PART A

1. INTRODUCTION

Many physical phenomena are due to the motion of energy, matter, electricity, etc. from one to other points of space with respect to matter distributed in the neighbourhood. This motion takes place on different scales in nature. The diffusion of the light across the intergalactic matter or in a Wilson chamber at the moment of the expansion; the transmission of accoustical waves through ionized gases or saturated vapors; the propagation of matter in the form of an aerosol; the diffusion of gamma quanta or neutrons through matter and others are examples of motions characterizing the transport phenomena. There are fundamental differences in the description of these transport phenomena deriving from the predominant physical mechanism according to which the transport takes place. We have for example:

(i) *free transfer* with no collisions neither between the particles or radiation quanta to be described nor between them and some other matter stationarily distributed in the same region of the space;

(ii) transfer with collisions between the particles or photons to be described and the non-moving matter distributed in the same region of the space and

(iii) transfer in which binary (or n-ary) collisions between the particles take place.

In cases (i) and (ii), since either there is no matter other than the free diffusing particles or it has a given constant distribution in space, the equation for the description of the transport phenomena is *linear* with respect to the distribution function.

In the case (iii) the transport equation is manifestly non-linear due to the collisions between the diffusing particles.

These differences are expressed in the transport equation by the form of *the scattering kernel*, entering the equation. In mathematical terms, if the kernel depends on the distribution of the particles to be described, then the phenomena are described by a non-linear equation.

It is a fundamental assumption of transport theory that all these phenomena associated with clouds of particles are described to a sufficient precision by the *Boltzmann equation* (1.1) in its general form

$$(\partial_t + \vec{\upsilon} \cdot \nabla + \vec{a} \cdot \nabla') \, \psi \, (\vec{x}, \vec{\upsilon}, t) = j \, (\psi, \psi_1). \tag{1.1}$$

In Eq. (1.1) $\psi(\vec{x}, \vec{\upsilon}, t)$ is the *distribution function* evaluated in the cell dx^3du^3dt around the 3-dimensional space point, $\vec{x}$, the 3-dimensional velocity space point $\vec{\upsilon}$, and the time instant t.

On the left side of Eq. (1.1) we have defined

$$\vec{\upsilon} \cdot \nabla = \sum_{i=1}^{3} \upsilon_i \, \partial_{x_i} \; ; \; \vec{a} \cdot \nabla' = \sum_{i=1}^{3} a_i \, \partial_{\upsilon_i} \tag{1.2}$$

where $a(\vec{x}, \vec{\upsilon}, t)$ is the force per unit mass acting on the particles at the point $(\vec{x}, \vec{\upsilon}, t)$ just defined.

On the right side we have the *collision integral* defined by the expression (1.3)

$$\begin{aligned} j\,(\psi, \psi_1) = \int [\, & \psi\,(\vec{x}, \vec{\upsilon}', t)\, \psi_1\,(\vec{x}, \vec{\upsilon}'_1, t) \\ & - \psi\,(\vec{x}, \vec{\upsilon}, t)\, \psi\,(\vec{x}, \vec{\upsilon}_1, t)\,] \,. \\ & W\,(\vec{\upsilon}, \vec{\upsilon}'; \vec{\upsilon}_1, \vec{\upsilon}'_1)\; d\upsilon'^3 d\upsilon_1^3 du'^3_1, \end{aligned} \tag{1.3}$$

where $W(\vec{\upsilon}, \vec{\upsilon}'; \vec{\upsilon}_1, \vec{\upsilon}_1')$ is the *scattering matrix* assuring conservation of energy and momentum during the collision process. The collision integral as defined in Eq. (1.3) takes account only of binary collisions. Thus the degree of non-linearity equals two. Eq. (1.1) was constructed phenomenologically by L. Boltzmann in the year 1872 and has since then been for long time the basis of the theory of gases. The validity of this equation was the subject of detailed discussions (1 - 6). It can in fact be derived from the Liouville theorem

$$[\partial_t + \sum_{n=1}^{N} (\vec{\upsilon} \cdot \nabla + \vec{a} \cdot \nabla')]\, \psi = 0 \; ; \; N = \text{number of particles.} \tag{1.4}$$

The conditions necessary for the derivation of Eq. (1.1) from Eq. (1.4) are physically not always easily realisable.

These conditions consist mainly in the explicit assumption that

— A function $F_2(z_1, z_2)$ is replaced by the product of two other functions $F_1(z_1) \cdot F_1'(z_2)$.

— Only binary collisions are considered.

— F_1, F_1' change slowly.

— A complete chaos characterizes the velocities of the particles.

The details of the derivation are given by Grad (ref. 3) and others. The knowledge of the distribution function allows us to calculate a series of quantities characterizing the properties of the system, e.g., diffusion coefficients, viscosity, specific heat, velocity distribution, mean velocity, pressure, temperature, etc.

There is to date no method for the construction of the general closed form solution to Eq. (1.1).

The most effective methods to find approximate distribution functions of sufficient precision are those of Chapman-Enskog (2) and of Grad (3). Exact particular distribution functions satisfying Eq. (1.1) exist, e.g. the Maxwell-Boltzmann distribution functions, known since longer than Eq. (1.1) and describing a gas of particles in a uniform state:

$$\psi = n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m\upsilon^2}{2kT}\right] \tag{1.5}$$

where k is the Boltzmann constant, T is the temperature and m is the mass of the particles. n_0 is a normalization constant. The non-uniform distribution function

$$\psi = n_0 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m}{2kT}\upsilon^2 + 2\omega\,(y\upsilon_1 - x\upsilon_2))\right] \tag{1.6}$$

where $\vec{\upsilon} = (\upsilon_1, \upsilon_2, \upsilon_3)$, $\vec{x} = (x, y, z)$ and ω is a constant. This distribution function is valid in the absence of forces (2) and describes a rotating gas. The non-uniform distribution functions

$$\psi_{\pm} = n_0 \exp\left[\pm \lambda\vec{\upsilon} \cdot \vec{a}\Lambda\vec{x}\right] \tag{1.7}$$

where λ is a constant describes also a rotational motion valid in the presence of Kaufmann force (7), $\vec{a}$, and satisfies also the Liouville equation rigorously.

All these functions are examples of distribution functions satisfying the nonlinear Boltzmann equation and may have applications in the linear transport theory. Thus, for example, Eq. (1.5) finds an important application in calculating the Doppler effect on the neutron resonance cross section.

The linearized form of the Boltzmann equation is obtained from Eq. (1.1) by convenient integrations under certain approximating assumptions in the collision term. In fact according to the definition of the scattering matrix the integral

$$\int \psi(\vec{x}, \vec{\upsilon}, t)\, \psi(\vec{x}, \vec{\upsilon}_1, t)\, W(\vec{\upsilon}, \vec{\upsilon}'\ ;\ \vec{\upsilon}_1, \vec{\upsilon}_1{}')\, d\acute{\upsilon}^3 d\upsilon_1^3 d\acute{\upsilon}_1^3 \tag{1.8}$$

gives the product of the distribution functions by the total cross section multiplied by the velocity, $\upsilon \cdot \sigma_t(\upsilon)$, where ψ_1 is supposed to be independent of $\vec{x}$ and representing particles with fixed distribution in x-space other than those to be described by the distribution function to be found. Proceeding in an analogous way for the term with positive sign in $j(\psi, \psi_1)$ we get the expression:

$$\int K(\vec{\upsilon}, \vec{\upsilon}_1)\, \psi(\vec{x}, \vec{\upsilon}_1, t)\, d\upsilon_1^3, \tag{1.9}$$

where $K(\vec{\upsilon}, \vec{\upsilon}_1)$ is the scattering kernel for collisions between particles distributed according to $\psi(\vec{x}, \vec{\upsilon}, t)$ and particles fixed in space.

Combining results of Eqs. (1.3), (1.8 - 1.9) we get the linear Boltzmann equation

$$\left[\partial_t + \vec{\upsilon} \cdot \nabla + \upsilon \cdot \sigma_t(\upsilon)\right] \psi(\vec{x}, \vec{\upsilon}, t) = \int K(\vec{\upsilon}, \vec{\upsilon}') \psi(\vec{x}, \vec{\upsilon}', t) d\upsilon'^3 . \qquad (1.10)$$

For details concerning the operations on $W(\vec{\upsilon}\,\vec{\upsilon}'; \vec{\upsilon}_1, \vec{\upsilon}_1')$ leading to Eq. (1.10) see, for example Waldmann (6) and Case and Zweifel (8). The properties and solutions of Eq. (1.10) occupy an enormous volume of the physical literature (9 - 20) developed mainly during the two last decades or so.

Historically, first the *approximation methods* were developed, e.g., the Fermi *age theory* (21) or the Placzek theory (22) in connection with neutron distributions. Today there exists an enormous number of approximation methods. We shall not review them but shall give a few recent references (23 - 37).

After a time of recognition of the real difficulties in obtaining precise practical distribution functions and the appearance of large capacity electronic computers the numerical methods gained considerable importance. These may be subdivided broadly into *statistical methods* (38 - 67) and direct attempts to solve the Boltzmann equation or some of its *approximations* (68 - 102). To the approximation methods belong also a set of solutions, the so-called *synthetical*, (103 - 105) in which the problem is simplified by considering only a part of its dimensions the rest being treated either numerically or with some kind of approximation and combining appropriately the final results. To the approximation methods belong, although not generally, also some *iterative methods* (106 - 118) of which some correspond to the resolvent expansion. Various varieties of the *variational method* have also been developed and adopted to the linear Boltzmann equation (119 - 127).

Perturbation methods (128 - 130) have also found some application but to some limited extent.

An important part of transport theory is based on the *invariant imbedding* procedure developed originally by Ambarzumian (131). This approach so important for the theoretical understanding of some transport aspects (132 - 136) has induced a rather extensive literature. Extensive application in transport theory find also using the *integral transforms* (137 - 159). The Laplace transformation is very useful in time-dependent problems, while Fourier transformations are usually applied with respect to the spatial variables in infinite media. Other types of integral transforms are not of general interest in transport theory.

The methods of *linear analysis* have found in recent years important applications in connection with the Boltzmann equation. They make possible the formal study of the spectral properties (160 - 179). An important question is the one regarding the problem of the existence of a smallest eigenvalue arising in time-dependent problems involving pulsed sources in systems of small spatial extent. Using linear analysis Albertoni and Montanini (180) were able to prove a theorem according to which there is no lower limit to the eigenvalue if the linear spatial extent of the system tends to zero (181 - 185).

It is not the purpose of the present section to dwell on all the above methods. The interested reader is referred for more extensive literature to the report of Lathrop and W.L. Hendry; K.D. Lathrop; S. Vantervoort; and J. Wooten; LASL, June 1970 (182p), (186).

In addition several of them make use of expansion of the distribution function in terms of certain systems of orthogonal functions (e.g. spherical harmonics) and, therefore, they have the limitations characterizing them in particular with respect to the satisfaction of boundary conditions. Instead, it is proposed to compare our method with the principal known methods and stress the differences and their complementarity. We shall restrict, therefore, the comparison of our method with those methods, which are of considerably general character and find a sufficiently wide application in the physics of the linear transport without being derivates of other more fundamental methods. The general problem of the linear transport theory consists in finding exact solutions to Eq. (1.10) satisfying given boundary and initial conditions in a space of given dimensions and in the three dimensional velocity space. Despite the apparent simplicity of Eq. (1.10) in view mainly of its linearity, there is still no unique method for obtaining the desired solutions of the general transport problem. Thus many different particular methods have been developed.

Methodically, the approximate methods were mainly of two different types. Either they neglected the variation of the distribution function in one or more dimensions or else they transformed Eq. (1.10) in a differential equation with respect to the relevant variables.

In this way the various diffusion approximations were obtained in the form of second-order-partial differential equations. These kinds of approximate equations shall not be considered here at all.

In what follows in this section we shall give a short account of a couple of established methods selected according the criteria just described. Thus we shall have to compare our results with those of

a) the spherical harmonics method,

b) the normal approach of Case,

c) the harmonic polynomials' approach of Birkhoff and Shumays.

a) Spherical Harmonics

We shall consider first the equations of the spherical harmonics approximation. This will be done for simplicity in a space of one dimension. In addition we shall consider here the time-independent situation, because the time can easily be integrated out by one dimensional Laplace transformation.

Upon writing for the distribution the series representation

$$\psi(x, z) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \psi_n(x) P_n(z) , \tag{1.11}$$

with $P_n(z)$ the Legendre polynomials, we get for the case of isotropic scattering the spherical harmonics equations in the form:

$$\partial_x \psi_1(x) + \frac{1}{2} \psi_o(x) = 0$$

$$\frac{2}{3} \partial_x \psi_2(x) + \frac{1}{3} \partial_x \psi_o(x) = 0$$

$$\frac{3}{5} \partial_x \psi_3(x) + \frac{2}{5} \partial_x \psi_1(x) = 0,$$

where
$$\psi_n(x) = \int d\Omega \psi(x,z) P_n(z) . \tag{1.12}$$

To demonstrate the difference between the spherical harmonics representation, Eq. (1.11) and the method to be developed in this paper let us write $\psi(x, z)$ with the help of the polynomials $S_n(x, z)$ to be given in this work (see Def. III, 1⁰, 5⁰):

$$\psi_+(x, z) = \sum_{n=o}^{\infty} q_n \left[S_n(x-a, z) - (-z)^n e^{-\frac{a-x}{z}} \right] \quad ; \; z \geq 0 \tag{1.13}$$

and

$$\psi_-(x, z) = \sum_{n=o}^{\infty} p_n S_n (x-b,z) \quad ; \quad z < 0 . \tag{1.14}$$

Inserting Eqs. (1.13 - 1.14) into the transport equation we get:

$$\sum_{n=o}^{\infty} q_n \frac{(x-a)^n}{n!} = \frac{\lambda}{2} \sum_{n=o}^{\infty} [V_n(x-a) q_n + W_n(x-b) p_n] \tag{1.15}$$

$$\sum_{n=o}^{\infty} p_n \frac{(x-b)^n}{n!} = \frac{\lambda}{2} \sum_{n=o}^{\infty} [V_n(x-a) q_n + W_n(x-b) p_n] , \tag{1.16}$$

where V_n and W_n are defined in Def. IV.

