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ΑΝΑΛΥΤΙΚΗ ΓΕΩΜΕΤΡΙΑ. — **Contributions to Curve Tracing,** *by* ***Christos B. Glavas*** *****. Ανεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰωάνν. Ξανθάκη.

Introduction: Besides the existing methods on curve tracing another one is presented in this paper, which may be proved useful in certain cases By the proposed method a curve C_1, represented by the equation $f(a_1, b_1) = 0$ in the coordinate system (a_1, b_1), is transformed to another curve C_2 represented by the equation $f(a_2, b_2) = 0$ in the system (a_2, b_2). If the latter curve is much simpler or well known then one can transform C_2 into C_1 by means of a finite number of Euclidean constructions.

Two relations $f_1(a_1, b_1) = 0$ and $f_2(a_2, b_2) = 0$ expressed in the systems (a_1, b_1), (a_2, b_2) respectively are analytically equivalent if there are formulae of transformation between the two systems such that each can be transformed to the other. In a previous report it is shown that two curves $f(a_1, b_1) = 0$ and $f(a_2, b_2) = 0$ corresponding to the same analytical relation represent «geometrically equivalent» curves if each can be transformed geometrically to the other[1]. This is possible if the systems (a_1, b_1), (a_2, b_2) are geometrically equivalent, i.e. if $a_1 = a_2$, $b_1 = b_2$ and given a point A de-

* Χ. Β. ΓΚΛΑΒΑ: Συμβολὴ εἰς τὴν χάραξιν καμπύλων γραμμῶν.

[1] C. B. Glavas, «The Principle of Geometrical Equivalence and Some of its Consequences to the Theory of Curves», *Proceedings of the Academy of Athens*, 32 (1957), p. 122-124.

termined by the first system one can go to a second B determined by the other system by a finite number of geometrical constructions.

The procedure which is followed in this paper consists first of a brief review of certain geometrically equivalent coordinate systems. Then two cases of the application of the proposed method of curve tracing are examined. The first and simpler one uses two coordinate systems while the second and more complicated more than two systems. The purpose of the given examples is not to present specific cases but only to illustrate the new method.

It should be noted that a certain method used by Frost has nothing to do with the one proposed here, for the former is not general and is based on auxiliary curves and not on coordinate systems.[2]

1. The two well known coordinate systems, the Cartesian (x,y) and the polar (r,θ), are not geometrically equivalent since the latter contains an angular measure, the angle θ, while the first has two linear ones. In the cathetic system (Fig. 1.1) a point P is defined by the polar angle θ and the segment OG=g, where G is the intersection of the perpendicular on OP at P with the polar axis[3].

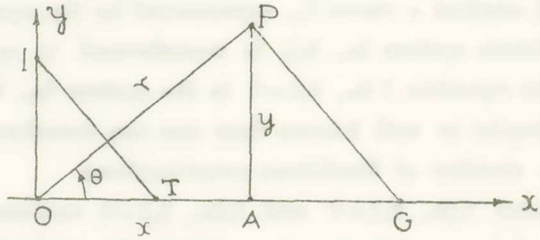

Fig 1.1.

It is not difficult to find the formulae of transformation from the Cartesian system to the cathetic and vice versa, which are:

$$x = g\cos^2\theta \ , \ y = g\sin\theta\cos\theta$$

(1a)
$$g = \frac{x^2+y^2}{x} \ , \ \tan\theta = \frac{y}{x}$$

[2] P. Frost, *An Elementary Treatise on Curve Tracing*, New York, Chelsea Publishing Co., Fifth Ed., 1960, p. 177-183.

[3] C. B. Glavas, «Plane Coordinate Systems in Mathematics Study», Doctoral Dissertation, New York, Teachers College, Columbia University, 1956, Ch. III.

Also, since θ is common to the polar and cathetic systems, the formula of transformation between these systems is $r = g \cdot \cos\theta$. If on the Oy axis we take $OI = 1$ and from I we draw a perpendicular on the radius vector OP, we determine the point T as the intersection of this perpendicular with the Ox axis. Then we see that $OT = \tan\theta$. If we put $\tan\theta = t$,

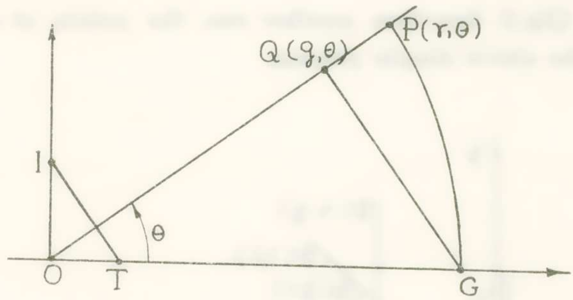

Fig. 1.2.

and we define t positive to the right of the origin O and negative to the left of it, we have the «tangential» forms of the polar and cathetic systems. Each point P may be determined by the pair of numbers (r,t) or (g,t). Then formulae (1a) become:

$$x = \frac{g}{1+t^2}, \quad y = \frac{gt}{1+t^2}$$

(1b)

$$g = \frac{x^2+y^2}{x}, \quad t = \frac{y}{x}$$

For the polar system we have $x = r\cos\theta = \dfrac{r}{\sqrt{1+t^2}}$, $y = r\sin\theta = \dfrac{rt}{\sqrt{1+t^2}}$.

It is important to observe that by the formulae (1a), (1b) the coordinates x,y are expressed as rational functions of g,t and vice versa.

