ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

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ΜΑΘΗΜΑΤΙΚΑ.— **Tensor products of algebras,** *by George F. Nassopoulos**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ὀθ. Πυλαρινοῦ.

1. Introduction. The purpose of this note is to give a realization to coproducts in the category of noncommutative algebras, a fact which on the other hand implies its cocompleteness, and to extend certain well known results for the commutative case [1], [3] to the noncommutative one, by constructing an appropriate tensor product of algebras. In this way, it is pointed out that this new tensor product algebra behaves, in generally, in the same manner as the usual one in the special case that the ingredient algebras are commutative, in which case the said (tensor product) algebras actually coincide within an isomorphism.

The motivation to the present study was a realization of the usual tensor product of two algebras given in [6], the results reported in the following being obtained in an attempt to put previous results into a more general setting as an application of the technique developed in [6], [5] with respect to the universal construction, as well as the limit process. Thus, the structure of tensor algebras over R-modules (resp. of free algebras over sets) is studied and a canonical decomposition of these algebras is obtained.

A more detailed exposition and the proofs of the results presented herein will be given elsewhere [7], [8]. Further applications along the

* ΓΕΩΡΓΙΟΥ Φ. ΝΑΣΟΠΟΥΛΟΥ, **Τανυστικὰ γινόμενα ἀλγεβρῶν.**

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lines of this note to certain topological algebras will also be considered in [9].

2. Weakly multiplicative bilinear maps and tensor products. The category *Al* of algebras we are dealing with in the sequel is that of (linear) associative, unitary, not (necessarily) commutative ones over a fixed commutative ring R with an identity.

We start with the definition of a class of bilinear maps among R-algebras, which is of fundamental importance in what follows:

Definition 2.1. *Let E_1, E_2 and F be three R-algebras with identities e_1, e_2 and e respectively. Then, a map f from the Cartesian product $E_1 \times E_2$ into F is called a weakly multiplicative bilinear map (abbreviated to w.m.b. map) if the following two conditions are satisfied:*

i) The partial maps $f_\varkappa$ from $E_\varkappa$ into F, for $\varkappa = 1, 2$, sending every $x_1 \in E_1$ on $f(x_1, e_2)$ and every $x_2 \in E_2$ on $f(e_1, x_2)$ respectively, are morphisms of unitary algebras.

ii) The multiplication π on F factorizes f in the sence that the relation $f = \pi o (f_1 \times f_2)$ holds true.

One may view the w.m.b. maps of a fixed pair (E_1, E_2) of R-algebras together with the range algebra as the objects of a category of *pointed* algebras, denoted by $Al(E_1, E_2)$. Indeed, if (F, f) and (G, g) are two objects in $Al(E_1, E_2)$ then a morphism $h : (F, f) \to (G, g)$ is a homomorphism (purely algebraically) $h : F \to G$ such that furthermore, we have $g = hof$. This category gives rise to a universal problem, the resolution of which is given in the following:

Theorem 2.2. *(Existence Theorem). For any two R-algebras E_1 and E_2 there always exists an initial object in the respective category $Al(E_1, E_2)$.*

Clearly, an initial object is unique within an isomorphism. The following definition is therefore legitimate:

Definition 2.3. *An initial object in $Al(E_1, E_2)$ is called the tensor product of E_1 by E_2 and it is denoted by $(E_1 \otimes E_2, \tau)$. Moreover, the arguments $\tau_\varkappa$ of τ, for $\varkappa = 1, 2$, are called the canonical morphisms.*

We remark, however, that the R-algebra $E_1 \otimes E_2$ is constructed as the quotient of the free R-algebra $(LA(E_1 \times E_2), \varphi)$ based on the set $E_1 \times E_2$, modulo its 2-sided ideal $M(E_1, E_2)$ generated by certain expressions of elements $\varphi(x_1, x_2)$, for all $(x_1, x_2) \in E_1 \times E_2$, these expressions being subjected to the respective conditions to those of Definition 2.1, while the tensor map τ is defined as the composition of the canonical injection φ with the canonical epimorphism p of $LA(E_1 \times E_2)$ onto $E_1 \otimes E_2$ (cf. also [6]).

Now, let (E', τ') denote the *usual* tensor product of the R-algebras E_1 and E_2 that is, E' stands for the tensor product of the R-modules underlying to the given R-algebras equipped with the argumentwise multiplication, and τ' for the canonical bilinear map [1]. Then, the connection of these two tensor product algebras is given in the following:

Theorem 2.4. *There exists a unique epimorphism j of $E_1 \otimes E_2$ onto E' such that $\tau' = j \circ \tau$. Moreover, if the factor algebras involved are commutative, then j is an isomorphism.*

Thus, in contrast to the category *Alc* of commutative R-algebras, where there exists *just one* tensor product, in the category *Al* there exist *two different ones*, the usual tensor product and a second one, as the latter is defined in Definition 2.3. Henceforth, the term tensor product will refer to Definition 2.3, unless otherwise specified.