From Eqs. (1.11) and (1.13) we see that if we consider, for example, the boundary condition

$$\psi(a, z) = 0 \quad ; \quad z > 0 \tag{1.17}$$

(i) Eq. (1.11) does not satisfy the boundary condition, Eq. (1.17), for any finite value of n, while Eq. (1.13) satisfies this boundary condition termwise.

(ii) Eqs. (1.11) are coupled differential equations, while Eqs. (1.15) and (1.16) yield coupled algebraic equations, if we equate to zero coefficients of equal x-powers.

(iii) The secular equation of the system, Eqs. (1.15 - 1.16), yields the spectral equation for the determination of the eigenvalues λ.

These features observed in the case of isotropic scattering are strongly accentuated in the other more complicated cases of anisotropic scattering and stratified systems.

b) Normal Mode Approach

It was in 1960 that a fundamental paper by Case (187) appeared. This paper opened up in fact a real avenue leading to the discovery of unexpected properties of the Boltzmann equation in one space dimension. In subsequent papers Case and others (188 - 191) have succeeded in proving a number of important theorems in the cases of isotropic as well as anisotropic scattering.

This new approach developed so rapidly that only a few years later a book based on it could appear (8) collecting the most important developments. The main idea characteristic to this approach derives from the ansatz

$$\psi(x, z) = \varphi_\nu(z) e^{-\frac{x}{\nu}} \tag{1.18}$$

and is related to the use of a distribution theoretical property (192) implying that from $(\nu - z)\varphi_\nu(z) = \frac{c\nu}{2}$ it follows that

$$\varphi_\nu(z) = P \frac{c\nu}{2(\nu - z)} + \lambda(\nu)\delta(\nu - z) , \tag{1.19}$$

where $\lambda(\nu)$ is a conveniently chosen function and $\delta(x)$ is the Dirac delta distribution.

One then uses the fact that either $\nu \in [-1, 1]$ or $\nu \notin [-1, 1]$.

If $\nu \notin [-1, 1]$, the normalization of $\varphi_\nu(z)$ leads to the condition

$$\frac{c\nu}{2} \ln \left(\frac{\nu + 1}{\nu - 1}\right) = 1, \tag{1.20}$$

from which the two discrete eigenvalues, $\pm\nu_0$, follow as functions of the parameter c.

If $\nu\in[-1, 1]$, the normalization allows to determine the function $\lambda(\nu)$ as

$$\lambda(\nu) = 1 - \frac{c\nu}{2}\int_{-1}^{1} \frac{dz}{\nu - z} \quad . \tag{1.21}$$

By defining $\lambda(\nu)$ one obtains that for every $\nu\in[-1, 1]$ Eq. (1.21) is satisfied and consequently $[-1, 1]$ delimits the continuum spectrum of eigenvalues of the Boltzmann equation, provided the ansatz, Eq. (1.18), is adopted.

We wish now to indicate briefly how the singular integral equation arises to which reduces solving the Boltzmann equation. It has been proved that the two eigenvalues $\pm\nu_0$ and the segment $[-1, 1]$ define a complete set of eigenvalues and corresponding eigenfunctions, denoted by $\varphi_{\mp\nu_0}(z)$ and $\varphi_\nu(z)$. The completeness theorem assures the existence of the representation of a given boundary function $\psi(z)$:

$$\psi(z) = \alpha\varphi_{\mp\nu_0}(z) + \beta\varphi_{-\nu_0}(z) + \int_{-1}^{1} A(\nu)\varphi_\nu(z)d\nu. \tag{1.22}$$

The unknown quantities in Eq. (1.22) are α, β, and $A(\nu)$. This Eq. (1.22) is singular because the kernel $\varphi_\nu(z)$ is a singular function of ν.

The theory of Eq. (1.22) has been completely developed by Case and Zweifel (8) and others and there exist in fact no problems with respect to solving Eq. (1.22). The same may be said also for the anisotropic scattering case.

Let us next consider the energy dependent kernel. This case has been investigated first by Bednarz and Mika (193).

Their method introduces a new variable, υ, defined by $\upsilon = zl(E)$, where z is the cosine of the scattering angle and $l(E)$ is the energy dependent mean free path. By introducing a new integration set, $M(\upsilon)$, such that the energy E belongs to it, $E\in M(\upsilon)$ for $\upsilon\in[-1, 1]$ if $l(E) > |\upsilon|$, they write the Boltzmann equation in the form:

$$(\upsilon\partial_x + 1)\psi(x,\upsilon,E) = \int_{-1}^{1} d\acute{\upsilon} \int_{M(\upsilon)} dE' K(\upsilon,E,;\acute{\upsilon},E')\psi(x,\acute{\upsilon},E') \tag{1.23}$$

If one again uses the ansatz

$$\psi(x,\upsilon,E) = \varphi_\nu(\upsilon,E)e^{-\frac{x}{\nu}} \tag{1.24}$$

one gets after multiplication of Eq. (1.23) by K(υ,E;ύ,E) and integration over ύ and E′ the equation

$$H(\nu,\upsilon,E) + \nu \int_{-1}^{1} \frac{d\upsilon'}{\upsilon-\nu} \int_{M(\acute{\upsilon})} dE' K(\upsilon,F;\upsilon',E') H(\nu,\upsilon',E')$$

$$= \int_{M(\upsilon)} dE' K(\upsilon,E;\upsilon',E') \lambda(\nu,E') , \qquad (1.25)$$

$$H(\nu,\upsilon,E) = \int_{-1}^{1} d\acute{\upsilon} \int_{M(\upsilon)} dE' \varphi_\nu(\upsilon',E') K(\upsilon,E;\upsilon',E') \qquad (1.26)$$

and λ(ν,E) is a function related to H(ν,υ,E).

Now it is obvious that the determination of the distribution function $\varphi_\nu(\upsilon, E)$ according to

$$\varphi_\nu(\upsilon,E) = P \frac{\nu H(\nu,\upsilon,E)}{\nu-\upsilon} + \lambda(\nu,E)\delta(\nu-\upsilon) \qquad (1.27)$$

requires the knowledge of H(ν,υ,E). Consequently Eq. (1.25) is a singular integral equation for the determination of H(ν,υ,E). A comparison of Eq. (1.25) with Eq. (1.19) reveals the degree of generalisation of the first. In addition it is observed that the definition of H(ν,υ,E) is directly based on the assumption that the kernel K(υ,E;υ′,E′) is L^2. This assumption excludes, for example, the applicability of this method to the case of cold neutrons in condensed matter. Because in this case the kernel is in general not L^2 (163).

c) Birkhoff's Method

An important new development to the understanding of the properties of the linear Boltzmann equation appeared with a paper by Birkhoff and Shumays (195). In that paper it was shown explicitly that if specific assumptions about the integrability of the distribution function $\psi(\vec{x},\vec{\upsilon})$, were made, e.g.,

$$\int_0^\infty dx e^{-x} \int_\Omega d\Omega \left| \varphi(\vec{x}) \right| < \infty , \qquad (1.28)$$

and

$$\int_0 dx e^{-x} \int_\Omega \int_{\vec{x}/x} \left| \psi(\vec{x},\vec{\upsilon}) \right| d\left(\frac{\vec{x}}{x}\right) d\Omega < \infty , \qquad (1.29)$$

$$\text{where} \quad \varphi(\vec{x}) = \int_{\Omega} d\Omega \psi(\vec{x},\vec{\upsilon}), \tag{1.30}$$

then polynomial solutions could be obtained. More rigorously the fundamental theorem stated and proved by the above authors, was as follows: Any function harmonic on R^p and satisfying the integrability conditions given in Eqs. (1.29) - (1.30) is a solution to the integral equation

$$\varphi(\vec{x}) = \int_0^{\infty} dx e^{-x} \int_{\Omega} d\Omega \varphi(\vec{x}-x\vec{u}). \tag{1.31}$$

This equation is, of course, equivalent to the integro-differential Boltzmann equation with isotropic scattering kernel.

Explicit polynomial solutions, $\psi(\vec{x},\vec{\upsilon})$, were given for example in the isotropic scattering case and in three dimensional space in the form

$$\psi(x,\upsilon) = xyz - [xy\zeta + yz\eta] + 2[x\eta\zeta + y\zeta\xi + z\xi\eta] - 6\xi\eta\zeta; \tag{1.32}$$

with corresponding integral $\varphi(\vec{x}) = xyz$. In Eq. (1.32) we have defined $\vec{x} \equiv (x,y,z]$ and $\vec{\upsilon} \equiv [\xi,\eta,\zeta]$. It is obvious that $\vec{x}$ being a vector in the three dimensional space, $\vec{x} \in R^3$, $\psi(\vec{x},\vec{\upsilon})$, as given by Eq. (1.32), cannot satisfy boundary conditions of physical interest. Due to this fact Birkhoff continued his researches and shortly later published another paper (196) extending considerably the results of ref. (195).

The extension consists in the use of functions satisfying the Helmholtz equation instead of harmonic polynomials satisfying the Laplace equation.

The main theorem runs as follows:

Let $\varphi(\vec{x})$ be any solution of the Helmholtz equation $\nabla^2\varphi + \lambda\varphi = 0$ in $R^P (\lambda > -1)$ for which $e^{-r}\varphi(\vec{x})$ is integrable. Then φ satifies the transport equation

$$\varphi(\vec{x}) = \frac{c}{4\pi} \int_0^{\infty} dr e^{-\sigma r} \int d\Omega \varphi(\vec{x}-r\vec{\upsilon}) \tag{1.33}$$

with c related to λ by the equation:

$$\begin{aligned} c &= \sqrt{1+\lambda} \,/ F(1/2, p/2; \lambda/1+\lambda); \; (\lambda > 0) \\ &= 1/F(1/2, 1, 1, p/2; -\lambda); \quad (-1 < \lambda < 0), \end{aligned} \tag{1.34}$$

where F is the hypergeometric function. Birkhoff and Shumays proceed then in the formulation of the solution, $\psi(\vec{x})$ in the forms:

$$\psi(\vec{x}) = \sum_n a_n j_n(kr) S_n(\partial,\varphi) \quad ; \quad (\lambda = k^2 > 0) \tag{1.35}$$

and

$$\psi(\vec{x}) = \sum_n a_n i_n(kr) S_n(\theta,\varphi) \quad ; \quad (\lambda = -k^2 > -1), \tag{1.36}$$

where $S_n(\theta,\varphi)$ are special angular functions and j_n and i_n are appropriate Bessel functions (196).

From Eqs. (1.35) - (1.36) one can directly assess the extent, to which boundary conditions like Eq. (1.17) may be satisfied for continuously changing variables θ and φ.

As stated in the introduction of ref. 196 the above solutions may be used as approximations to actual neutron distributions.

A number of theorems for time-dependent distributions are proved, all concerning situations of an infinitely extended system, where the problem of boundary conditions is simple. The above examples (subsections a, b, c) of analytical methods, give a rough idea of the different stages of development of the linear transport theory until recently. The impression is clearly, that this field of the theoretic investigation is still an open one and that further analytical methods are highly desirable.

d. Introduction and Notation to the Present Method

The purpose of what follows is to present a new approach for the treatment of the linear Boltzmann equation in one space dimension. It puts in the foreground the existence and the utility of the polynomial solutions.

The practical importance of the new polynomials lies in that the particular combination of the independent variables induced by the structure of the transport equation reveals new properties and implies rapid convergence of the rigorous solution in the series representation and in physical systems of finite extension.

This approach based largely on the structural properties of the Boltzmann equation might well be termed the *structural approach.* The basic idea underlying the *structural approach* is that the functions representing the solution of a physical problem should be as much simpler as the structure of the equation governing it is taken into account in constructing the solution. As such properties are considered in connection with our problem here the following:

(i) Analytical structural properties

(ii) Spectral properties of the operators involved

(iii) Transformation properties of the equation.

To make this introductory discussion more specific the notation to be used throughout the present work is first given.

Definition I.

1° The values of x for which Eq. (1.32) is studied will lie in the subset, A, of R^1 such that

$$x \in [a, b] \equiv A; \quad 0 < a < b < \infty,$$
$$]a, b[\equiv A^{\cdot}$$

2° The angular variable z is such that

$$z \in [-1, 1] \equiv B,$$
$$z \in]0, 1] \equiv B_+,$$
$$z \in [-1, 0[= B_-,$$
$$z \in [-1, 1]^{\cdot} \equiv B^{\cdot} = B - \{0\}$$

3° The scaling transformation $z \to z' = p \cdot z$ will be frequently applied.