The systems (r,θ) and (g,θ) are analytically and geometrically equivalent. The first is clear since we have already established before the formula $r = g\cos\theta$. For the second, take the point $P(r,\theta)$ (Fig. 1.2) and on the Ox axis $OG = OP = r$. Then from G draw a perpendicular OQ on OP. The point Q has cathetic coordinates $OG = g = r$ and θ. Conversely, given Q one can detemine P by applying the reverse process. The systems therefore (r,θ) and (g,θ) are geometrically equivalent.

The Cartesian (x,y) and the systems (r,t) and (g,t) may be proved to be geometrically equivalent. Given R(x,y) (Fig. 1.3) draw the perpendicular RT on Ox axis. Join I to T and from O draw OP perpendicular on IT. Take OP=TR=y. Since OT=x=t, the systems (x,y) and (t,r) are geometrically equivalent. Similarly, if we take OG=RT=y, and from G we draw a perpendicular GQ on OP, then the points Q(g,t) and R(y,x) have g=y and x=t, which shows that the systems are geometrically equivalent. If the point for example R(x,y) describes a curve, then the corresponding point P(r,t) or Q(g,t) describes another one, the points of which may be detemined by the above simple process.

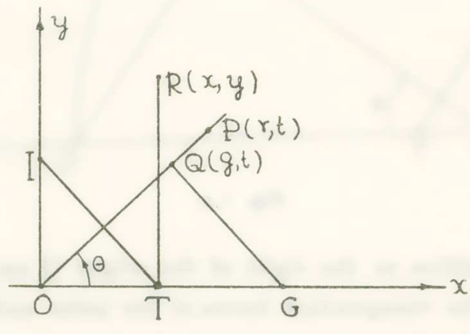

Fig. 1.3.

2. After the above remarks we proceed to the examination of the first case about curve tracing. This will be done by the presentation of certain examples.

Example 2.1. Trace the curve g=t. For one who is not familiar with cathetic coordinates this equation may be transformed to the Cartesian system by the application of formulae (lb). Then immediately one recognizes the kind of the curve the given equation represents. But, for the sake of illustrating the new method, equation g=t has as its geometrically equivalent the equation y=x in the Cartesian system. Therefore we may write $(g=t) \overset{g}{\sim} (y=x)$. But y=x is well known to be the line through the origin making with Ox axis an angle of 45° (Fig. 2.1).

From the known line y=x we may trace the corresponding one g=t. Take the point P (x,y) and draw the perpendicular PT on Ox. Join I to T and draw OQ perpendicular on IT. Take on Ox axis a segment equal to PT(=OT). From T draw a perpendicular to OQ. Their point of intersection coincides with point Q. Then we immediately see that Q is on a circumference with diameter $OI=1$. The origin O corresponds to itself, while it is not difficult to see that the infinite point of OP corresponds to I. Really, for the infinite point of OP, IT becomes parallel to Ox. Its intersection with the perpendicular from O is evidently the point I. The points of OP′ correspond to the semicircumference on the left of Oy and the infinite point of OP′ corresponds again to I.

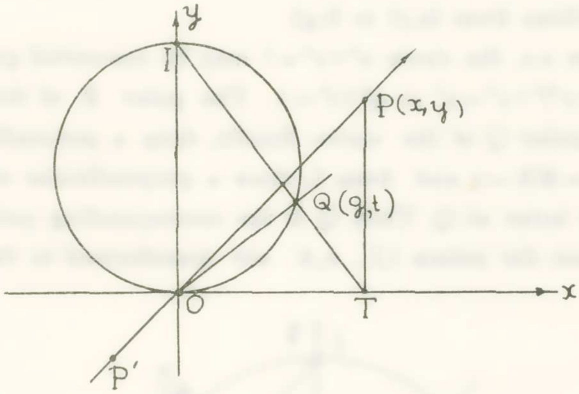

Fig 2.1.

This simple example shows the quick results of the new method. It is not necessary to find many points in order to trace the required curve. Here the branch OP of $y=x$ has as its «image» the branch OQI, while OP′ is transformed to the other one. The line $y=x$ «bends itself» from both sides around O with the infinite point falling on I. Thus the straight line OP is converted to its geometrically equivalent closed curve which is a circle. Since in this case the equations of the two curves have the form $f(x,y)=0$ and $f(t,g)=0$, it is clear that there exists a one-to-one correspondence between their points. To the point (x,y) satisfying the equation $f(x,y)=0$ their corresponds the point (t,g) satisfying the equation $f(t,g)=0$, while we have $x=t$ and $y=g$. The line $y=x$ may be considered as a closed curve with its two infinite points on both sides coinciding to one which corresponds to the point I of the curve $g=t$.

Example 2.2. Trace the curve $(x^2+y^2)^2+y^2=x^2$. In this second example the equation of a curve is given in the Cartesian system (x,y). If we write it in the form $\frac{(x^2+y^2)^2}{x^2}+\frac{y^2}{x^2}=1$ and apply the formulae (lb) we get $g^2+t^2=1$. The two latter equations represent one and the same curve, expressed in two different coordinate systems. We can immediately write the corresponding geometrically equivalent of $g^2+t^2=1$ in the system (x,y), which is $y^2+x^2=1$. But this is a well known curve, a circle through the origin with radius 1. Therefore, starting from an unknown curve in the system (x,y) we find a known curve again in the system (x,y). If we trace the latter we can trace the original also by applying the process of geometrical constructions from (x,y) to (t,g).