3. Coproducts and Colimits in the category Al. One of the more notable consequences of Definition 2.1 and Theorem 2.2 is the next Proposition, formulating the basic relation referring to morphisms between R-algebras, w.m.b. maps and the tensor product, which on the other hand is the extension of a well known property [1] to the noncommutative case.

Proposition 3.1. *Let E_1, E_2 and F be any three R-algebras and let $(E_1 \otimes E_2, \tau)$ be the tensor product of E_1 by E_2. Then we have*

$$\mathrm{Mor}(E_1, F) \times \mathrm{Mor}(E_2, F) = B(E_1 \times E_2, F) = \mathrm{Mor}(E_1 \otimes E_2, F)$$

within an isomorphism, where $B(E_1 \times E_2, F)$ stands for the set of the w.m.b. maps from $E_1 \times E_2$ into F.

In view of this result the triple $(E_1 \otimes E_2, \tau_1, \tau_2)$ is exactly the *coproduct* of E_1 and E_2 in *Al*. Our primary objective is now to give an extension of the above situation to any (not necessarily finite) family of R-algebras. For this purpose, we first observe that the tensor product defined is, as well as the usual one, *commutative and associative,* so that it can be extended to every finite family of R-algebras. Moreover, let $\{E_i \mid i \in I\}$ be an *infinite* family of R-algebras and denote by E_α the tensor product algebra corresponding to some finite subject α of I. Then the system $\{E_\alpha ; \tau_{\beta\alpha}\}$ is *inductive*, the *direct limit* of which being by definition the tensor product algebra of the family [**1**], [**5**].

We are now in a position to state the main result of this paper, which specializes to a well known theorem for the commutative case. [**3**]

Theorem 3.2. *Tensor products are the coproducts in the category Al of noncommutative R-algebras.*

The proof of this theorem makes use of the extension to every finite family of R-algebras of Proposition 3.1, as well as the property that the set of morphisms from a *direct limit* R-algebra E into any R-algebra F is isomorphic to the *inverse limit* of the respective sets of morphisms of the factors [**5**], [**10**] and related facts (cf. also [**4**], p. 208).

Next, since *Al* is a category with *difference cokernels* [**11**], the preceding theorem implies.

Theorem 3.3. *The category Al of noncommutative R-algebras is cocomplete.*

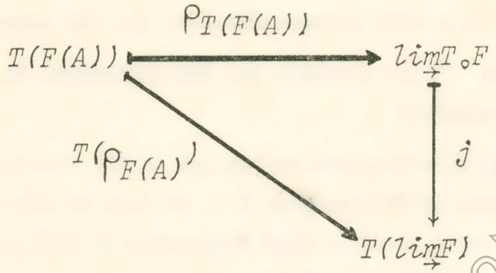
Corollary 3.4. *The category Rng of noncommutative unitary rings is cocomplete.*

According to the general construction of the *colimit* of a diagram over an abstract category [**11**], this one of a diagram *F* over *Al* is constructed as the quotient of the tensor product (: coproduct) algebra of the respective system of R-algebras, by one of its 2-sided ideal i.e., the ideal generated by the union of the images of the R-algebras of the diagram into the tensor product algebra under certain appropriate maps, the canonical «injections» being obtained as the obvious compositions.

4. Structure theorems and applications We wish now to state the next theorem regarding to the structure of the tensor R-algebra $T(U)$ on a given R-module U, in the case the latter is the colimit of a diagram *F* over the category *Mod* of R-modules, the proof of which rests upon the well known property that *left adjoint* functors are *cocontinuous* (: preserve colimits) [**10**]. We denote by *T* the left adjoint functor from *Mod* to *Al* assigning to every R-module U the tensor R-algebra T(U) on U [**4**].