In this case the corresponding sets become:

$$A \to \bar{A} = [pa, pb],$$

$$A^{\cdot} \to \bar{A}^{\cdot} =]pa, pb[,$$

$$B \to \bar{B} = [-p, p],$$

$$B_+ \to \bar{B}_+ =]0, p],$$

$$B_- \to \bar{B}_- = [-p, 0[,$$

$$B^{\cdot} \to \bar{B}^{\cdot} = [-p, p]^{\cdot}.$$

We propose to investigate the properties of the equation

$$z\partial_x \psi(x, \upsilon, z) + \sigma_t(\upsilon)\psi(x, \upsilon, z) = \lambda \int_{\upsilon} d\upsilon' \int_{B} dz' K(\upsilon, \upsilon'; z, z')\psi(x, \upsilon', z') \tag{1.37}$$

and its solutions under various particular circumstances, where the kernel will in general be defined by

$$K(\upsilon, \upsilon'; z, z') = \sum_{l=1}^{L} g_e\,(u, z) h_e\,(\acute{\upsilon}, z').$$

Following the aforementioned fundamental aspects it was possible to derive results of very far reaching simplicity and to easily prove new important properties of the Boltzmann equation. As an example it may be mentioned that the solution of the Boltzmann equation is represented by a superposition of very simple polynomials (197) $S_n(x-a, z)$ and $S_n(x-b, z)$ as well as exponentials of the form $\exp\left(\frac{x-a}{z}\right)(-z)^n$ and $\exp\left(\frac{x-b}{z}\right)(-z)^n$ with coefficients determined from the solution of an algebraic system of equations, of which the determinant is the spectral equation for the determination of the eigenvalues λ.

The polynomials $S_n(\xi, z)$ have the form

$$S_n(\xi, z) = (-z)^n e_n\left(-\frac{\xi}{z}\right), \tag{1.38}$$

where $e_n(x)$ is the sum of the $n+1$ first terms of the expansion of $\exp(x)$: One of the remarkable properties of the polynomials $S_n(x-s, z)$; $(s = a, b)$ is that they are transformed to a simple z — independent term by the operator $z \cdot \partial_x + 1$, i.e. (198 - 199),

$$(z \cdot \partial_x + 1) S_n(x-\alpha, z) = \left(\frac{x-\alpha}{n!}\right)^n, \tag{1.39}$$

where $\alpha = a$ for $z \in B_+$ and $\alpha = b$ for $z \in B_-$.

This property gives rise to a series of consequences of which the algebraization of the approach seems to be the most important. Since the polynomials S_n depend on the differences $x-a$, $x-b$, it turns out that linear superposition of these polynomials together with the exponentials $(-z)^n \exp\left(-\frac{x-\alpha}{z}\right)$ exhibit translational invariance on the x-axis. These superpositions remain still invariant against translations even when inhomogeneous Dirichlet boundary conditions are applied provided the imposed boundary functions are translationally invariant.

Moreover, owing to the homogeneity

$$S_n(\lambda\xi, \lambda z) = \lambda^n S_n(\xi, z) \tag{1.40}$$

$$\text{and} \quad \exp\left(-\frac{\lambda\xi}{\lambda z}\right) = \exp\left(-\frac{\xi}{z}\right) \tag{1.41}$$

the Boltzmann equation is invariant against simultaneous scaling transformations of the type $P_x\colon x \to \bar{x} = \rho x$ and $P_z\colon z \to \bar{z} = \rho z$, a fact justifying the introduction of the tilded sets in Def. I.

If in the above scaling transformations (P_x and P_z)λ is equal to -1, they become the parity transformations; under the simultaneous parity transformations P_x, P_z, P_λ

the solutions of the Boltzmann equation satisfying homogeneous Dirichlet boundary conditions are invariant.

$$P_x P_z P_\lambda \psi(x, z) = \psi(x, z) \ . \tag{1.42}$$

These are some examples of structural properties. Concerning the continuity properties of the solutions $\psi(x, z)$ of the Boltzmann equation the fact should be pointed out that they are uniformly continuous everywhere on $A \times B^{\boldsymbol{\cdot}}$. This important property is shared by all derivatives with respect to both x and z of $\psi(x, z)$ as well as of its integral $\varphi(x)$. However, the uniform continuity of $\psi(x, z)$ disappears at the points $(x = a,\ z = 0)$ and $(x = b,\ z = 0)$. This circumstance forces us to introduce some regularization procedure. Mathematical and physical arguments suggest to use the prescriptions

$$\lim_{x \to \alpha} \ \lim_{z \to \pm o} \ F(\psi(x, z))\,; \ \alpha = \begin{cases} a\,; z \to +0 \\ b\,; z \to -0 \end{cases} \tag{1.43}$$

but not inversely, where $F(\psi)$ is any functional of the kinds occurring in this theory. The mathematical motivation for the prescription given in Eq. (1.43) is the lack of equivalence of the variables x and z with respect to the structure of Eq. (1.37). In addition, taking the limits in the inverse order, i.e., $\lim\limits_{o \to z} \lim\limits_{x \to \alpha} F(\psi)$ is almost meaningless, because in a certain sense it would be equivalent to requiring the solutions of the equation $z \cdot \partial_x \psi(\alpha, z) + \psi(a, z) = \varphi(\alpha)$, which does not make much sense. From the physical point of view requiring $\lim\limits_{z \to o} \lim\limits_{z \to \alpha}$ is equivalent to requiring the flux in the direction $z = o$, which in fact is neither observable (measure zero) nor is its knowledge desirable. On the other hand, it is meaningless to include the case $z = 0$ at $x = a, b$ in the boundary conditions, because the incoming flux in the direction $z = 0$ does not influence in any way the system, for which Eq. (1.37) is considered. On the other hand it is clear that the limits given by Eq. (1.43) have a perfect physical meaning and lead to well defined results. These reasons made it necessary to introduce and use the sets $B_\pm$.

The application of the prescriptions (1.43) makes the solutions $\psi(x, z)$, their integrals $\varphi(x)$ and all their derivatives regular everywhere on $A \times B^{\boldsymbol{\cdot}}$. The theorems demonstrated previously enable us to prove and to generalize the integrability assumption introduced recently (195 - 196) for the generation of polynomial solutions. These solutions being harmonic polynomials on the whole R^p can possibly be used to construct solutions satisfying prescribed boundary conditions.

The study of the spectral properties of the Boltzmann equation is greatly facilitated by the property of the streaming operator given by Eq. (1.39). It turns out that in all cases examined, here the point spectrum is embedded in an annulus of the λ–plane having finite diameters.

In Sec. 2 we give a number of general theorems valid in the case of isotropic kernel. In Sec. 3 the spectral properties and the kinds of occurring solutions in R^1

are investigated. The case of velocity-independent, anisotropic kernel is discussed in Sec. 4, and the conditions are stated under which polynomial solutions exist. Sec. 5 is an application of the theory to the case of step-wise varying physical properties of the system. It turns out in fact that each region of constant physical properties can be treated separately if the adjacent regions are represented by the appropriate boundary conditions. However, coupled equations arise for the determination of the constant coefficients. Sec. 6 is a generalization to the velocity-and angle-dependent kernel. The most remarkable result is that the general solution is represented by superpositions of constant-kernel solutions each of which has velocity-independent coefficients. It turns out that in the case of degenerate kernels the eigenvalues do not depend on the velocity.

Sec. 7 gives an introduction of the present approach to the many-dimensional case. In Sect. 8 the particular case of a constant kernel is briefly discussed. Sect. 9 gives an example of application in the case of symmetrical convex systems. Finally, in Sect. 10 the completeness aspect of the polynomials is discussed in conjunction with representing any solution of the transport equation in terms of the proposed polynomials.

PART B

2. GENERAL STRUCTURAL PROPERTIES — ISOTROPIC SCATTERING

In this section we shall give attention to the simplest case of the one-velocity stationary equation with isotropic scattering in a homogeneous system. Therefore, the Boltzmann equation takes the form

$$z\partial_x\psi(x, z) + \psi(x, z) = \lambda\int_{-1}^{1}\psi(x, z')dz' . \tag{2.1'}$$

It exhibits obviously besides translational invariance, $x \to x' = x + \alpha$, also form invariance against simultaneous scaling transformations $x \to \bar{x} = \rho x$, $z \to \bar{z} = \rho z$.

Remark I. If we introduce the tilded sets $\bar{A}, \bar{B}, \bar{B}_{\pm}, \bar{B}_{-}, \bar{B}^{\cdot}$ with $\rho = \lambda$ then equation (2.1') takes the form

$$\bar{z}\partial_{\bar{x}}\ \psi(\bar{x}, \bar{z}) + \psi(\bar{x}, \bar{z}) = \int_{\bar{B}}\psi(\bar{x}, \bar{z}')d\bar{z}' . \tag{2.1}$$

In this case we shall be speaking of the eigensets $\mathscr{E}_i$ on which the various operations with regard to $\bar{z}$ are defined. In particular one is led to consider integrations $\int_{\mathscr{E}_i} \psi_i(\bar{x}, \bar{z})d\bar{z}$, where $\psi_i(\bar{x}, \bar{z})$ is the i-th eigenfunction of Eq. (2.1). For simplicity the tildes will be dropped confusion being avoided through the absence of the factor λ in Eq. (2.1).

Remark II. The eigensets, $\mathscr{E}_i$, are restricted to the domain of existence of solutions of Eq. (2.1).

Definition II.

1° $$\varphi(x) = \int_{B}\psi(x, z)dz, \tag{2.2}$$

2° $$\varphi_{\pm}(x) = \int_{B_{\pm}}\psi(x, z)dz. \tag{2.3}$$

3° The boundary conditions at $x = a$ and $x = b$ will be defined by the functions

$\psi_+(z)$, $\psi_-(z)$ supposed to be finite and unique on B_+ and B_- respectively with all their derivatives.

4° The integrals

$$\chi_+(x) = \int_{B_+} \psi_\pm(z) e^{-\frac{x-a}{z}} dz, \tag{2.4}$$

$$\chi_-(x) = \int_{B_-} \psi_-(z) e^{-\frac{x-b}{z}} dz \tag{2.5}$$

exist for all x in A.

Theorem I. Let $\psi_\pm(x, z)$ be a solution of Eq. (2.1) satisfying the boundary conditions

$$\psi_+(a, z) = \psi_+(z); \ \forall(z \mid z\in B_+) \tag{2.6}$$

$$\text{and} \quad \psi_-(b, z) = \psi_-(z); \ (\forall z \mid z\in B_-) \tag{2.7}$$

Let further $\varphi(x)$ be bounded on A with $\inf\limits_{x\in A} \varphi(x) = L$ and $\sup\limits_{x\in A} \varphi(x) = U$. Then:

1° $\varphi(x)$ is differentiable $(\forall x \mid x\in V)$.

2° $\psi_\pm(x, z)$ is uniformly differentiable with respect to both variables x and z such that $x\in A$ and $z\in B^\cdot$.

3° The derivatives of $\psi_\pm(x, z)$ at $x = a, b$ and $z = 0$ become also finite and unique, if the conditions $\lim\limits_{x\to a}\lim\limits_{z\to+0} F(\psi_+(a, z))$ and $\lim\limits_{x\to b}\lim\limits_{z\to-0} F(\psi_-(x, z))$ are imposed, where $F(\psi)$ is any linear continuous functional of ψ.

Proof: It follows directly from the transport equation that

$$\psi_+(x, z) = \psi_+(z)\exp\left(-\frac{x-a}{z}\right) + \int_a^x \varphi(x')\exp\left(-\frac{x-x'}{x}\right)\frac{dx'}{x}, \tag{2.8}$$

$$\psi_-(x, z) = \psi_-(z)\exp\left(-\frac{x-b}{z}\right) - \int_x^b \varphi(x')\exp\left(-\frac{x-x'}{z}\right)\frac{dx'}{z}. \tag{2.9}$$

We first show that $\varphi(x)$ is continuous and differentiable $(\forall x \mid x \in A)$. From Eqs. (2.8) - (2.9) we get by addition and integration over B the expression

$$\varphi(x) = \int_{1/\lambda}^{\infty} \frac{dt}{t} \int_{A} \varphi(x') e^{-|x-x'|t} dx' + \int_{B_+} \psi_+(z) e^{-\frac{x-a}{z}} dz + \int_{B_-} \psi_-(z) e^{-\frac{z-d}{z}} dz. \quad (2.10)$$

The continuity behavior of $\varphi(x)$ will be studied on the assumption that $\psi_+(z)$ and $\psi_-(z)$ are such that their integrals in Eq. (2.10) are equal to the continuous and differentiable functions $\chi_+(x)$ and $\chi_-(x)$ respectively.

Let us suppose that $\varphi(x)$ has one discontinuity at $x = x_0$ with a saltus $C_0 = \varphi(x_0 + 0) - \varphi(x_0 - 0)$. On approaching x_0 from above we have (200) in Eq. (2.10)

$$\varphi(x_0 + \varepsilon) = \int_{1/\lambda}^{\infty} \frac{dt}{t} \left\{ \int_{a}^{x_0 - \varepsilon} \varphi(x') e^{-(x_0 - x' + \varepsilon)t} dx' + \int_{x_0 + \varepsilon}^{b} \varphi(x') e^{-(x' - x_0 - \varepsilon)t} dx' \right\}$$

$$+ \int_{1/\lambda}^{\infty} \frac{dt}{t} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \varphi(x') e^{-(x_0 + \varepsilon - x')t} dx' + \chi_+(x_0) + \chi_-(x_0); \ \varepsilon > 0. \quad (2.11)$$

The third term is

$$\int_{1/\lambda} \frac{dt}{t} \int_{x^0 - \varepsilon} \varphi(x') e^{-(x_0 + \varepsilon - x't} dx' = \lambda \left\{ M_+ \left[1 - E_2\left(\frac{\varepsilon}{\lambda}\right) \right] + M_- \left[E_2\left(\frac{\varepsilon}{\lambda}\right) - E_2\left(\frac{2\varepsilon}{\lambda}\right) \right] \right\}$$

$$\equiv C_0^+ \quad (2.12)$$

where $L \leqq M$, $M_+ \leqq U$,

$$E_2(x) = \int_{1}^{\infty} \frac{dt}{t} e^{-xt}, \quad (x \geqslant 0). \quad (2.13)$$

Therefore,

$$\varphi(x_0 + \varepsilon) = F_a(x_0) + F_b(x_0) + \chi_+(x_0) + \chi_-(x^0) + C_0^+, \quad (2.14)$$

where $F_a(x)$ and $F_b(x)$ are defined by the first two integrals in Eq. (2.11). On approaching x_0 from lower x-values we get similarly

$$\varphi(x_0-\varepsilon) = F_a(x_o) + F_b(x_o) + \chi_+(x_o) + \chi_-(x_o) + C_o^-, \tag{2.15}$$

where

$$C_0^- = \lambda\left\{M_-\left[E_2\left(\frac{\varepsilon}{\lambda}\right) - E_2\left(\frac{2\varepsilon}{\lambda}\right)\right] + M_+\left[1 - E_2\left(\frac{\varepsilon}{\lambda}\right)\right]\right\}.$$

From Eqs. (2.14) - (2.15) it follows that

$$\lim_{\varepsilon\to 0}[\varphi(x_0+\varepsilon) - \varphi(x_0-\varepsilon)] = 0$$

and the saltus is equal to zero contrary to the assumption.