In Figure 2.2. the circle $x^2+y^2=1$ may be converted geometrically to the curve $(x^2+y^2)^2+y^2=x^2$ or $g^2+t^2=1$. The point R of this circle has as its image the point Q of the curve. Really, drop a perpendicular ON on IM. Take OL=MR=y and from L draw a perpendicular on ON intersecting with the latter at Q. Then Q is the corresponding point to R. It is easily found that the points I,I′, A,A′ are transformed to the points A,A′,

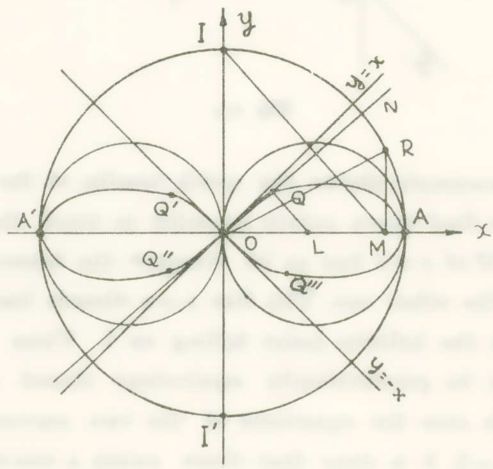

Fig. 2.2.

O,O respectively. Therefore it is immediately seen that the arc AI is transformed to the arc OQA and the arcs I′A, I′A′, IA′ to the arcs OQ″A′, OQ′A′, OQ‴A. Since the four arcs of the circle are equal, then the corresponding arcs of the curve must be equal. Thus the required tracing of the given equation is very quick, based on the transformation of certain important points of the circle.

In order to fully appreciate the importance of this method we must remember that «geometrically equivalent curves have those properties in common which are the by-product of the same analytical operations on their common equation»[4]. To the tangents $x=\pm 1$ at the points A,A′ of the circle $x^2+y^2=1$ there must correspond the tangent lines $t=\pm 1$ or $y=\pm x$ of the curve under tracing, which is true. Also to the tangents $y=\pm 1$ at I,I′ of the circle there must correspond the tangent lines $g=\pm 1$ or $x^2+y^2=\pm x$ of the curve at the points A,A′. The circle $x^2+y^2=1$ is symmetrical with respect to $x=0$, $y=0$ and the origin ($x=0$ and $y=0$). Then the curve must be symmetrical to the lines $t=0$, $g=0$, which is equivalent to saying that the curve is symmetrical to both axes $x=0$, $y=0$ and the origin Θ.

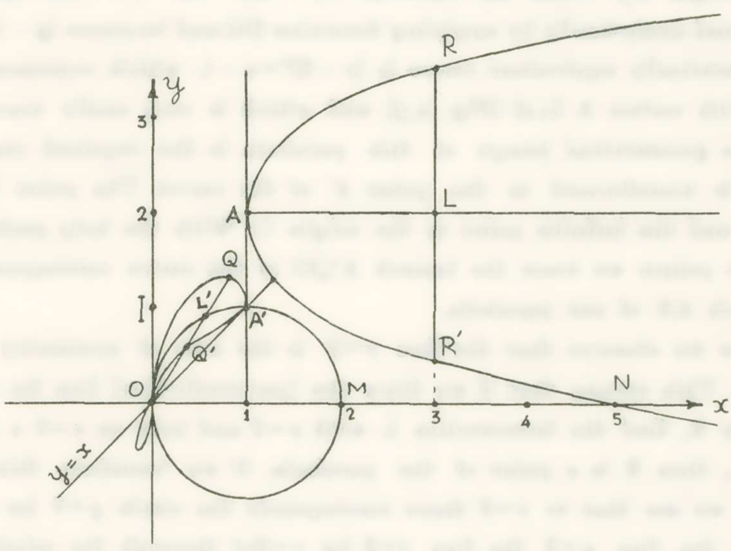

Fig. 2.3

We remarked in the previous example 2.1 that there exists a one-to-one correspondence between the points of two geometrically equivalent curves. But in our case we found that the points A,A′ are transformed to one point O. However the one-to-one correspondence may be preserved by observing that to the point A with coordinates $x=1$, $y=0$ there must correspond a point with coordinates $t=1$, $g=0$ which is the origin

[4] C. B. Glavas, «The Principle of Geometrical Equivalence and Some of its Consequences to the Theory of Curves», *Proceedings of the Academy of Athens*, 32 (1957), p. 126.

O, lying on the line $t=1$ or $y=x$. On the other hand the point A′ ($x=-1$, $y=0$) corresponds to the point O lying on the line $t=-1$ or $y=-x$.

Also, «geometrically equivalent curves represented by the same equation have maxima and minima corresponding to common values of the variables»[5]. To the maximum point I (0,1) of the circle there corresponds the maximum point $t=0$, $g=1$ or $y=0$, $x=1$, i.e. the point A of the curve, which is true, etc. Thus we see that by the new method we may quickly trace an unknown curve by transforming to it its geometrically equivalent. At the same time we may discover many properties of that curve by properly «translating» the corresponding ones to the known curve.

