ΜΑΘΗΜΑΤΙΚΑ.— **Products and lengths in halfgroupoids (third part). Remarks,** *by S. P. Zervos* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κ. Π. Παπαϊωάννου.

The present paper forms a continuation to our two preceding papers, published under the same title (resp. first and second part) in the present ΠΡΑΚΤΙΚΑ of the same year; so, it presupposes their terminology, notations and results. However, our last section 12 is independent of what precedes it, except the very first definitions.

X. APPLICATIONS OF THE PRECEDING RESULTS

Let $(A_0, \cdot) \underset{f,g}{\sqsubseteq} (A, \cdot)$. In the special case, where no equality of the form $b \cdot c = a_0$ holds in $(A_0, \cdot)$, they say that $(A, \cdot)$ is a *free* hgr, with *free basis* A_0. From what has already been said, it is obvious that A_0 is, then, the set of all elements of A strictly prime in $(A, \cdot)$.

The following lemma 1.5 is obtained in R. H. B. as a corollary of the (partially inexact, as we have already shown) lemma 1.4: «The following conditions are necessary and sufficient in order that a hgr $(A, \cdot)$ be free: (1) If $a \in A$ and if a is not prime in $(A, \cdot)$, then $a = b \cdot c$ in $(A, \cdot)$ for one and only one ordered pair $(b, c) \in A^2$. (2) Every divisor chain in $(A, \cdot)$ is finite. Moreover, if A is free, then A has one and only one free basis, namely, the set of all primes in $(A, \cdot)$». (R.H.B.'s term «prime» coïncides with our term «strictly prime».)

Happily, lemma 1.5 is valid, because it can be considered as a corollary of the last result of the second part of the present study, namely, of: $\{2), C.5), 9)\} \Longleftrightarrow (A_0, \cdot) \underset{f,g}{\sqsubseteq} (A, \cdot)$. To see it, call A_0 the set of all elements of A strictly prime in $(A, \cdot)$. Then, C.5) is satisfied trivially, (1) is 2)

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and (2) implies 9). And, A_o is characterized by its very definition. |

Theorem 1.4 and lemma 1.6 in R.H.B. are valid, because they are direct corollaries of lemma 1.5; also, his theorem 1.6. Let us, now, consider his theorem 1.7 and the proof he gives of this useful result.

«**Theorem 1.7.** Let the groupoid $(G,\cdot)$ be freely generated by a subhgr $(S,\cdot)$ and let $(H,\cdot)$ be a subgroupoid of $(G,\cdot)$. If P is the set of all primes (read, here: strictly primes) of H which are not in $S \cap H$ [the latter being either the empty set or the carrier of a subhgr of $(G,\cdot)$], then one of the following possibilities must occur: (ι) $P = \emptyset$, $S \cap H \neq \emptyset$ and $(S \cap H,\cdot) \underset{f,\,g}{\sqsubset} (H,\cdot)$. (ιι) $S \cap H = \emptyset$, $P \neq \emptyset$ and P is a free basis of $(H,\cdot)$. (ιιι) Neither P, nor $S \cap H$ is empty and $(H,\cdot) = (F,\cdot) * (K,\cdot)$ ($*$ notation for the *free product,* defined in R.H.B.), where $(F,\cdot)$ is a free subgroupoid of $(H,\cdot)$ with free basis P and $(K,\cdot)$ is a subgroupoid of $(H,\cdot)$, which is freely generated by $(S \cap H,\cdot)$.»

On closer examination of the proof of this theorem in R.H.B, one sees that our conditions 2), C.5) and 9) are satisfied by $(H,\cdot) \sqsubset (W,\cdot)$, so that, truly, $(H,\cdot) \underset{f,\,g}{\sqsubset} (W,\cdot)$. Hence, the validity of this theorem also is reestablished.

The «refinement theorem for free decomposition» in R.H.B. is a corollary of theorem 1.7 and, therefore, it also holds as well; etc.

Final remarks 1. The replacement of lemma 1.4 by $[(A_o,\cdot) \sqsubset (A,\cdot) \Longleftrightarrow \{2), C.5), 9)\}]$ made it possible to save *all* the results in [R.H.B. p. 1 - 8] depending on it, with, essentially, the same proofs (for which reason we ommitted them). 2. In the first part of this study, we insisted on the usefulness of the notion of prod (A_o, A). Many other examples of this usefulness could be presented. For instance, one can easily prove, by induction on the number of parentheses, that, *if* $(A,\cdot)$ *is a free groupoid with the free basis* A_o, *every element of* A *can be written in one and only one way as prod* (A_o, A). This gives, in particular, an immediate proof of the Free Representation Theorem (theorem 1.2 in R.H.B.).