Since x^0 was taken arbitrarily in A we conclude that $\varphi(x)$ cannot have discontinuities provided $\chi_+(x)$ and $\chi_-(x)$ are continuous. Now since $\varphi(x)$ is continuous and $\{\varphi(x')[\exp(x-x')t - \exp(x'-x)t]\}_{x'=x} = 0$, the interchange of the integrations $\int\frac{dt}{t}\int dx'$ and the differentiation $\frac{d}{dx}$ is allowed. Hence the right-hand side of Eq. (2.10) possesses a derivative for all x in A and therefore $\varphi(x)$ has a derivative at all x in A. This proves 1^o.

To prove 2^o we introduce the new variable $x' = x - z\xi$ and write Eqs. (2.8) - (2.9) in the form

$$\psi(x,z) = \psi_\pm(z)e^{-\frac{x-\alpha}{z}} + \int_0^{(x-\alpha)/z} \varphi(x-z\xi)e^{-\xi}\,d\xi\ ;\ \alpha = \begin{cases} a; z\in B_\pm \\ b; z\in B_- \end{cases}. \tag{2.16}$$

Next we observe that for a given positive number ε a number $\rho > 0$ can be found such that the inequality

$$|\varphi(x + r\cos\theta - (z - r\sin\theta)\xi) - \varphi(x - z\xi)| < \varepsilon$$

holds true for r such that $0 < r < \rho$ and every θ such that $0 \leqslant \theta \leqslant 2\pi$. Hence, the continuity of $\varphi(x)$ implies the uniform continuity of $\varphi(x - z\xi)$ and therefore the following formulas hold true:

$$\text{(i)}\ \partial_x\psi_+(x,z) = \frac{1}{z}[\varphi_+(x) - \psi_+(z)]e^{-\frac{x-a}{z}} + \int_0^{(x-a)/z} \varphi^{(1)}(x-z\xi)e^{-\xi}d\xi \tag{2.17}$$

$$\{(\forall z \mid z\in B_+)\Lambda(\forall x \mid x\in A)\},$$

$$\text{(ii)}\ \partial_x\psi_+(x,z) = \left[\frac{x-a}{z_2}(\psi_+(z) - \varphi(a) + \frac{d\psi_+(z)}{dz}\right]e^{-\frac{x-a}{z}} - \int^{(x-\mathrm{v}/z} \varphi^{(1)}$$

$$(x-z\xi)e^{-\xi}d\xi\ \{(Az \mid z\in B_+)\Lambda(Ax \mid x\in A)\}. \tag{2.18}$$

The expressions (i) and (ii) are finite and unique uniformly on $A \otimes B_+$.
For $z \in B_-$ we have similarly

$$\text{(iii)}\quad \partial_x \psi_-(x, z) = \frac{1}{z}\,[\varphi(b) - \psi_-(z)]e^{-\frac{x-b}{z}} - \int_0^{(x-b)/z} \varphi^{(1)}(x - z\xi)e^{-\xi}d\xi$$

$$\{(\forall z \mid z \in B_-) \wedge (\forall x \mid x \in A)\}, \tag{2.17a}$$

$$\text{(iv)}\quad \partial_z \psi_-(x, z) = \frac{1}{z_2}\left[\psi_-(z) - \varphi(b) + \frac{d\psi_-(z)}{dz}\right]e^{-\frac{x-b}{z}} - \int_0^{(x-b)/z} \varphi^{(1)}(x - z\xi)e^{-\xi}d\xi;$$

$$\{(\forall z \mid z \in B_-) \wedge (\forall x \mid x \in A)\}. \tag{2.18a}$$

The expressions (iii) and (iv) are finite and unique uniformly on $A \otimes B_-$.
Hence, the derivatives $\partial_x \psi_\pm(x, z)$ and $\partial_z \psi_\pm(x, z)$ exist uniformly on $A \otimes B$.
and this proves 2°.
Furthermore, by applying the operation $\lim\limits_{x \to a}\lim\limits_{z \to +o}$ on Eqs. (2.17a)-(2.18a) we get

$$\partial_x \psi_\pm(x, z)\,|_{z=o} = \varphi^{(1)}(x)\ ;\ (\forall x \mid x \in A) \tag{2.17b}$$

$$\partial_z \psi_\pm(x, z)\,|_{z=o} = -\varphi^{(1)}(x)\ ;\ (\forall x \mid x \in A) \tag{2.18b}$$

The expressions in Eqs. (2.17b) - (2.18b) are according to 1° finite and unique and this proves 3°.

Q.E.D.

Remark III. It follows immediately from Eqs. (2.8) - (2.10) that $\psi_\pm(x, z)$ and $\varphi(x)$ fall off exponentially for $x \to \infty$. Therefore, the integrability assumption (195 - 196) which is a particular case of Eq. (2.16) follows from the Boltzmann equation itself.

Corollary I. Eqs. (2.17a) imply that

$$\lim_{x \to 0} z\partial_x \psi(x, z) = 0\ ;\ (\forall x \mid x \in A^\cdot). \tag{2.19}$$

Corollary II. From Eq. (2.19) it follows that

$$\psi(x, 0) = \int_B dz\psi(x, z)$$

$$\text{or}\qquad \psi_+(x, 0) = \psi_-(x, 0) \tag{2.20}$$

Theorem II. The derivatives of any order of $\psi(x,z)$; $\{(\forall x \mid x \in A) \wedge (\forall z \mid z \in B)\}$ with respect to x are bounded and unique provided this is true for the derivatives of any order, $\varphi^{(n)}(x)$ of $\varphi(x)$; $(\forall x \mid x \in A)$.

Proof (by induction). $\partial_x^n \psi(x, z)$ is finite and unique for $n = n_0$, where n_0 is any positive integer. Then, $\partial_x^{n+1} \psi(x, z)$ is finite and unique. From Eq. (2.1) we have after multiplication by ∂_x^n

$$z\partial_x^{n+1}\psi(x, z) + \partial_x^n\psi(x, z) = \varphi^{(n)}(x) \; ; z \neq 0 \, . \tag{2.21}$$

By putting $n = n_0$ we have immediately the proof of the assertion for $n = n_0$, because $\partial_x^{n+1}\psi(x, z)$ is the difference of finite and unique quantities. For $n = 0$ Eq. (2.1) satisfies the induction assumption $(\forall z \mid z \in B.)$. The existence and uniqueness of $\partial_x^n\psi(x,z)$ for $z = 0$ follows directly from Eq. (2.20) and the definition given in Eq. (2.2).

Q.E.D.

Corollary I. The converse of Theorem II is also true, i.e.,

$$\{ \mid \partial_x^n\psi(x, z) \mid < \infty \} \rightarrow \{ \mid \partial_x^n\varphi(x) \mid < \infty \}.$$

It does not follow from Eq. (2.10) because the integrand of the first integral is not uniformly continuous in x and t, and the differentiation of the integrals gives rise to singularities at the boundaries $x = a, b$. The regularization of $\partial_x^n \psi(x, z)$ is obtained by Theorem I,3°; they are by construction finite.

Corollary II. From Eq. (2.1) and from Corollary I it follows that

$$\psi(x,z) = \sum_{\nu=0}^{n-1} (-z)^\nu \varphi^{(\nu)}(x) + (-z)^n \partial_x^n y(x, z), \; \{(\forall x \mid x \in A) \wedge (\forall z \mid z \in B)\} \, . \tag{2.22}$$

Corollary III. Theorem I and Theorem II, Corollary I imply that

$$\partial_z^n\psi(x, z) \mid_{z=0} = (-)^n n! \; \partial_x^n\psi(x, z) \mid_{z=0} . \tag{2.23}$$

Corollary IV. We consider the identity (201):

$$\partial_t \left[\sum_{\nu=2}^{n-1} \frac{z^\nu}{\nu!} (1-t)^\nu \, \partial_t^\nu \psi(x, s) \right] = z^n \frac{(1-t)^{n-1}}{(n-1)!} \partial_t^n\psi(x,s) - z\partial_t\psi(x,s) \, , \tag{2.24}$$

where $s = zt$ and $t \in [0, 1]$. By integrating Eq. (2.24) over t and omitting vanishing sums we get

$$\psi(x, z) = \sum_{\nu=0}^{n-1} \frac{z^\nu}{\nu!} \partial_z^\nu \psi(x, z) \mid_{z=0} + \frac{z^n}{(n-1)!} \int_0^1 (1-t)^{n-1} \partial_t^\nu \psi(x,s) dt \ . \tag{2.25}$$

From Corollary III we are allowed to equate the coefficients of equal powers of z in Eqs. (2.22) and (2.25).

By equating the last terms of Eqs. (2.22) and (2.24) we get the relation

$$\partial_x^n \psi(x, z) = \frac{(-)^n}{(n-1)!} \int_0^1 (1-t)^{n-1} \partial_s \psi(x, s) dt, \tag{2.26}$$

$n = 0, 1, 2, \ldots\ldots$

Multiplication of Eq. (2.1) by ∂_z^n and appropriate iteration lead to the very important relation

$$z\partial_x \partial_z^n \psi(x, z) + n\partial_x \partial_z^{n-1} \psi(x, z) = -\partial_z^n \psi(x, z). \tag{2.27}$$

Repeated application of Eq. (2.27) leads to

Corollary IV.

$$\psi(x, 0) = \sum_{\nu=0}^{n-1} \frac{(-z)^\nu}{\nu} \partial_z^\nu \psi(x, z) + (-)^{n-1} \frac{z^n}{(n-1)!} \partial_x \partial_z^{n-1} \psi(x,z) \tag{2.28}$$

This is the expansion of $\psi(x,z)$ in a Taylor series ($n \to \infty$) at $h = -z$, and it results from Eq. (2.27).

Theorem III. Let $\psi(x,z)$ and $\varphi(x)$ possess maxima at $x = x_1$ for $z \neq 0$ and at $x = x_2$ respectively. Then

1°. $x_1 \neq x_2$

2°. $\mathrm{sign}\{\partial_x \varphi(x) \mid_{x=x_1}\} = -\mathrm{sign}(z_1)$.

Remark IV. The value x_1, at which the maximum of $\psi(x,z)$ occurs, is in general a function of z. Let call it z_1, and therefore $x_1 = x_1(z_1)$.

Proof: From $z\partial_x \psi(x,z) + \psi(x,z) = \varphi(x)$ we have that $x = x_1(z_1)$

$$\psi(x_1(z_1), z_1) = \varphi(x_1(z_1)) \ . \tag{2.29}$$

On the other hand there holds

$$z_1 \partial_{x1}^2 \psi(x_1, z_1) = \varphi^{(1)}(x_1)\,. \tag{2.30}$$

Since $\psi(x_1,z_1)$ is a maximum, $\varphi^{(1)}(x_1) \neq 0$ for $z_1 \neq 0$ and this proves 1°. Furthermore, from the maximum assumption it follows that $\partial_{x1}^2\psi(x_1,z_1) < 0$ and we therefore have the validity of 2°.

Corollary I. At the particular point $z_1 = 0$ we have

$$\varphi^{(1)}(x) = 0, \qquad x = x_1(0), \tag{2.31}$$

i.e., the maximum of $\psi(x,0)$ and of $\varphi(x)$ coincide.

Theorem IV. Let $\psi(x, z)$ satisfy Eq. (2.1) $\{(\forall x \mid x \in A) \wedge (\forall z \mid z \in B)\}$ and let B_0 be the neighbourhood of $z = 0$. If $\partial_x^n\psi(x, z)$ $(n = 0, 1, 2, \ldots)$ is decreasing for increasing $x \in A$ and fixed $z \in B_0$, then $\partial_x^n\psi(x, z)$ is increasing for increasing $z \in B_0$ and fixed $x \in A$, and vice-versa.

Proof: From Eqs. (2.1) — (2.2) and (2.20) it follows that

$$\partial_x^n\psi(x, z) - \partial_x^n\psi(x, z) = -z\partial_x^n\psi(x, z) \tag{2.32}$$

and therefore $z^{-1}[\partial_x^n\psi(x, z) - \partial_x^n\psi(x, 0)] = -\partial_x^{n+1}\psi(x, z)$. The limit for $z \to 0$ defines the derivative $\partial_x^n(x, z) \mid z = 0$, i.e.,

$$\lim_{z\to 0}\left[\frac{\partial_x^n\psi(x, z) - \partial_x^n\psi(x, 0)}{z}\right] = -\partial_x^{n+1}\psi(x, z) \mid_{z=0}\,. \tag{2.33}$$

Hence,

$$\pm\partial_z(\partial_x^n\psi(x, z)) \mid_{z=0} = \mp\partial_x(\partial_x^n\psi(x, z)) \mid_{z=0} \tag{2.34}$$

for $n = 0, 1, 2, \ldots\ldots$ Q.E.D.