Example 2.3. Trace the curve $(x^2+y^2-2x)^2=xy-x^2$. This equation is transformed analytically by applying formulae (1b) and becomes $(g-2)^2=t-1$. Its geometrically equivalent curve is $(y-2)^2=x-1$, which represents a parabola with vertex A (1,2) (Fig. 2.3), and which is very easily traced.

The geometrical image of this parabola is the required curve. The point A is transformed to the point A′ of the curve. The point R to the point Q and the infinite point to the origin O. With the help perhaps of a few more points we trace the branch A′QO of the curve corresponding to the branch AR of our parabola.

Now we observe that the line $y=2$ is the axis of symmetry for the parabola. This means that if we draw the (perpendicular) line for example $x=3$ from R, find the intersection L with $y=2$ and take on $x=3$ a segment R′L=RL, then R is a point of the parabola. If we translate this phrase properly we see that to $y=2$ there corresponds the circle $g=2$ (or $x^2+y^2=2x$), to the line $x=3$ the line $t=3$ (or $y=3x$) through the origin. Their intersection is the point L′. If we take on $t=3$ from Q a segment QL′=Q′L′, then Q′ is a point of the curve. Therefore, it is enough to find the symmetrical branch of A′QO with respect to the circle $g=2$ (OA′MO) in the above sense.

Note also that the line $y=2$ intersects with the parabola at the point A ($x=1$,$y=2$) and at the infinite point. The corresponding line $g=2$ (the circle OA′MO) intersects with the curve at the point A′ ($t=1$, $g=2$) and the origin O corresponding to the infinite point. The line $x=1$ is tangent to the parabola at the point A ($x=1$, $y=2$) while the corresponding line

[5] Loc. cit.

$t=1$ or $y=x$ must be tangent to our curve at the point A' $(t=1, g=2)$. The part of the parabola below the Ox axis is transformed into the small part of our curve lying in the third quadrant, while the point N (5,0) is transformed to the origin lying on the line $t=5$ $(y=5x)$.

If the given equation is $g^2=t$, then the corresponding curve is the image of the parabola $y^2=x$. The two branches of $g^2=t$ are symmetrical with respect to the origin O and the tracing is much easier.

3. Now we are going to apply our method to a more complicated case where more than one coordinate systems are used. One may readily see that the proposed method has infinite possibilities for curve tracing. The few examples, which are given below have again the purpose of illustrating the second case.

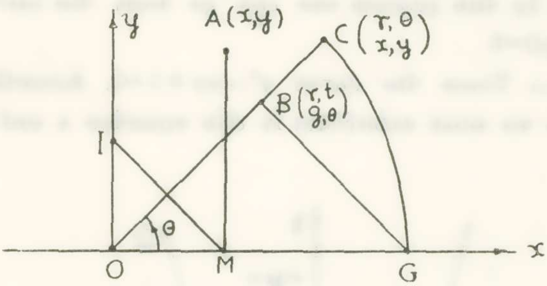

Fig. 3

Let be given the equation $f(x,y)=0$. Transforming this analytically into the (r,θ) system we get $f(r\cos\theta, r\sin\theta)=0$. The geometrically equivalent of the latter in the system (g,θ) is $f(g\cos\theta, g\sin\theta)=0$. The analytically equivalent of the latter in the system (r,t) is found if we put $g=\dfrac{r}{\cos\theta}$, $t=\dfrac{\sin\theta}{\cos\theta}$. Hence we get $f(r,rt)=0$. Finally transforming geometrically $f(r,rt)=0$ to the (x,y) system we get $f(y,yx)=0$. This process is symbolically expressed as follows:[6]

$$f(x,y)=0 \overset{a}{\sim} f(r\cos\theta, r\sin\theta)=0 \overset{g}{\sim} f(g\cos\theta, g\sin\theta)=0 \overset{a}{\sim} f(r,rt)=0 \overset{g}{\sim} f(y,yx)=0$$

[6] C. B. GLAVAS, «On Geometrical Equivalence and on a Certain Group of Plane Curves», *Proceedings of the Academy of Athens*, 35 (1960), p. 120.

An immediate conclusion is that the curve $f(x,y)=0$ is finally transformed to the curve $f(y,xy)=0$ by a series of transformations. One can go directly from the first equation to the last by putting y and yx in $f(x,y)=0$ for x and y respectively. The converse is applied for $f(y,xy)=0$. If $f(y,xy)=0$ is a known curve or very easy to trace, then the problem is reduced to transforming geometrically $f(y,xy)=0$ to $f(x,y)=0$.

Let be given the point $A(x,y)$ of the curve $f(y,xy)=0$ (Fig. 3). Its geometrically corresponding in the system (r,t) is found if we drop the perpendicular AM from A on Ox axis and join I to M. On the perpendicular from O on IM we take $OB=r=MA=y$. Hence we find the point $B(r,t)$ which has cathetic coordinates $OG=g$ and θ. To find the geometrically corresponding to B from the system (g,θ) to (r,θ) we take $OC=r=OG=g$. But the curve $f(r\cos\theta, r\sin\theta)=0$ is exactly the same with the original $f(x,y)=0$. Hence by this process one can go from the curve $f(y,yx)=0$ to the original $f(x,y)=0$.