Theorem 4.1. *Let* Σ *be a diagram scheme, let* $F : \Sigma \rightarrow Mod$ *be a dia-*

gram and let $(\varinjlim F, \varrho_F)$, $(\varinjlim T \circ F, \varrho_{T \circ F})$ *be the colimits of the corresponding diagrams in Mod and Al respectively. Then, there exists a unique isomorphism* j *from* $\varinjlim T \circ F$ *onto* $T(\varinjlim F)$ *such that the following diagram*



commutes for all $A \in Ob\Sigma$, *where* $T(\varrho_{F(A)})$ *stands for the tensorial extension of the respective canonical linear map.*

In particular, if the category Σ is *discrete* then one gets from the foregoing theorem a *canonical decomposition* of the tensor R-algebra on a *direct sum* R-module, as the tensor product algebra of the tensor algebras on the direct summand R-modules. On the other hand, in the special case that the R-modules under consideration are *free* on some sets, in which case the respective tensor R-algebras are also *free* on the same sets [2], then one derives analogous *canonical decomposition* of *free* R-algebras and recaptures a previous result in [6] for the commutative case and the usual tensor product.

We conclude this paper with the formulation of two applications of Theorem 4.1. Thus, we have:

1. Every tensor R-algebra T(U) on a R-module U is isomorphic to the inductive limit of its finitely generated admissible subalgebras.

2. Every mixed tensor algebra on a vector space V [2] has a *representation* as the tensor product algebra $T(V) \otimes T(V^*)$, where V^* stands for the dual vector space of V, a fact which on the other hand explains the factorization of any «geometric mixed tensor» to a covariant and a contravariant one.

ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν ἐργασίαν ταύτην ἐπιτυγχάνεται διὰ τῆς κατασκευῆς καταλλήλου τανυστικοῦ γινομένου ἀλγεβρῶν πραγματοποίησις καὶ εἰς τὴν κατηγορίαν τῶν μὴ (κατ' ἀνάγκην) μεταθετικῶν ἀλγεβρῶν τῶν συν-γινομένων, ἡ ὁποία πρὸς τοῖς ἄλλοις συνεπάγεται καὶ τὴν συμπληρότητα τῆς κατηγορίας ταύτης. Ἐπίσης ἐπιτυγχάνεται ἡ ἐπέκτασις τῆς ἰσχύος καὶ εἰς τὴν περίπτωσιν τῶν μὴ μεταθετικῶν ἀλγεβρῶν προτάσεων γνωστῶν ὡς ἰσχυουσῶν εἰς τὴν περίπτωσιν καθ' ἣν αἱ ἄλγεβραι εἶναι μεταθετικαί. Οὕτω διαπιστοῦται ὅτι τὸ νέον τανυστικὸν γινόμενον συμπεριφέρεται ἐν γένει ἐξ ἴσου καλῶς ὡς τὸ σύνηθες, εἰς τὸ ὁποῖον αἱ ἄλγεβραι - παράγοντες εἶναι μεταθετικαί. Πρὸς τούτοις διερευνᾶται ἡ μεταξὺ τοῦ νέου καὶ τοῦ συνήθους τανυστικοῦ γινομένου σχέσις καὶ ἀποδεικνύεται ὅτι αἱ δύο αὗται ἔννοιαι συμπίπτουν (ὡς πρὸς ἰσομορφισμὸν) εἰς τὴν περίπτωσιν, καθ' ἣν αἱ ἄλγεβραι - παράγοντες εἶναι μεταθετικαί.

Ἐν τέλει ὡς ἐφαρμογὴ μελετᾶται ἡ δομὴ τῶν τανυστικῶν ἀλγεβρῶν ἐπὶ δεδομένου προτύπου καὶ διατυποῦται θεώρημα παραστάσεως τῶν ἀλγεβρῶν τούτων.

REFERENCES

1. N. Bourbaki, Algèbre, Chap. III, 3rd ed., Herman. Paris, 1970.
2. L. Chambadal and J. Ovaert, Algèbre linéaire et algèbre tensorielle, Dunod. Paris, 1968.
3. S. Lang, Algebra, Addison und Wesley. London, 1967.
4. S. Maclane, Categories for the Working Mathematician, GTM 5, Springer-Verlag. Berlin, 1971.
5. A. Mallios, Inductive limits and tensor products of topological algebras, Math. Ann. 170 (1967), 214 - 220.
6. G. Nassopoulos, On the local behavior of A-algebras and their cotangent spaces, Ph. D. Thesis, Uuiversity of Athens. Athens, 1972 (Greek ; English abstract).
7. ——, On the existence of coproducts in the category of noncommutative algebras, Linear and Multil. Algebra (to appear).
8. ——, Concerning colimits in the category of noncommutative algebras, Bull. Soc. Math. Grèce (to appear).
9. ——, Colimits in certain categories of topological algebras (to appear).
10. B. Pareigis, Categories and Fuctors, PAM 39, Academic Press. New York, 1970.
11. H. Schubert, Categories, Springer-Verlag. Berlin, 1972.