For certain classes of function $\psi_+(x, z)$ and $\psi_-(x, z)$ the conditions for convergence of the series in Eq. (2.22) are stated in

Theorem V. Let $\psi_+(x, z)$ be a solution of Eq. (2.1) satisfying the boundary condition $\psi_+(a, z) = \psi_+(z)$, where $\psi_+(z)$ is a polynomial of degree m. Let further $\psi_+(x, z)$ possess of Taylor series expansion. Then, $\psi_+(x, z)$ is either a (double) polynomial in z and x and satisfies the relations:

$$\deg_z\psi_+(x, z) = \deg_x\psi_+(x, z) = \deg_z\psi_+(z), \ \{(\forall x \mid x \in A) \wedge (\forall z \mid z \in B_+)\}$$

or its derivatives of order higher than m vanish at $x = a$.

Proof: From Eq. (2.23) and by the hypothesis of existence of a power series expansion we have by Cauchy's inequality

$$\lim_{n\to\infty} \left|\frac{z^n}{n!}\, \partial_x^n \psi_\pm(x, z)\right| \leqslant \lim_{n\to\infty} \frac{M}{r^n} = 0, \tag{2.35}$$

where r is the convergence radius of the Taylor series representation of $\psi_+(z, x)$ and $M = \sup_{x\in A} \sup_{z\in B+} \psi(x, z)$.

By the uniqueness theorem for Taylor series the expansion in question is that given by Theorem II, Corollary II for $n = \infty$, i.e.,

$$\psi_+(x, z) = \sum_{v=0}^{\infty} (-)^v z^v \varphi^{(v)}(x), \{(\forall z \in B_+) \wedge (\forall x \mid x \in A)\}. \tag{2.36}$$

This expansion must satisfy the boundary condition at $x = a$.
By assumption, $\psi_+(z)$ is given by

$$\psi_+(z) = q_0 + q_1 z + \dots\dots\dots\dots q_m z^n . \tag{2.37}$$

From the boundary condition it follows that

$$\varphi^{(v)}(a) = 0 \ (\forall v \mid v > m) . \tag{2.38}$$

If Eq. (2.38) is satisfied identically, then $\varphi^{(v)}(x) = \partial_x^v \psi(x, z) \mid_{z=0}$

$$\partial_x^v \psi_+(x, z) \mid_{z=0} = 0, \{(\forall x \mid x \in A) \wedge (v > m)\} . \tag{2.39}$$

If Eq. (2.38) does not hold identically, then

$$\partial_x^v \psi_\pm(x, z) \mid_{x=\alpha} = 0, \{(\forall z \mid z \in B_\pm) \wedge (v > m)\} .$$

Therefore, there are certainly polynomial solutions of Eq. (2.1) or entire functions satisfying the polynomial boundary condition of degree m and having either the property expressed by Eq. (2.39) or

$$\partial_x^v \psi_+(x, z) \mid_{z=a} = 0, \{(\forall z \mid z \in B_+) \wedge (v > m)\}, \tag{2.40}$$

$\partial_x^v \psi_+(x, z) \neq 0, \{(\forall x \mid a \neq x \in A) \wedge (\mathrm{A} z \mid z \in B_+) \wedge (v > m)\}.$ Q.E.D.

3. EIGENFUNCTIONS AND EIGENVALUES

The lack of uniform continuity of $\psi(x, z)$ forced us to use the prescription introduced in Theorem I,3°. This prescription, which is physically fully justified will be now applied to exclude any ambiguity of the solution appearing at the point $(x = a,\ z = o)$ or $(x = b,\ z = o)$. One of the most important observations is that the operator $z\partial x + 1$ acting on functions $\{\psi(x, z)\}$ of two variables transforms them to the functions $\{\varphi(x)\}$ depending only on one variable. This property is valid only on a particular part of the spectral plane. This part of the spectrum and the corresponding eigenfunctions will be determined presently. First is given

Definition III.

1° The polynomials given below satisfy inhomogeneous Dirichlet boundary conditions (198 - 199):

$$\psi^1_{n+}(x, z) = \sum_{\nu=o}^{\infty} \lambda^\nu q_\nu S_n(x - a, z)\,;\ \{(\forall x \mid x \in A) \wedge (\forall z \mid z \in B_+)\} \tag{3.1}$$

and

$$\psi^1_{n-}(x, z) = \sum_{\nu=o}^{n} \lambda^\nu \varphi_\nu S_\nu(x - b, z)\,;\ \{\forall x, z \mid x \in A \mid z \in B_-\} \tag{3.2}$$

i.e.,

$$\psi^1_{n+}(a, z) = \sum_{\nu=o}^{n} \lambda^\nu q_\nu (-z)^\nu \tag{3.3}$$

and

$$\psi^1_{n-}(b, z) = \sum_{\nu=o}^{n} \lambda^\nu p_\nu (-z)^\nu\,. \tag{3.4}$$

2°. Homogeneous Dirichlet boundary conditions satisfy the functions

$$\psi^2_-(x, z) = \sum_{n=o}^{\infty} \lambda^n q_n \left[S_n(x-a, z) - (-z)^n e^{-\frac{z-a}{z}}\right]\,;\ \{\forall x, z \mid x \in A,\ z \in B+\} \tag{3.5}$$

and

$$\psi^2_-(x, z) = \sum_{n=o}^{\infty} \lambda^n p_n \left[S_n(x-b, z) - (-z)^n e^{-\frac{x-b}{z}}\right]\,;\ \{\forall x, z \mid x \in A,\ z \in B-\} \tag{3.6}$$

i.e.,

$$\psi^2_+(a, z) = 0;\ (\forall z \mid z \in B_+) \tag{3.7}$$

and

$$\psi^2_-(b, z) = 0;\ (\forall z \mid z \in B_-) \tag{3.8}$$

3°. $$\psi^1_n(x, z) = \psi^1_{n+}(x, z) + \psi^1_{n-}(x, z) \tag{3.9}$$

4°. $$\psi^2_n(x, z) = \psi^2_{n-}(x, z) + \psi^2_-(x, z). \tag{3.10}$$

Eqs. (3.1) — (3.8) follow from the properties of the polynomials S_n.

$$S_n(0, z) = (-z)^n$$

$$S_n(x, 0) = \frac{x^n}{n!}$$

$$\partial_x S_n(x, z) = S_{n-1}(x, z)$$

$$S_n(\lambda x, \lambda z) = \lambda_n S^n(x, z) \, .$$

5°. A few examples of the polynomials are given:

$$S_0(x, z) - 1$$

$$S_1(x, z) = -z + x$$

$$S_2(x, z) = z^2 - zx + \frac{x^2}{2!}$$

$$S_3(x, z) = -z^3 + z^2x - \frac{zx^2}{2!} + \frac{x^3}{3!}$$

$$S_4(x, z) = z^4 - z^3x + \frac{z^2x^2}{2!} - \frac{zx^3}{3!} + \frac{x^4}{4!} \, .$$

Definitions IV.

1° $$V_{kn}(x) = \partial_x^k \int_{B+} S_n(x, z) \, dz, \tag{3.11}$$

2° $$W_{kn}(x) = \partial_x^k \int_{B_-} S_n(x, z) dz, \tag{3.12}$$

3°. $$\beta_{kn}(x) = \partial_x^k \left[V_n(x) - E_{n+2}(x) \right] , \tag{3.13}$$

4°. $$\gamma_{kn}(x) = \partial_x^k \left[W_n(x) - E_{n+2}(x) \right] , \tag{3.14}$$

where

$$E_n(x) = \int_1^\infty \frac{dt}{t} e^{-xt} \, . \tag{3.15}$$

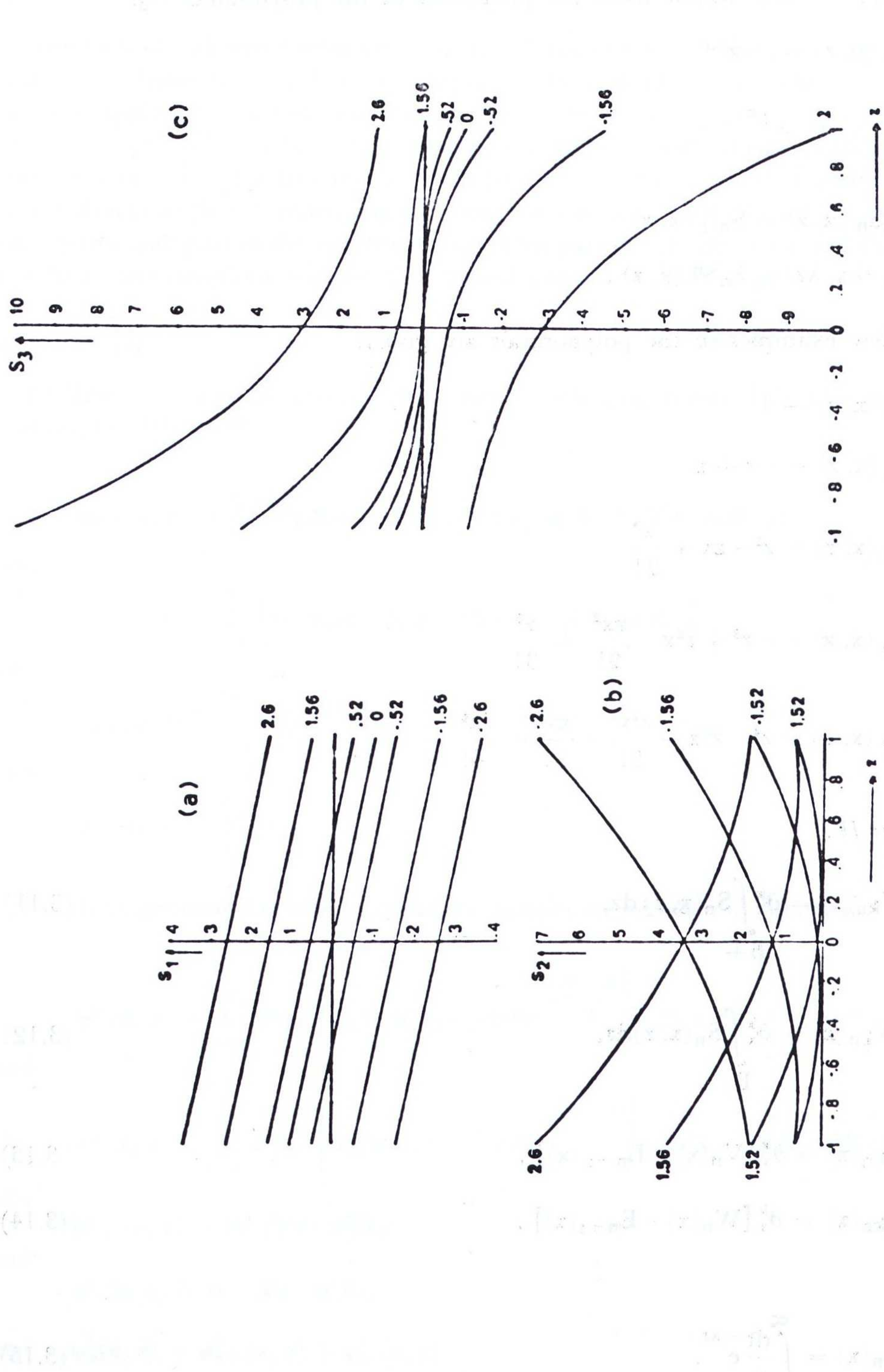


Fig. 1a,b,c. The polynomials S_1, S_2, S_3 of the independent variables x and z=cosθ are represented as functions of z for the indicated values of x.