Example 3.1. Trace the curve $y^2-xy+1=0$. According to the previous discussion we must substitute in this equation x and y for y and xy

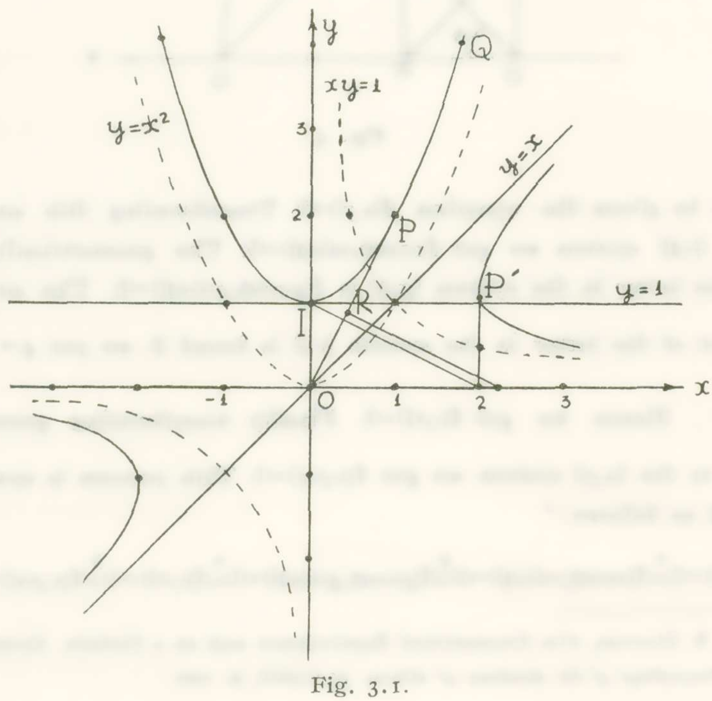

Fig. 3.1.

respectively. Thus we get the equation $x^2-y+1=0$, which represents a well known curve, i.e. a parabola with its vertex at I (0,1) and symmetrical with respect to Oy axis. For the geometrical transformation of our parabola to the required curve, it is enough to find first the corresponding branch of the curve to the branch IPQ of the parabola. It is easily seen, by following the reverse process to the one described in the previous section 3, that to the point I of the parabola there corresponds the infinite point of the curve on Ox axis.

Also the infinite point of the parabola corresponds to the infinite point of the curve. Now take the point P of the parabola and draw the line OP. On Ox axis we take a segment equal to OP and from the end of this segment we draw a perpendicular line on OP intersecting at R. From I we draw perpendicular on OP and from its intersection with Ox axis we raise a perpendicular on this axis. If we take on this perpendicular a segment equal to OR, then P′ is the point of the curve corresponding to the point P of the parabola.

We observe that the parabola $y=x^2$ is an asymptote line of the parabola $x^2-y+1=0$. But to the parabola $y=x^2$ there corresponds another asymptote line of our curve which we can find if we substitute in the latter equation y and xy for x and y respectively. Thus we get $xy=y^2$ or $y(x-y)=0$. This gives $y=0$ and $y=x$ as asymptote lines of our curve, which is true. Also to the tangent line $y=1$ at I (0,1) of $y=x^2+1$ there must correspond as tangent the line $xy=1$ at the point determined by $y=0$ and $xy=1$, i.e. at the infinite point of Ox axis. But $xy=1$ is the rectangular hyperbola being an asymptote to our curve along the Ox axis.

Example 3.2. Trace the curve $x^3+y=2x$. Putting in this equation y and xy for x and y respectively we take $y(y^2+x-2)=0$, which consists of two curves, $y=0$, and $y^2+x-2=0$. Both are well known and easy to construct. In order to transform geometrically these curves to the original one $x^3+y=2x$, we must apply the process from A through B to C (Fig. 3). It is immediately found that $y=0$ is transformed to the origin O (Fig. 3.2). The parabola $y^2+x-2=0$ has its vertex A (2,0) on Ox axis and intersects with Oy axis at the points $B(0,\sqrt{2})$ and $B'(0,-\sqrt{2})$. It is enough to transform only the branch ABC above Ox axis. This gives one branch of the curve under tracing, the other one being symmetrical to it with respect to the origin.

The point A is transformed to the origin O, while the point B to B_1. By the help of a few only points we see that the branch APB of our parabola is transformed to ORB_1 of our curve. Also BC goes to B_1C_1. The rest of the tracing is much easier for $OR'B_2$ and B_2C_2 are symmetrical to ORB_1 and B_1C_1 respectively. Note also that to the tangent $x=2$ or $xy=2y$

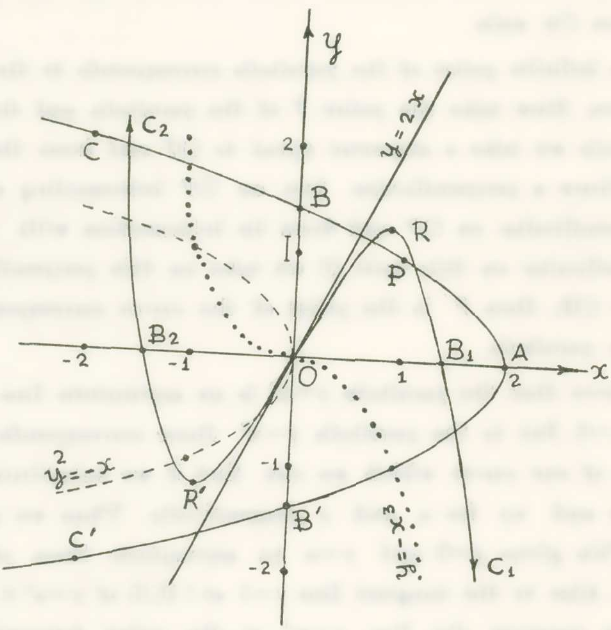

Fig. 3.2.

of the parabola there must correspond the tangent line $y=2x$ of the curve, which is easily checked to be correct. Finally to the asymptote parabola $y^2=-x$ to our parabola there must correspond the asymptote cubic $y=-x^3$ of our curve. This asymptote shows the direction of the infinite branch to our curve in the third and fourth quadrants.