XI. THE LENGTH OF AN ELEMENT OVER $(A_o, \cdot)$, AS A SPECIAL CASE OF THE GENERALIZED LIMINF

We have given, in two papers [3], [4], the following definition of liminf (analogous definition of limsup and, also, of infinf and supsup): Let E and P be nonempty sets, R a nonempty subset of E^2, $\leqslant$ a relation of partial order on P and g a mapping $E \to P$. For all $x \in E$, $E_x = \{x' \mid x' \in E \text{ and } (x', x) \in R\}$; $E^{(2)} = \{x \mid x \in E \text{ and } E_x \neq \emptyset\}$; since $R \neq \emptyset$, $E^{(2)} \neq \emptyset$. If, for some $x \in E^{(2)}$, $\inf\limits_{x' \in E_x} g(x')$ exists, we shall denote it by $\underline{\mu}_x$; if, for all $x \in E^{(2)}$, $\underline{\mu}_x$ exists and if, moreover, $\sup\limits_{x \in E^{(2)}} \underline{\mu}_x$ exists, this last will be called *liminfg, with respect to (E, R).* In the special case where R is a relation of order *directing* E, these definitions of liminfg and limsupg coïncide with those already used by Mc Shane [1]); our larger generalization had been inspired by P. Zervos's treatment of the classical ones [2]).

It was shown, in [3]) and [4]), that mathematical notions apparently so different as, for example, linear dimension, Lebesgue's topological dimension and Darboux's superior integral may, in a rather natural way, be put in the form of a liminfg. We, now, show that, in the case of hgr, also, $l_{A_o}(\gamma)$ may be written as a liminfg. Let E be the set of all ordered pairs $x = (S, \nu)$, where S is a divisor chain for γ, in $(A, \cdot)$, finite over $(A_o, \cdot)$ and ν is the order in S of a term γ_λ of S, such that, for all $\varrho \geqslant 0$, $\gamma_{\lambda+\varrho} \in A_o$; let R be the order relation on E, defined by $(S_1, \nu_1) \prec (S_2, \nu_2)$ iff $S_1 = S_2$ and $\nu_1 \leqslant \nu_2$ (this order does not, in general, direct E); let $P = \mathbb{N}_\infty$, with the natural order, and define g by $g(x) = \nu + 1$. Then, obviously, $l_{A_o}(\gamma) = liminfg$. ❙ (There must be no confusion between g_γ and g.)

Similarly, $l(\gamma)$ may be written as supsupg (with g properly defined).

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1. E. J. Mc Shane. Order-preserving maps and integration processes, Princeton, Annals of Mathematics Studies, 1953, p. 14 - 16.

2. P. Zervos. 'Απειροστικὸς Λογισμός, 'Αθῆναι, 1928, σελ. 38 - 60.

3. S. P. Zervos. Comptes Rendus de l'Acad. des Sc. de Paris, t. 261, 1965, p. 859.

4. S. P. Zervos. Séminaire Delange - Pisot - Poitou, 1965/66.

XII. A CONDITION FOR A HALFGROUPOID TO BE FREE OVER A SUBHALFGROUPOID OF IT

Let $(A_0, \cdot) \subseteq (A, \cdot)$. By definition, $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$, iff, for *every* groupoid $(K, \cdot)$ and *every* homomorphism $\varphi : (A_0, \cdot) \to (K, \cdot)$, there exists a homomorphism $\sigma : (A, \cdot) \to (K, \cdot)$, extending φ. On the other side, if $(A_0, \cdot) \underset{f, g}{\subseteq} (C, \cdot)$, $(C, \cdot)$ is easily defined (to an isomorphism), by $C = \text{Prod}$ (A_0, C) and relations $b . c = a$, only these in $(A_0, \cdot)$ and those obtained from them in prod A_0, C). Hence, the idea of using this «known» $(C, \cdot)$, for characterizing in a finite manner $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$. This is easily obtained by the following

Proposition 18. *Let* $(A_0, \cdot)$ *and* $(A, \cdot)$ *be hgr, with* $(A_0, \cdot) \subseteq (A, \cdot)$. *Then,* $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$ *iff there exists a homomorphism* $\sigma : (A, \cdot) \to (C, \cdot)$, *extending the identity* $A_0 \to A_0$. *Special case:* $(A_0, \cdot)$ *is a groupoid; then,* $(C, \cdot) = (A_0, \cdot)$.