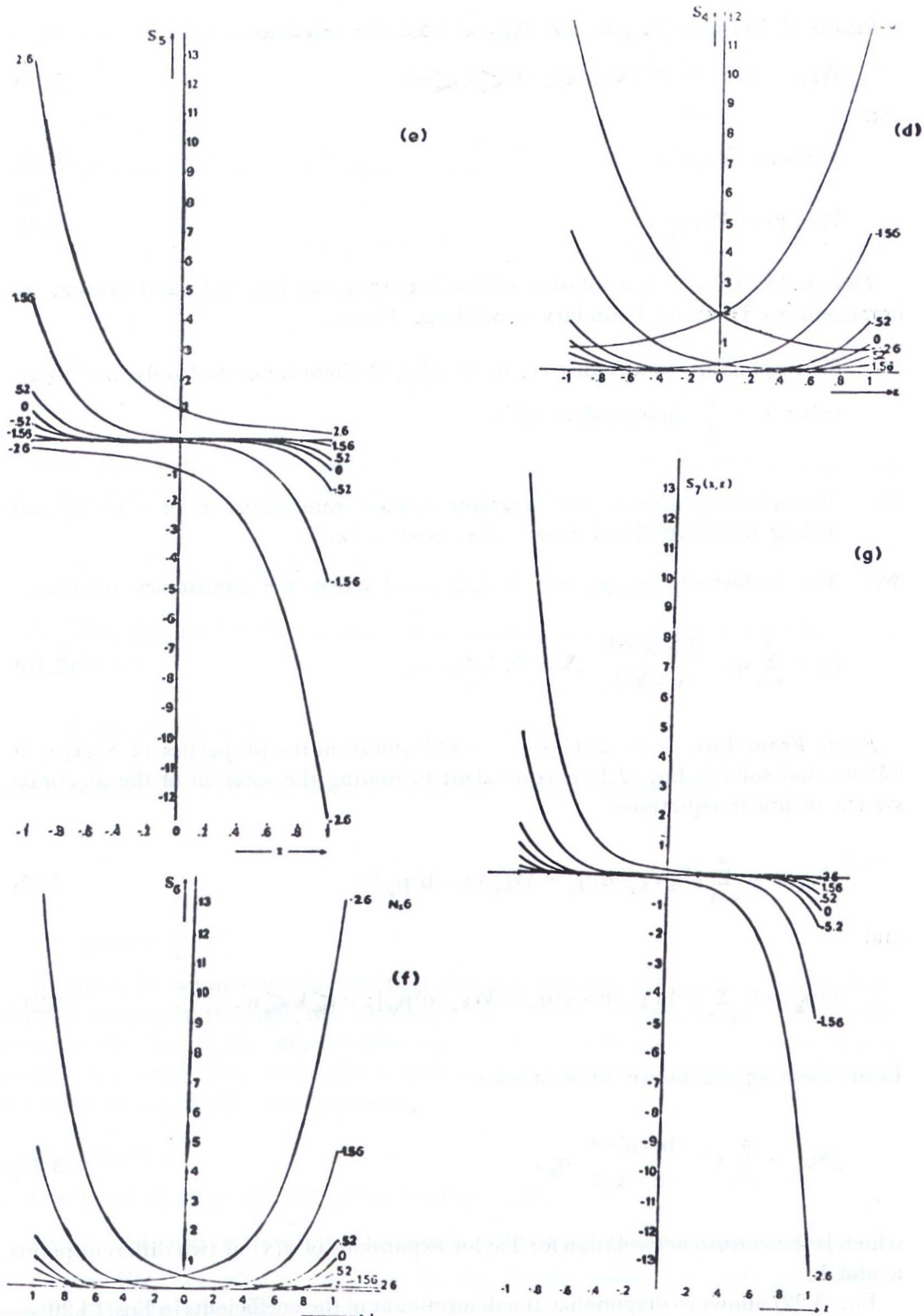


Fig. 1d,e,f,g. The polynomials S_4 , S_5 , S_6 , S_7 represented as functions of z for the indicated values of x.

Remark V. Between $V_{kn}(x)$ and $W_{kn}(x)$ hold the relations

$$W_{kn}(-x) = (-)^{n-k} V_{kn}(x)\,; \quad (0 \leqslant k \leqslant n) \tag{3.16}$$

and

$$V(x)_{on} = V_n(x)\,, \tag{3.17}$$

$$W_{on}(x) = W_n(x)\,. \tag{3.18}$$

Theorem VI. $\psi_n^1(x, z)$ is a solution of the homogeneous Eq. (2.1) and satisfies inhomogeneous Dirichlet boundary conditions. Then:

1°. Corresponding to any degree, n, of $\psi_n^1(x, z)$ there is one and only one eigenvalue $\lambda = \dfrac{1}{2}$ independent of n.

2°. The solutions $\psi_n^1(x, z)$ are invariant against translations $x \to x' = x + x_0$ and scaling transformations $x \to x' = \lambda x$, $z \to z' = \lambda z$.

3°. The coefficients $\{q_n, p_n \mid n = 0, 1, 2, \dots\}$ satisfy the consistency relations

$$p_n = \sum_{\nu=k}^{n} q_\nu \frac{(b-a)^{\nu-k}}{(\nu-k)!}\,; \; k = 0, 1, 2, \dots \tag{3.19}$$

Proof: From Eqs. (2.1) and (3.1) — (3.2) and from the properties of $S_n(x, z)$ it follows that solving Eq. (2.1) is equivalent to finding the solution of the algebraic system of linear equations

$$\lambda^k q_k = \lambda \sum_{\nu=k}^{n} \lambda^\nu \{V_{k_\nu}(o) q_\nu + W_{k_\nu}(a-b) p_\nu\} \tag{3.20}$$

and

$$\lambda^k p_k = \lambda \sum_{\nu=k}^{n} \lambda^\nu \{V_{k_\nu}(b-a) q_\nu + W_{k_\nu}(o) p_\nu\}; \quad o \leqslant k \leqslant n\,. \tag{3.21}$$

From these equations we have at once

$$\lambda^k p_k = \sum_{\nu=k}^{n} \lambda^\nu \frac{(b-a)^{\nu-k}}{(n-k)!} q_\nu\,, \tag{3.22}$$

which is the consistency relation for Taylor expansions of $\varphi(x)$ at two different points a and b.

Eq. (3.22) allows to diagonalize the determinant of the coefficients in Eqs. (3.20) —

(3.21) and in fact it can be used to eliminate Eq. (3.21). From Eqs. (3.11) — (3.12) and (3.20) - (3.22) we get the triangular matrix equation for $\{q_v \mid n = 0, 1, \ldots n\}$:

$$\mathbf{D}^{(1)} \underline{q} = 0 , \tag{3.23}$$

where $\underline{\mathbf{q}}$ is the column vector $\{q_0 q_1 \cdot \ldots \cdot q_n\}$
and

$$(D^{(1)})_{kk+1}(\lambda) = \lambda^{k+1}\{\lambda[V_{kk+1}(o) + \Sigma\, (-)^{v-k}\, V_{kv}(d)\zeta_{vk+1}] - \delta_{kk+1}\} , \tag{3.24}$$

$$\zeta_{kv} = \begin{cases} \dfrac{d^{n-k}}{(n-k);} \;;\; k \leqslant n \\ & ;\; d = b - a \; . \\ 0 \quad ;\; k > n \end{cases}$$

Now, the value of the triangular determinant in Eq. (3.23) equals the product of the diagonal elements and, therefore,

$$\det D^{(1)}(\lambda) = \prod_{k=o}^{n} (D^{(1)})_{kk}(\lambda) = \prod_{k=o}^{n} \lambda^k (2\lambda - 1) \; . \tag{3.25}$$

In order that Eq. (3.23) has non-trivial solutions it is required that det D(1) = o and therefore

$$\prod_{k=o}^{n} \lambda^k . (2\lambda - 1) = 0 ,$$

whence, since $\lambda = o$ is incompatible with Eq. (2.1), it follows that

$$\lambda = \frac{1}{2} \; . \tag{3.26}$$

This proves 1°.

To prove 2° we merely observe that the polynomials $S_n(x-a, z)$, $S_n(x-b, z)$ depend on x only through the difference $(x-a)$ or $(x-b)$ and that the coefficients of the Eq. (3.23) depend also on $(b-a)$. Furthermore, it is clear that a scaling transformation multiplies x and z by p and λ by p^{-1} and consequently the roots of Eq. (3.25) are invariant.

This proves 2°.

The proof of 3° is already given by Eq. (3.22).

Remark VI. It will be shown in Theorem XI that n can take only the values o or 1.

Theorem VII. Let $\psi_i^2(x, z)$ be given by Eq. (3.10), satisfy Eq. (2.1) and homogeneous Dirichlet boundary conditions. Then:

1o. There is a countable infinite number of open eigensets $\mathscr{E}^{\cdot}_i$ such that $\mathscr{E}^{\cdot}_{i+1} \subset \mathscr{E}^{\cdot}_i \subset \mathscr{E}^{\cdot}$; $i > o$, where $\mathscr{E}^{\cdot} = \bigcup_{i=1}^{\infty} \mathscr{E}^{\cdot}_i$ is an open bounded set.

2o. Corresponding to each complex eigenset $\mathscr{E}^{\cdot}_i$ there is also its complex conjugate $\mathscr{E}^{\cdot *}_i$ for which a solution of Eq. (2.1) exists.

3o. Each real eigenset $\mathscr{E}^{\cdot}_i$ defines an eigenfunction of Eq. (2.1) with a positive eigenvalue, λ_i, satisfying the relation

$$\underline{\lambda} \leqslant \lambda_i \leqslant \overline{\lambda} \quad ; \text{ for all i,}$$

where the positive numbers $\underline{\lambda}$ and $\overline{\lambda}$ are given by

$$\underline{\lambda} = \inf_{i,j} \{ \int_A dx \int_B dz\, \psi_i^2(x, z) \psi_j^2(x, z) \,/\, (\int_B dz \psi_i^2(x, z)) \ (\int_B dz \psi_j^2(x, z)) \} \leqslant 1$$

and

$$\overline{\lambda} = 1 + \sup_{i,j} \{ \int_B dzz \left[\psi_i^2(b, z) \psi_j^2(b, z) - \psi_i^2(a, z) \psi_j^2(a, z) \right] \}.$$

Proof:

1o. From Eqs. (2,1), (3.5) - (3.6) it follows that in order that $\psi^2(x, z)$ be a solution, the following equation must be satisfied;

$$D^{(1)}(\lambda)\, Q = o, \tag{3.27}$$

where the elements of the matrix $D^{(1)}(\lambda)$ are given by

$$D^{(1)}_{kn}(\lambda) = \frac{\lambda^n}{k!\,n!} \left[\lambda \theta_{kn} - \alpha_{kn} \right]. \tag{3.28}$$

and

$$\theta_{kn} = \beta_{kn} + (-)^k \sum_{\nu=o}^{n} (-) \beta_{k\nu} \zeta_{kn},$$

$$\alpha_{kn} = \begin{cases} \left(\dfrac{d}{2}\right)^{n-k} / (n-k)!\ ; n \geqslant k \\ o \qquad\qquad ; n > k. \end{cases}$$

and Q is the column vector

$$Q = (\psi(a, o), \psi^{(1)}(a, o), \ldots\ldots). \tag{3.29}$$

The n-th component of Q is defined by

$$\psi^{(n)}(a, o) = \lim_{x\to a} \lim_{z\to +o} \partial_z^n \psi(x, z). \tag{3.30}$$

In deriving Eq. (3.28) use was made of the structural property given in Eq. (2.23) and also of the property

$$\beta_{kn}(\xi) = (-)^{n-k}\, \gamma_{kn}(-\xi)\,. \tag{3.31}$$

For the Eq. (3.27) to have meaning it is necessary and sufficient that the following limit of the N-th order determinant $D_N(\lambda)$ exists:

$$|\lim_{N\to\infty} D_N{}^{(2)}(\lambda)| = |\lim_{N\to\infty} \det D_N{}^{(2)}(\lambda)| = |\det D^{(2)}(\lambda)| < \infty\,, \tag{3.32}$$

where $D_N{}^{(2)}(\lambda)$ is the matrix obtained from $D^{(2)}(\lambda)$ by omitting the rows and columns of order, N', higher than $N+1$.

The proof that the limit in Eq. (3.32) does indeed exist will not be given here (199).

Based on the existence of this limit we can now study the eigenvalue spectrum resulting from the solubility condition of Eq. (3.27) and prove 1°. With the help of Eq. (3.28) we construct the determinant $D_N{}^{(2)}(\lambda)$ and expand it in a polynomial in powers of λ.

In doing so we exclude the value $\lambda = o$, which in fact is not an eigenvalue of Eq. (2.1), because its structure changes for $\lambda = 0$. Due to the structure of Eq. (3.28) we can divide the 1-th column of $D_N(2)(\lambda)$ by λ^l; $(l = o, 1, 2, \ldots n)$. In this way we get the polynomial

$$\begin{aligned} D_N(\lambda) \equiv\ & \lambda^{N+1} D_N(\bar\theta_{\cdot 0}\,\bar\theta_{\cdot 1}\ldots\bar\theta_{\cdot N}) - \lambda^N \sum_{l=0}^{N} PD_N(\bar\theta_{\cdot 0}\ldots\bar\theta_{\cdot i-1}\,\bar\alpha_{\cdot i}\,\bar\theta_{\cdot i+1}\ldots\bar\theta_{\cdot N}) \\ & + \lambda^{N-1} \sum_{i\neq j=0}^{N} PD_N(\bar\theta_{\cdot 0}\ldots\bar\theta_{\cdot i-1}\,\bar\alpha_{\cdot i}\,\bar\theta_{\cdot i+1}\ldots\bar\theta_{j-1}\,\bar\alpha_{\cdot j}\,\bar\theta_{\cdot j+1}\ldots\bar\theta_{\cdot N}) \\ & + \ldots\ldots\ldots\ldots\ldots \\ & + (-)^{\nu}\lambda^{N+1-\nu} \sum_{i\neq j\ldots\neq l=0} PD_N(\bar\theta_{\cdot 0}\ldots\bar\theta_{\cdot i-1}\,\bar\alpha_{\cdot i}\,\bar\theta_{\cdot i+1}\ldots\bar\theta_{\cdot j-1}\,\bar\alpha_{\cdot j}\,\bar\theta_{\cdot j+1}\ldots \\ & \ldots\bar\theta_{\cdot l-1}\,\bar\alpha_{\cdot l}\,\bar\theta_{\cdot l+1}\ldots\bar\theta_N) + (-)^{N+1} D_N(\bar\alpha_{\cdot 0}\,\bar\alpha_{\cdot 1}\ldots\bar\alpha_{\cdot N}) = o\,. \end{aligned} \tag{3.33}$$

In Eq. (3.33) $D_N(\bar\theta_{\cdot 0}\ldots\bar\theta_{\cdot i-1}\bar\alpha_{\cdot i}\bar\theta_{\cdot i+1}\ldots\bar\theta_{\cdot j-1}\bar\alpha_{\cdot j}\bar\theta_{\cdot j+1}\ldots\bar\theta_{\cdot N})$ is equal to the determinant resulting from $D_N(\bar\theta_{\cdot 0}\bar\theta.\bar\theta_{\cdot 1}\ldots\bar\theta_{\cdot N})$ if we replace the i-th, j-th. . . . . l-th columns by the $\tilde\alpha_{\cdot i}, \tilde\alpha_{\cdot j}, \tilde\alpha_{\cdot l}$ respectively, where $\alpha_{\cdot i}$ is the column $(\tilde\alpha_{1i}\tilde\alpha_{2i}\tilde\alpha_{3i}\ldots\tilde\alpha_{Ni})$.