Example 3.3. Trace the curve $y=x^2+x^3$. This final example will be a combination of the methods of sections 2 and 3. The given equation is written $\frac{y}{x}=x+x^2$. Then putting $\frac{y}{x}=t$ we transform analytically the originally given equation to $t=x+x^2$ in the (x,t) system of coordinates. But the geometrically equivalent of the latter in the system (x,y) is $y=x+x^2$.

Applying the transformation of section 3 we substitute xy and y for y and x respectively. Thus get $xy=y+y^2$ or $y(x-y-1)=0$. This relation consists of the two equations $y=0$ and $x-y-1=0$, both of which represent well known lines.

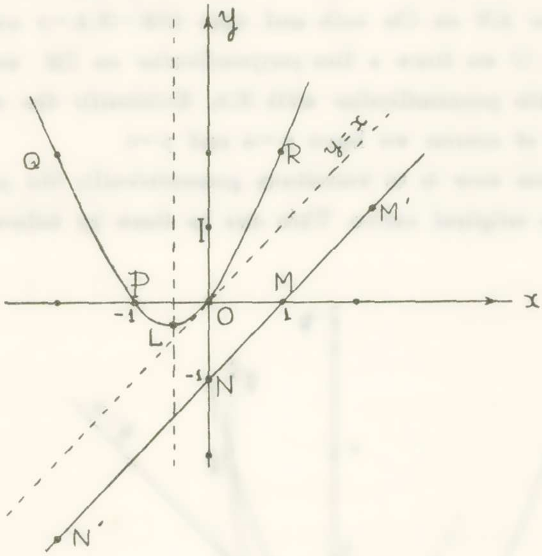

Fig. 3.3.a

We shall not go into details for tracing the original curve. It is enough to state that to trace the geometrically equivalent of $y=0$ and

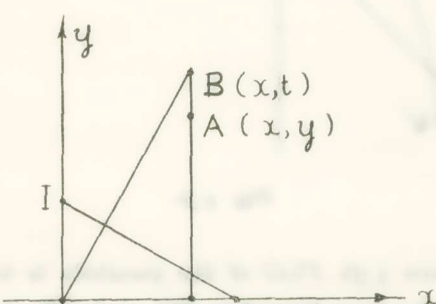

Fig. 3.3b.

$x-y-1=0$, which is the curve $y=x+x^2$, we must follow the process from A through B to C of Figure 3. The axis $y=0$ is transformed to O and the straight line MN of Figure 3.3a to the parabola $y=x+x^2$. MN is transformed to PLO while parts MM′, NN′ to the branches OR,PQ of the parabola respectively. The next step is to transform the parabola to the curve $y=x^2+x^3$, which is exactly the original curve. This means that we must be able to transform a point A(x,y) to B(x,t) (Fig. 3.3b). From A we drop the perpendicular AN on Ox axis and take OM=NA=y and we draw the line IM. If from O we draw a line perpendicular on IM we find B as the intersection of this perpendicular with NA. Evidently the coordinates of B are x and t and of course we have x=x and y=t.

The problem now is to transform geometrically the parabola QPOR (Fig. 3.3c) to the original curve. This can be done by following the process

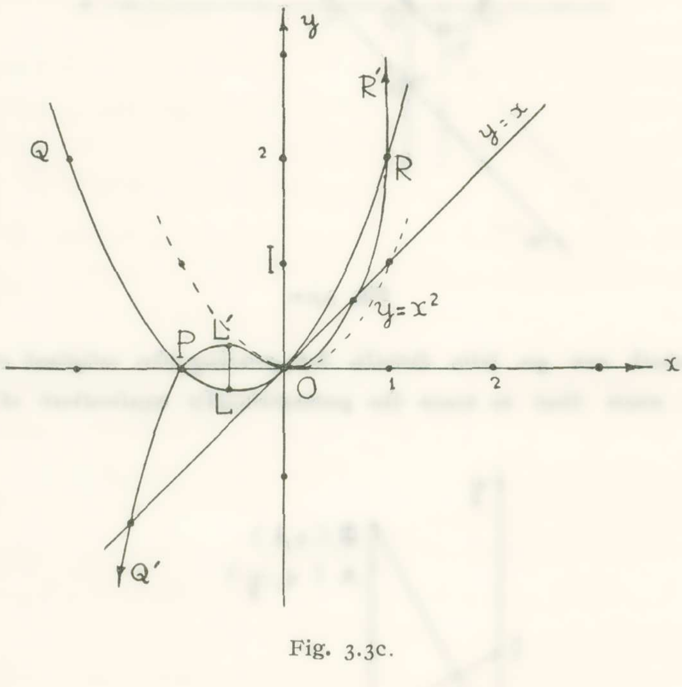

Fig. 3.3c.

from A to B of Figure 3.3b. PLO of the parabola is transformed to PL′O of our curve, while the two branches PQ and OR to the branches PQ′, OR′ respectively. It is not difficult to see that to the tangent line y=x to our parabola there corresponds the tangent parabola $y=x^2$ to our curve. This shows that our curve is tangent also at the origin O to the Ox axis. By this example we see that the problem of tracing the given curve is finally reduced to the one of tracing just a straight line.