Proof. 1) The «only if» part is obvious. 2) Let σ be a homomorphism, $(A, \cdot) \to (C, \cdot)$, extending the identity $A_0 \to A_0$; let $(B, \cdot)$ be a groupoid and φ a homomorphism $(A_0, \cdot) \to (B, \cdot)$. Then, $(A_0, \cdot) \underset{f}{\subseteq} (C, \cdot)$ implies that φ may be extended to a homomorphism $\bar{\varphi} : (C, \cdot) \to (B, \cdot)$. Then, $\bar{\varphi} \circ \sigma$ is a homomorphism $(A, \cdot) \to (B, \cdot)$, extending φ; hence, $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$. ∎

C o r o l l a r y. *A necessary condition for* $(A_0, \cdot) \underset{f}{\subseteq} (A, \cdot)$ *is that all equalities of the form* $a_0' . a_0'' = a_0'''$, *with* $(a_0', a_0'', a_0''') \in A_0^3$, *true in* $(A, \cdot)$, *are already true in* $(A_0, \cdot)$.

Proof. It suffices to consider the effect of σ on the equalities of this form. ∎

Π Ε Ρ Ι Λ Η Ψ Ι Σ

Διὰ τῆς παρούσης συμπληροῦται ἡ σειρὰ τῶν τριῶν ἀνακοινώσεών μας ἐπὶ τῶν δομῶν μιᾶς μερικῶς ὡρισμένης ἐσωτερικῆς πράξεως. Ἐπὶ πλέον τῆς συμπληρώσεως αὐτῆς, διαπιστοῦται ὅτι ἡ εἰς τὴν πρώτην τῶν ἀνακοινώσεων αὐτῶν εἰσαχθεῖσα ἔννοια μήκους στοιχείου ὑπεράνω τῆς $(A_0, \cdot)$ ἐντάσσεται εἰς τὸ γενικὸν πλαίσιον τῆς ἐννοίας τοῦ «μικροτέρου ὁρίου», τὴν ὁποίαν εἰσηγάγομεν εἰς παλαιοτέραν ἐργασίαν μας, τὴν ὁποίαν ἐνεπνεύσθημεν ἀπὸ τὴν σπουδὴν τῆς κλασσικῆς μορφῆς τοῦ θέματος τούτου εἰς τὸ σύγγραμμα «Ἀπειροστικὸς Λογισμὸς» τοῦ Παναγιώτου Ζερβοῦ. Τέλος, ἀποδεικνύομεν πρότασιν ὁδηγοῦσαν εἰς «πεπερασμένον» χαρακτηρισμὸν τῆς δομῆς, τῆς ἐλευθέρας ὑπεράνω ὑποδομῆς της.

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Ὁ Ἀκαδημαϊκὸς κ. **Κ. Π. Παπαϊωάννου** ἀνακοινῶν τὴν ὡς ἄνω ἐργασίαν εἶπε τὰ ἑξῆς:

Διὰ τῆς παρούσης ἀνακοινώσεώς του, εἰς τὴν ὁποίαν ἐκτίθενται νέα ἀποτελέσματα, συμπληρώνει καὶ τερματίζει ὁ καθηγητὴς κ. Σ. Π. Ζερβὸς τὴν σειρὰν τῶν τριῶν ἀνακοινώσεών του ἐπὶ τῶν δομῶν μιᾶς μερικῶς ὡρισμένης ἐσωτερικῆς πράξεως. Εἰς τὴν παροῦσαν ἀνακοίνωσιν, συνδυάζει τὴν ἔννοιαν τοῦ μήκους στοιχείου ὑπεράνω ὑποδομῆς πρὸς τὴν γενικὴν ἔννοιαν τοῦ μικροτέρου ὁρίου, τὴν ὁποίαν εἰσήγαγε πρὸ ἐτῶν. Τὴν γενίκευσιν αὐτήν, ὡς ἀναφέρει, εἶχεν ἐμπνευσθῆ ἐκ τῆς σπουδῆς τῆς κλασσικῆς μορφῆς τοῦ αὐτοῦ θέματος εἰς τὸ σύγγραμμα «Ἀπειροστικὸς Λογισμὸς» τοῦ ἀειμνήστου Παναγιώτου Ζερβοῦ.