of three different sets-groups of the «abscissae» (x) and the «ordinates» (y) of the max. or min. It is noteworthy that these groups have certain pairs in common. Thus in examples 3.2 and 3.3 the pairs $\left(\frac{1}{2}, \frac{1}{4}\right)$ and (0, 0) are common to both.

The groups of (y) 's of 3.2 and 3.3 are respectively:

$$(y) \quad \ldots, 91, \frac{121}{4}, 10, \frac{13}{4}, 1, \frac{1}{4}, 0, -\frac{1}{12}, -\frac{1}{9}, -\frac{1}{108}, \ldots$$

$$(y) \quad \ldots, \frac{36}{4}, \frac{25}{4}, \frac{16}{4}, \frac{9}{4}, 1, \frac{1}{4}, 0, \frac{1}{4}, 1, \frac{9}{4}, \ldots$$

From the above considerations one can draw the following conclusion. Given any finite number of pairs of numbers and an initial curve then one may determine certain different coordinate systems and produce groups of the «abscissae» (x) and the «ordinates» (y) representing max. or min. points of the corresponding groups of curves. The remarkable thing is that the given arbitrary pairs of numbers may become elements of different groups of numbers under different laws of composition. The latter is theoretically assured and verified by the previous three examples.

ΠΕΡΙΛΗΨΙΣ

Εἰς προηγουμένην ἀνακοίνωσιν ἡμῶν (*Πρακτ. τῆς Ἀκαδημ. Ἀθηνῶν*, τόμ. 32 (1957), σ. 507 - 517) ἐτέθη τὸ πρόβλημα τῆς ἐξετάσεως τῶν ἀριθμητικῶν τιμῶν τῶν συντεταγμένων τῶν μεγίστων ἢ ἐλαχίστων σημείων μιᾶς ὁμάδος καμπυλῶν, παραγομένων δυνάμει τῆς ἀρχῆς τῆς γεωμετρικῆς ἰσοδυναμίας. Διεπιστώθη ἐκεῖ, ὅτι εἰς δοθεῖσαν ἀρχικὴν καμπύλην καὶ εἰς δοθέντα γεωμετρικῶς καὶ ἀναλυτικῶς ἰσοδύναμα συστήματα συντεταγμένων ἀντιστοιχοῦν ὡρισμένα σύνολα τῶν «τετμημένων» καὶ τῶν «τεταγμένων» τῶν μεγίστων ἢ ἐλαχίστων τῆς ὡς ἄνω ὁμάδος τῶν καμπυλῶν. Ἐδείχθη ἀκόμη, ὅτι, ἂν ὑπάρχουν μέγιστα ἢ ἐλάχιστα, τὰ σύνολα τῶν συντεταγμένων των, χωριστὰ λαμβανομένων, συνιστοῦν καὶ ταῦτα ὁμάδας ὑπὸ ὡρισμένον ἑκάστοτε νόμον συνθέσεως.

Ἀπομένουν νὰ ἐξετασθοῦν δύο ἀκόμη συναφῆ θέματα. Πρῶτον, ἄν, δοθέντων δύο γεωμετρικῶς καὶ ἀναλυτικῶς ἰσοδυνάμων συστημάτων συντεταγμένων καὶ ἀριθμοῦ τινος μεγίστων ἢ ἐλαχίστων σημείων διὰ τῶν συντεταγμένων των, εἶναι δυνατὸς ὁ προσδιορισμὸς μιᾶς ἀρχικῆς καμπύλης μὲ τὰ ἐν λόγῳ σημεῖα ὡς μέγιστα ἢ ἐλάχιστα. Τὸ δεύτερον πρόβλημα εἶναι ὁ προσδιορισμὸς δύο συστημάτων συντεταγμένων γεωμετρικῶς καὶ ἀναλυτικῶς ἰσοδυνάμων, ἱκανῶν νὰ παραγάγουν ὁμάδα μεγίστων ἢ ἐλαχίστων σημείων ἔκ τινος δοθείσης ἀρχικῆς καμπύλης καὶ ἀριθμοῦ τινος μεγίστων ἢ ἐλαχίστων σημείων.