Summary and *Remarks*. In this paper a new method is presented for curve tracing. The central aim of this method is to reduce a given curve to another which is known or easy to trace. This «model» curve constitutes the basis to trace the original curve by a finite number of geometrical steps. In conclusion the proposed method has the following characteristic advantages:

1. Using the model curve as a guide it only requires the transformation of a few basic points of that curve, giving therefore quick results·
2. It does not generally need the use of the Calculus or Analytic Geometry or the performance of calculations.
3. The properties of the model curve, properly translated, may contribute to the discovery of the corresponding ones to the original curve.
4. It presents many possibilities for further research by an ingenious combination of transformations through geometrically equivalent coordinate systems.

The realization of the last point 4, however, depends upon the original form of a given equation of an unknown curve as well as upon the possibility of using mainly geometrically equivalent coordinate systems to obtain a suitable model curve. These restrictions point to the need of investigation to increase the area of its application. Especially the discovery of other substitutions of the type of section 3 is very desirable.

ΠΕΡΙΛΗΨΙΣ

Ὡς γνωστόν, ἐὰν δύο ἐξισώσεις, ἐκπεφρασμέναι εἰς δύο διάφορα συστήματα συντεταγμένων, δύνανται νὰ μετατραποῦν ἡ μία εἰς τὴν ἄλλην βάσει τύπων μετασχηματισμοῦ μεταξὺ τῶν συστημάτων τούτων, τότε αἱ ἐξισώσεις αὗται παριστῶσι μίαν καὶ τὴν αὐτὴν καμπύλην, εἰς τὴν ὁποίαν ἀντιστοιχοῦν δύο διάφοροι ἀναλυτικαὶ σχέσεις. Ἐν τοιαύτῃ περιπτώσει αἱ ἐξισώσεις τῆς καμπύλης χαρακτηρίζονται ὡς ἀναλυτικῶς ἰσοδύναμοι.

Εἰς προηγουμένην ἀνακοίνωσιν διετυπώθη ἡ ἀρχὴ τῶν γεωμετρικῶς ἰσοδυνάμων καμπύλων (*Πρακτικὰ τῆς Ἀκαδημίας Ἀθηνῶν, 32 (1957), σ. 122-124)*. Κατ' αὐτὴν μία καὶ ἡ αὐτὴ ἀναλυτικὴ σχέσις, ἀναφερομένη εἰς δύο διάφορα συστήματα συντεταγμένων, παριστᾷ διαφόρους καμπύλας. Αὗται εἶναι γεωμετρικῶς ἰσοδύναμοι (ἢ μετατρέψιμοι), ἂν ὑπάρχῃ ἡ δυνατότης γεωμετρικοῦ μετασχηματισμοῦ τῆς μιᾶς εἰς τὴν ἄλλην. Ἔχει ἤδη δειχθῆ, ὅτι τὸ τελευταῖον ἐξαρτᾶται ἐκ τῆς ὑπάρξεως γεωμετρικῆς ἰσοδυναμίας μεταξὺ τῶν εἰς ἑκάστην περίπτωσιν χρησιμοποιουμένων συστημάτων συντεταγμένων.

Δύο συστήματα (α_1, β_1) καὶ (α_2, β_2) εἶναι γεωμετρικῶς ἰσοδύναμα, ἂν καθίσταται δυνατὴ διὰ πεπερασμένου ἀριθμοῦ Εὐκλειδείων γεωμετρικῶν κατασκευῶν ἡ μετάβασις ἐξ ἑνὸς σημείου ὁριζομένου ὑπὸ τοῦ ἑνὸς συστήματος εἰς ἄλλο ὁμοίως ὑπὸ τοῦ ἄλλου καὶ ἂν $\alpha_1=\beta_1$ καὶ $\alpha_2=\beta_2$.

Εἰς τὴν ἀνακοίνωσιν ταύτην γίνεται ἡ παρουσίασις μιᾶς νέας μεθόδου χαράξεως καμπύλων διὰ τῆς ἐφαρμογῆς κυρίως τῆς ἀρχῆς τῶν γεωμετρικῶς ἰσοδυνάμων καμπύλων. Ἂν ἡ ἐξίσωσις $\varphi(\alpha_1, \beta_1)=0$ μιᾶς πρὸς χάραξιν καμπύλης, ἐκπεφρασμένης εἰς τὸ σύστημα συντεταγμένων (α_1, β_1), γραφῇ $\varphi(\alpha_2, \beta_2)=0$ εἰς τὸ γεωμετρικῶς ἰσοδύναμον τοῦ πρώτου σύστημα (α_2, β_2), τότε ἡ τελευταία ἐξίσωσις παριστᾷ καμπύλην γεωμετρικῶς ἰσοδύναμον τῆς πρώτης. Ἂν ἡ καμπύλη αὕτη, δυναμένη νὰ κληθῇ καὶ πρότυπος ἢ ὁδηγὸς καμπύλη, τυγχάνῃ νὰ εἶνε γνωστὴ ἢ εὔκολος πρὸς χάραξιν, τότε ἐκ ταύτης καθίσταται δυνατὴ διὰ τῆς προτεινομένης μεθόδου ἡ χάραξις τῆς ἀρχικῆς ἀγνώστου καμπύλης διὰ πεπερασμένου ἀριθμοῦ γεωμετρικῶν κατασκευῶν.