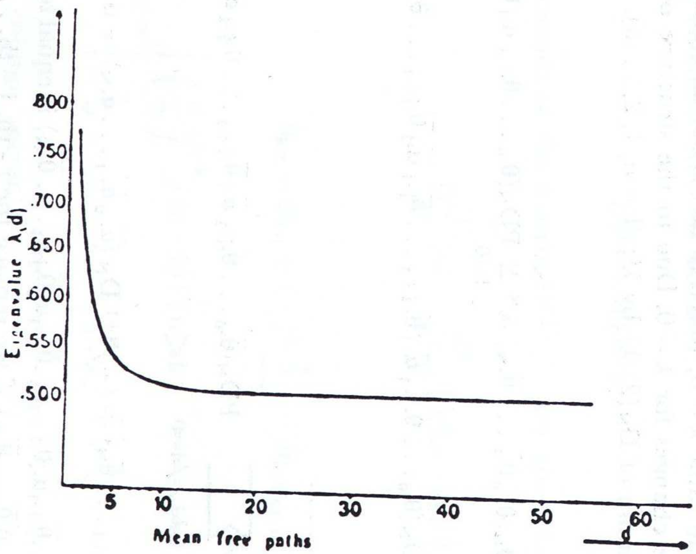


Fig. 2. The fundamental eigenvalue 1 as function of the linear dimension of the system, d. For values of d larger than approximately 5 mfp the eigenvalue changes very slowly with increasing value of d.

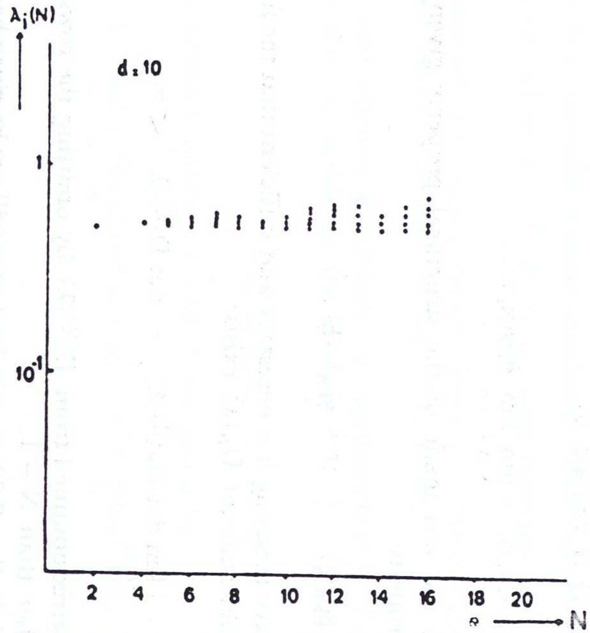


Fig. 3. The set of the positive eigenvalues (l) for various orders, N, of approximation up to N = 16. It is seen that the fundamental eigenvalue is independent of the order of approximation. This is an indicator for the quality of the convergence of the series representation of the distribution functions ψ_+ and ψ_-.

P is the "index-ordering"(202) operator acting on the column indices, whereby the columns are considered as commuting objects. The summation extends over all resulting different determinants. The quantities $\bar{\theta}_{kn}$ are defined by

$$\bar{\theta}_{kn} = (k!n!)^{-1}\,\theta_{kn}\,. \tag{3.34}$$

From Eq. (3.33) we see at once that the product of the $n+1$ roots is given by

$$P_N = \prod_{i=o}^{N} \lambda i = \det_N(\tilde{\alpha}_{kn})\,\det_N{}^{-1}(\bar{\theta}_{kn})\,. \tag{3.35}$$

According to the analysis given earlier the right-hand side of Eq. (3.35) has a limit for $N \to \infty$ and therefore the set $\{\lambda_i \,|\, i = 1, 2, \ldots\}$ is bounded(199). Consequently, there is at least an accumulation point λ_∞ which may belong ($\lambda_\infty \neq o$) or may not belong ($\lambda_\infty = o$) to the eigenvalue spectrum of Eq. (2.1).

From the definition of $\det_N(\tilde{\alpha}_{kn})$ it follows that

$$\det_N(\tilde{\alpha}_{kn}) = \prod_{\nu=o}^{N} (\nu!)^{-2} \tag{3.36}$$

and, therefore, by Eq. (3.35)

$$\lim_{N\to\infty} \prod_{\nu=o}^{N} \lambda^{\nu} = \lim_{N\to\infty} \det_N^{-1}\left(\frac{n!}{k!}\,\theta_{kn}\right). \tag{3.37}$$

Since $\lim_{N\to\infty} \det_N\left(\frac{n!}{k!}\,\theta_{kn}\right) = \infty$, Eq. (3.37) shows that $\prod_{\nu=o}^{N} \lambda^{\nu} \to \infty$ as $N \to \infty$ and consequently $\lambda_N \to o$ asymptotically as $N \to \infty$. This implies the existence of at most a finite number of eigenvalues λ_m satisfying the relation.

$$|\,\mathrm{Re}\lambda'_m\,| > 1\ ;\ (m = 1, 2, \ldots m_o). \tag{3.38}$$

This result allows us to construct the sets $\mathcal{E}^{\cdot}{}_i = [-\lambda_i, \lambda_i]^{\cdot}$ where the "·" indicates that the set has a hole at $\lambda_i = o$, i.e. $\mathcal{E}^{\cdot}{}_i = [-\lambda_i, o[U]o, \lambda_i]$.

This proves 1°.

The proof of 2° is now obvious.

To prove 3° let us write Eq. (2.1) for the two eigenfunctions ψ_i and ψ_j with $\lambda_i \neq \lambda_j$. According to Eqs. (3.5) - (3.6) and (3.10) we have

$$z\partial_j\,\psi_j(x, z) + \psi_j(x, z) = \lambda_j\varphi_j(x) \tag{3.39}$$

and

$$z\partial_x \psi_j(x, z) + \psi_j(x, z) = \lambda_j\varphi_j(x), \tag{3.40}$$

where

$$\varphi_i(x) = \int_B dz\ \psi_i(x, z). \tag{3.41}$$

From Eqs. (3.39) - (3.41) it follows that

$$\begin{aligned} &2\int_B dz \int_A dx\ \psi_j(x, z)\partial_x\psi_j(x, z) = \\ &(\lambda_i - \lambda_j)\int_A dx\varphi_i(x)\varphi_j(x) + \int_B dzz\,[\psi_i(b, z)\psi_j(b, z) - \psi_i(a, z)\psi_j(a, z)]. \end{aligned} \tag{3.42}$$

Using the invariance of $\psi_i(x, z)$ against the transformations

$$\text{a)}\begin{cases} P_X: x \rightarrow \bar{x} = -x \\ \\ P_Z: z \rightarrow \bar{z} = -z \end{cases} \quad \text{and b)}\ T_X:\ x \rightarrow x' = x - \frac{d}{2}, \tag{3.43}$$

as well as the boundary conditions satisfied by ψ_i, ψ_j one easily verifies that

$$\int_B dzz\,[\psi_i(b, z)\psi_j(b, z) - \psi_i(a, z)\psi_j(a, z)] \equiv G_{ij} > o. \tag{3.44}$$

Hence,

$$2\int_A dx \int_B dz\,[\lambda_i\varphi_i(x) - \psi_i(x, z)]\psi_j(x, z) > (\lambda_i - \lambda_j)\int_A dx\varphi_i(x)\varphi_j(x) \tag{3.45}$$

or

$$\int_A dx\,[(\lambda_i + \lambda_j)/2(\int_B dz\psi_i(x, z)\,(\int_B dz\psi_j(x, z)) - \int_B dz\psi_i(x, z)\psi_j(x, z)] > o, \tag{3.46}$$

whence

$$(\lambda_i + \lambda_j)\,/\,2 > \underline{\lambda} \tag{3.47}$$

and

$$\underline{\lambda} = \inf(\int dx \int dz\psi_i(x, z)\psi_j(x, z))\,/\int dx\varphi_i(x)\varphi_j(x) < 1.$$

On the other hand it follows from the inequality

$$\int_B dz\psi_i(x, z)\psi_j(x, z) \leqslant (\int_B dz\psi_i(x, z))\,(\int_B dz\psi_j(x, z)), \tag{3.48}$$

which is valid for positive functions ψ_i and ψ_j, and from Eqs. (3.45) and (3.47), upon taking the $\sup\limits_{i,j}$ of Eq. (3.46) that

$$(\lambda_i + \lambda_j) / 2 \leqslant 1 + \sup \; G_{ij} = \bar{\lambda}. \tag{3.49}$$

Eqs. (3.47) and (3.49) prove the assertion 3° of the theorem.

Remark VII. Eq. (3.49) is in agreement with Eq. (3.38) according to which some eigenvalues λ'_m may be larger than unity.

Remark VIII. When ψ_i, ψ_j are positive and the boundary conditions given in Eqs. (3.7) - (3.8) apply, then G_{ij} is a measure for the outflow through the boundaries of the system.

Theorem VIII. Let $\mathscr{E}^{\cdot}$, ($\mathscr{E}^{\cdot} \subseteq \mathscr{E}$), be the set of real eigenvalues corresponding to the eigenfunctions ψ^2 (x. z). The P_λ defined on $\mathscr{E}$, ($\lambda \in \mathscr{E}$), together with the parity transformation on the space $A \otimes B$ leaves $\psi^2(x, z)$ invariant.

Proof: Let us apply on $\psi(x,z)$ the translation transformation

$$T_x : x \to x' = x - x_0 \; ; \; x^0 = \frac{a+b}{2}. \tag{3.50}$$

The transformed Eqs. (3.5) - (3.6) take then the form

$$\psi^2_+(x, z) = \sum_{n=0}^{\infty} \lambda_n q_n \left[S_n(x+a, z) - (-z)^n e^{-\frac{-x+a}{z}} \right] \tag{3.51}$$

and

$$\psi^2 (x, z) = \sum_{n=0}^{\infty} \lambda^n p_n \left[S_n(x-a, z) - (-z)^n e^{-\frac{x-a}{z}} \right], \tag{3.52}$$

where now $A = [-a, a] = \left[-\frac{d}{2}, \frac{d}{2} \right]$; $d = 2a$.

Let us apply the transformation $P_\lambda : \lambda \to \lambda' = -\lambda$ on $\psi^2(x, z)$. It follows that

$$P_\lambda \lambda^n \left[S_n\, (x+a,z) - (-z)^n e^{-\frac{x-a}{z}} \right] = \lambda^n \left[S_n\, (x-a, -z) - z^n e^{-\frac{x+a}{z}} \right] \tag{3.53}$$

and

$$P_\lambda \lambda^n \left[S_n\, (x-a, z) - (-z)^n e^{-\frac{x-a}{z}} \right] = \lambda^n \left[S_n\, (-x+a, -z) - z^n e^{-\frac{x-a}{z}} \right]. \tag{3.54}$$

Application of the transformations (3.43a) on Eqs. (3.53) — (3.54) brings the right-hand sides of them to

$$P_xP_zP_\lambda\lambda^n\left[S_n(x+a,z)-(-z)^n e^{-\frac{x+a}{z}}\right] = \lambda^n\left[S_n(x-a,z)-(-z)^n e^{-\frac{x-a}{z}}\right] \tag{3.55}$$

and

$$P_xP_zP_\lambda\lambda^n\left[S_n(x-a,z)-(-z)^n e^{-\frac{x-a}{z}}\right] = \lambda^n\left[S_n(x+a,z)-(-z)^n e^{-\frac{x-a}{z}}\right] \tag{3.56}$$

One recognizes that Eqs. (3.55) - (3.56) are simply the interchanged brackets of Eqs. (3.51) - (3.52). Consequently,

$$P_xP_zP_\lambda\lambda^n(x, z) = \psi^2(x, z). \tag{3.57}$$

The structure of the matrices $D^{(1)}(\lambda)$ and $D^{(2)}(\lambda)$ determine largely the properties of the solutions $\psi^1(x, z)$ and $\psi^2(x, z)$. Further specification of $\psi^1(x, z)$ is provided by Theorem IX.

Theorem IX. Let $\omega_m(x)$ be a polynomial on A of degree m and $\psi^1(x, z)$ be a solution of the equation

$$z\partial_x\psi(x, z) + \psi(x, z) = \int_B dz'\psi(x, z') + \omega_m(x) \tag{3.58}$$

satisfying inhomogeneous boundary conditions of the Dirichlet type. Then:

1°. $n = o$ or 1, if $\omega^n(x) \equiv o$

2°. $n = m$, if $m \geqslant o$.

Proof: Let us consider first the case $\omega_m(x) \equiv o$. Theorem V implies that $\psi^{(2)}(x, z)$ is a polynomial and the matrix $D^{(1)}(\lambda)$ has the form given by Eq. (3.24).

Since $V_{kn}(d) = V_{n-k}(d)$ and $V_{n-k}(o) = \dfrac{(-)^{n-k}}{n-k+1}$ we have from Eqs. (3.11) - (3.12) - what is easily verified-that $\sum\limits_{\nu=k}^{n} (-)^\nu V_{\nu-k}(d)\zeta_{\nu n} = (n-k+1)^{-1}$.