Ἡ ἀπάντησις καὶ εἰς τὰ δύο προβλήματα εἶναι καταφατικὴ ὑπὸ ὡρισμένας προϋποθέσεις ἀλγεβρικῶν κυρίως δυνατοτήτων. Οὕτω διὰ τὴν πρώτην περίπτωσιν λαμβάνεται μία ἀρχικὴ ἐξίσωσις ὑπό τινα μορφὴν μὲ ὡρισμένον ἀριθμὸν προσδιοριστέων συντελεστῶν. Ἐὰν π.χ. ὁ ἀριθμὸς τῶν συντελεστῶν τούτων εἶναι τέσσαρες, τότε διὰ τῆς μεθόδου τῆς γεωμετρικῆς ἰσοδυναμίας παράγονται τέσσαρες ἐξισώσεις. Αἱ τελευταῖαι ἀποτελοῦνται ἀπὸ τὴν ἀρχικήν, τὴν μερικὴν παράγωγον ταύτης ὡς πρὸς x, ἐξισουμένην μὲ τὸ μηδέν, καὶ τὰς δύο ὁμοίως παραγομένας καὶ ἀμέσως ἑπομένας τῶν προηγουμένων. Εἰς τὰς δύο πρώτας ἀντικαθίστανται τὰ x καὶ y διὰ τῶν δοθεισῶν συντεταγμένων τοῦ πρώτου δοθέντος μεγίστου ἢ ἐλαχίστου. Παρομοία ἀντικατάστασις γίνεται εἰς τὰς δύο ὑπολοίπους ἐξισώσεις τῶν συντεταγμένων τοῦ δευτέρου μεγίστου ἢ ἐλαχίστου. Αἱ οὕτω προκύπτουσαι τέσσαρες ἐξισώσεις λυόμεναι δίδουν τὰς ζητουμένας τιμὰς τῶν συντελεστῶν τῆς ἀρχικῆς ἐξισώσεως.

Δύο συγκεκριμένα παραδείγματα τῆς ἄνω περιπτώσεως ὡδήγησαν εἰς τὴν σκέψιν, ὅτι ὁ προσδιορισμὸς τῆς ἀρχικῆς καμπύλης εἶναι ἀνεξάρτητος τῆς ἀρχικῶς λαμβανομένης μορφῆς της καὶ ἑπομένως ἡ οὕτω πως προσδιοριζομένη καμπύλη εἶναι μοναδική. Τοῦτο ἀπεδείχθη ἐν συνεχείᾳ κατὰ τρόπον γενικόν.

Διὰ τὸ δεύτερον πρόβλημα ἐλήφθη ὡρισμένη ἀρχικὴ μορφὴ τῶν τύπων μετασχηματισμοῦ, ἐχόντων ὡρισμένον ἀριθμὸν προσδιοριστέων συντελεστῶν. Ἐὰν ὁ ἀριθμὸς τούτων εἶναι π.χ. τέσσαρες, τότε ἀπαιτοῦνται τέσσαρες ἐν ὅλῳ ἐξισώσεις πρὸς προσδιορισμόν των καὶ κατὰ συνέπειαν πρέπει νὰ ἔχουν δοθῆ δύο μέγιστα ἢ ἐλάχιστα. Κατὰ τὸν ἴδιον, ὡς καὶ εἰς τὸ προηγούμενον πρόβλημα, τρόπον παράγονται τέσσαρες ἐξισώσεις, αἱ ὁποῖαι εἶναι ἀρκεταὶ διὰ τὸν ἐπιζητούμενον προσδιορισμὸν συντελεστῶν.

Εἶναι ἀξιοσημείωτον, ὅτι ὑπὸ τὰς αὐτὰς ἀρχικῶς συνθήκας, δηλ. τὴν αὐτὴν ἀρχικὴν καμπύλην καὶ τὰ αὐτὰ μέγιστα ἢ ἐλάχιστα σημεῖα διδόμενα διὰ τῶν συντεταγμένων των, εἶναι δυνατὸς ὁ προσδιορισμὸς διαφόρων συστημάτων συντεταγμένων, προσδιοριζομένων, ὡς ἀνωτέρω, ἐκ διαφόρου μορφῆς τύπων μετασχηματισμῶν. Ἐκ τῶν δοθέντων ἀκόμη παραδειγμάτων διεπιστώθη, ὅτι, δοθέντων οἱωνδήποτε ἀριθμῶν ὡς συντεταγμένων τῶν μεγίστων ἢ ἐλαχίστων σημείων, εἶναι ἐνίοτε δυνατὸς ὁ προσδιορισμὸς διακεκριμένων μὲν ὁμάδων συντεταγμένων, ἀλλ' ἐχουσῶν κοινὰς τὰς συντεταγμένας τῶν ἀρχικῶς διδομένων μεγίστων ἢ ἐλαχίστων σημείων.