Ἡ παρουσίασις τῆς νέας ταύτης μεθόδου γίνεται εἰς δύο περιπτώσεις μετὰ παραδειγμάτων. Εἰς τὴν πρώτην περίπτωσιν χρησιμοποιοῦνται δύο, ἐνῷ εἰς τὴν δευτέραν περισσότερα τῶν δύο συστήματα συντεταγμένων. Ἡ χάραξις τῆς ἀρχικῆς καμπύλης ἀπαιτεῖ ἐν γένει τὸν γεωμετρικὸν μετασχηματισμὸν ὀλίγων σημείων τῆς ὁδηγοῦ, πολλαὶ δὲ ἰδιότητες τῆς τελευταίας, ἥτις τυγχάνει γνωστὴ καμπύλη, καταλλήλως ἑρμηνευόμεναι, ὁδηγοῦν εἰς τὴν ἀνακάλυψιν καὶ διατύπωσιν ἀντιστοίχων ἰδιοτήτων τῆς ἀρχικῆς. Τὸ τελευταῖον στηρίζεται εἰς τὸ γεγονός, ὅτι «γεωμετρικῶς ἰσοδύναμοι καμπύλαι ἔχουν κοινὰς ἐκείνας τὰς ἰδιότητας, αἵτινες εἶνε τὸ προϊὸν τῶν ἰδίων ἀναλυτικῶν πράξεων ἐπὶ τῆς κοινῆς τούτων ἐξισώσεως» (βλ. *Πρακτικὰ τῆς Ἀκαδημίας Ἀθηνῶν, 32 (1957), σ. 126)*.

Τὰ πλεονεκτήματα τῆς προτεινομένης μεθόδου πρὸς χάραξιν καμπύλων εἶναι τὰ ἀκόλουθα: (1). Μὲ βάσιν τὴν ὁδηγὸν ἀπαιτεῖται ὁ μετασχηματισμὸς ἐλαχίστου ἀριθμοῦ σημείων ταύτης πρὸς χάραξιν τῆς ἀρχικῆς καμπύλης, ὡς προκύπτει ἐκ τῶν διδομένων παραδειγμάτων. Εἶναι ὅθεν ἡ μέθοδος ταχεῖα. (2). Δὲν παρίσταται ἀνάγκη χρησιμοποιήσεως τῆς Ἀναλυτικῆς Γεωμετρίας ἢ τοῦ Διαφορικοῦ Λογισμοῦ ἢ ἐκτελέσεως ἄλλων ὑπολογισμῶν, ὡς συμβαίνει εἰς ἄλλας μεθόδους χαράξεως καμπύλων. (3). Αἱ ἰδιότητες τῆς προτύπου καμπύλης ὁδηγοῦν εἰς τὴν ἀνακάλυψιν ἰδιοτήτων τῆς ἀρχικῆς τοιαύτης. Καὶ (4). Ἡ προτεινομένη μέθοδος, ὡς ἐκ τῆς φύσεώς της, παρουσιάζει πολλὰς δυνατότητας ἐπεκτάσεως εἰς εὐρύτερον πεδίον ἐφαρμογῶν.

Ἀκριβῶς λόγῳ τοῦ ἀνωτέρω τελευταίου χαρακτηριστικοῦ ἡ προτεινομένη μέθοδος περιέχει τὰ σπέρματα περαιτέρω ἐρεύνης διὰ τὴν ἀνακάλυψιν νέων συνδυασμῶν γεωμετρικῶς ἰσοδυνάμων συστημάτων συντεταγμένων διὰ νὰ χρησιμοποιηθοῦν ὡς ὄργανα ἀναγωγῆς καμπύλων εἰς ἁπλουστέρας μορφάς, μὴ δυναμένων νὰ χαραχθοῦν βάσει τῶν ὑπαρχουσῶν δυνατοτήτων.

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1. **L'idée et le mode de comparaison.**

La comparaison comme notion et méthode de raisonnement est très fréquente dans la vie, presque générale. On la rencontre très souvent à des conséquences et à des conclusions diverses déduites en comparant deux situations par leur similitude ou bien par leur contraste. En mathématiques tant plus, on rencontre la comparaison dans toute sa généralité et comme notion et comme méthode. Il suffit de mentionner par exemple que chaque relation mathématique regardée comme notion, n'est qu'une comparaison. Il est naturel qu'une classification concrète dans cet ensemble est possible pour déterminer de plus près le caractère ou l'espèce de la comparaison. Comme nous connaissons on compare les éléments par l'ordre, par la grandeur, par la position mutuelle etc. On peut trouver des exemples mathématiques où la comparaison n'est pas seulement la notion, mais aussi et la méthode de laquelle proviennent les conclusions.

On envisage un ensemble d'expressions, autrement un ensemble de formes (au sens mathématique). On suppose que parmi eux il existe au moins une forme qui possède une ou plusieurs propriétés. Elle sert alors comme la forme typique, le modèle, plus court comme le type. L'accommodement d'une élément quelconque de l'ensemble à ce type fait, que les propriétés apartenant au type se transportent immédiatement sur l'élément accomodé. Dans un ensemble de formes il est possible d'avoir plusieurs

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