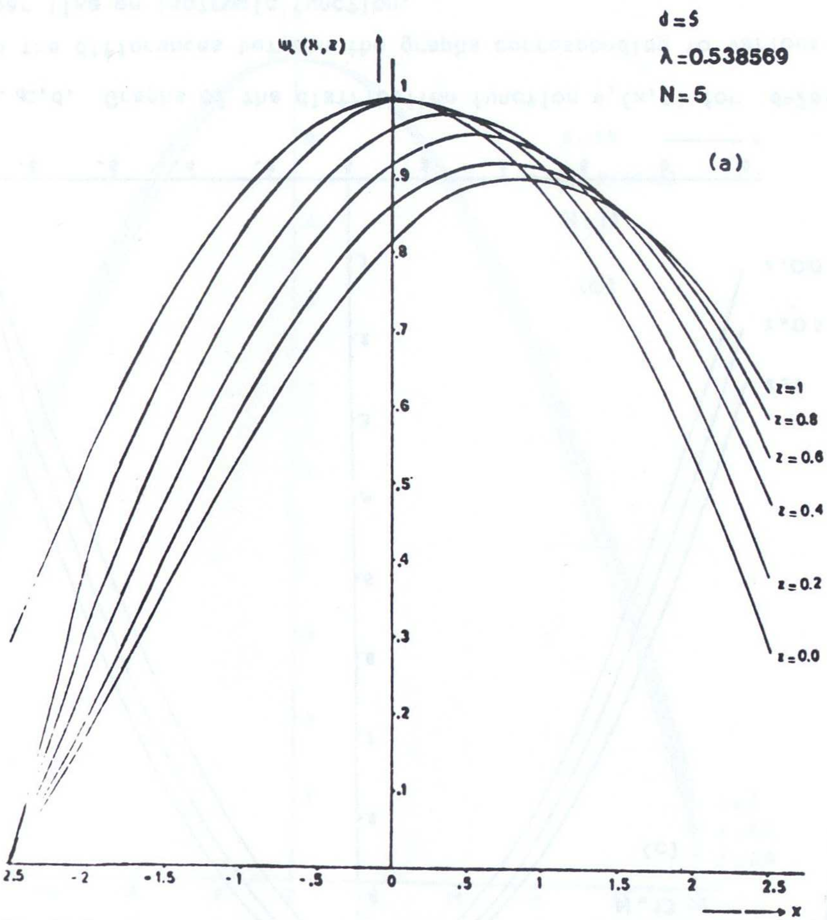
It follows that

$$D^{(1)}(\lambda) = o \; ; k > n,$$

$$D_{kn}{}^{(1)}(\lambda) = \lambda^n(2\lambda - 1) \; ; k = n$$

and

$$D_{kn}{}^{(1)}(\lambda) = \frac{\lambda^{n+1}}{n-k+1}\left[1 + (-)^{n-k}\right] \; ; n > k, \tag{3.59}$$



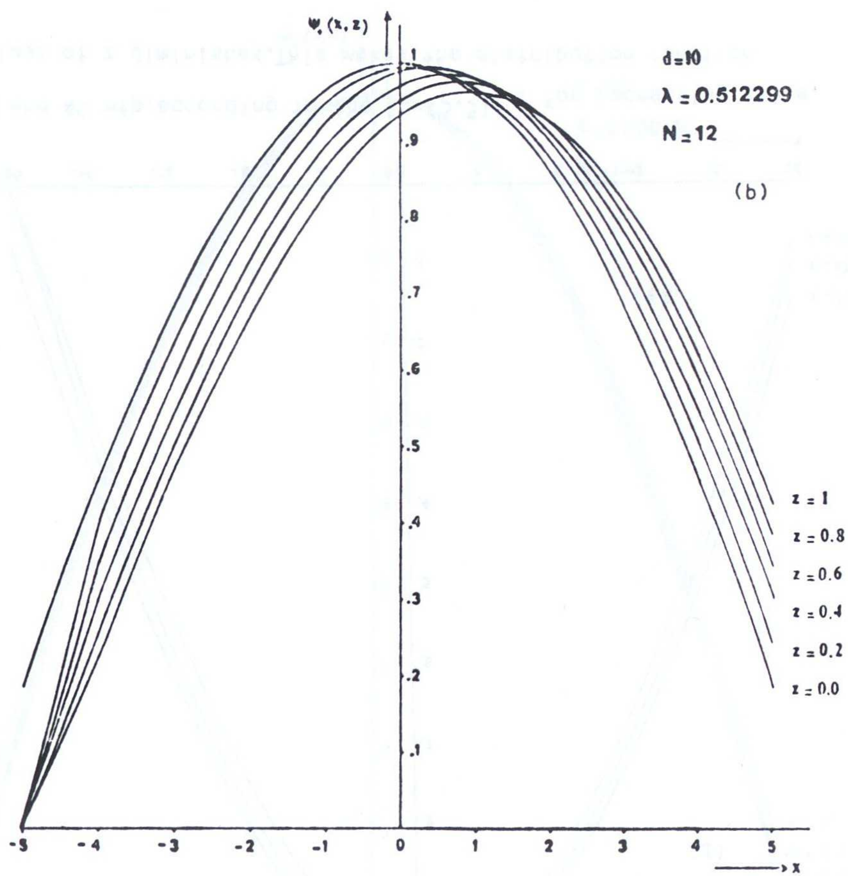


Fig. 4a,b. Graphs of the angular distribution functions $\psi_+(x,z)$ and $\psi_-(x,z)$ in systems for $-a<x<+a$ with $d=2a=5$ and 10 mfp. The number, N, of the terms taken in Eq. (3.51), is indicated. The distribution function $\psi_-(x,z)$ is obtained from $\psi_+(x,z)$ by putting $x \to x' = -x$ in the graphs above.

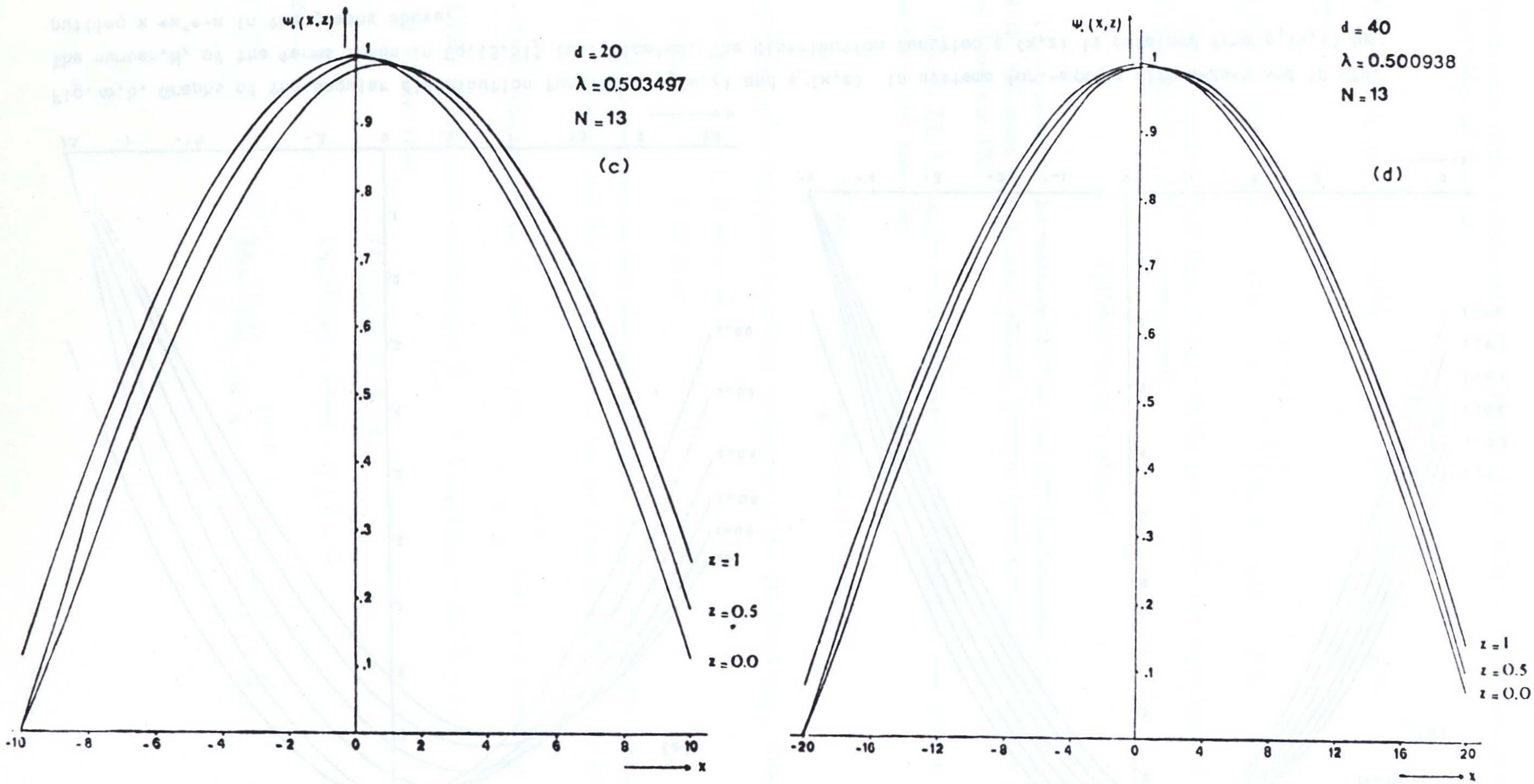
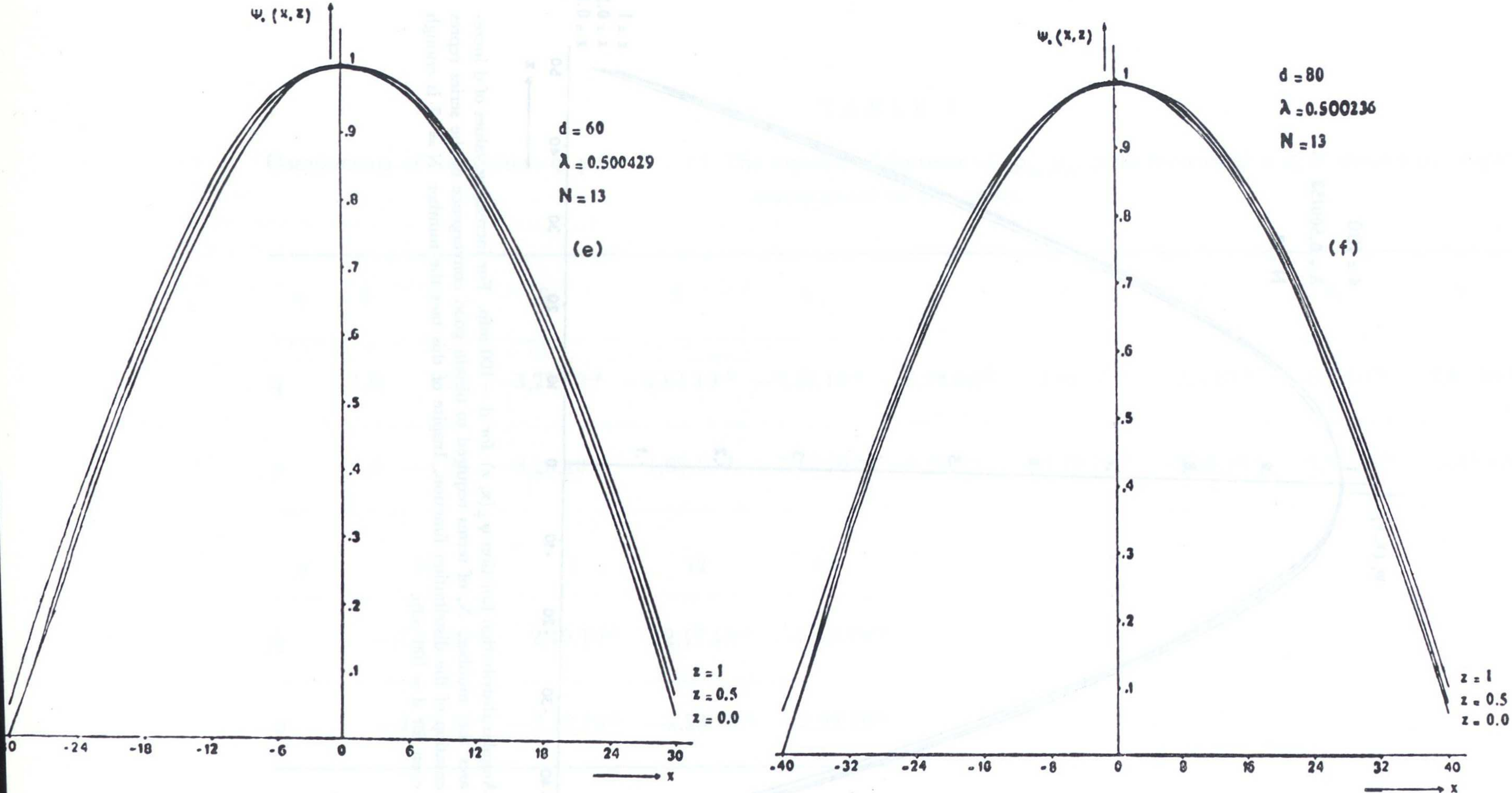


Fig. 4c,d. Graphs of the distribution function $\psi_+(x,z)$ for d=2a=20 and 40 mfp.according to the Eq.(3.51) . For increasing value of d the differences between the graphs corresponding to various values of z diminishes.This makes the distribution function appear like an isotropic function.



ig. 4 e,f. Graphs of the angular distribution function $\psi_+(x,z)$ in systems with d =60 and 80 mfp. It appears that the distribution unction becomes more and more isotropic far from the boundaries of the system.

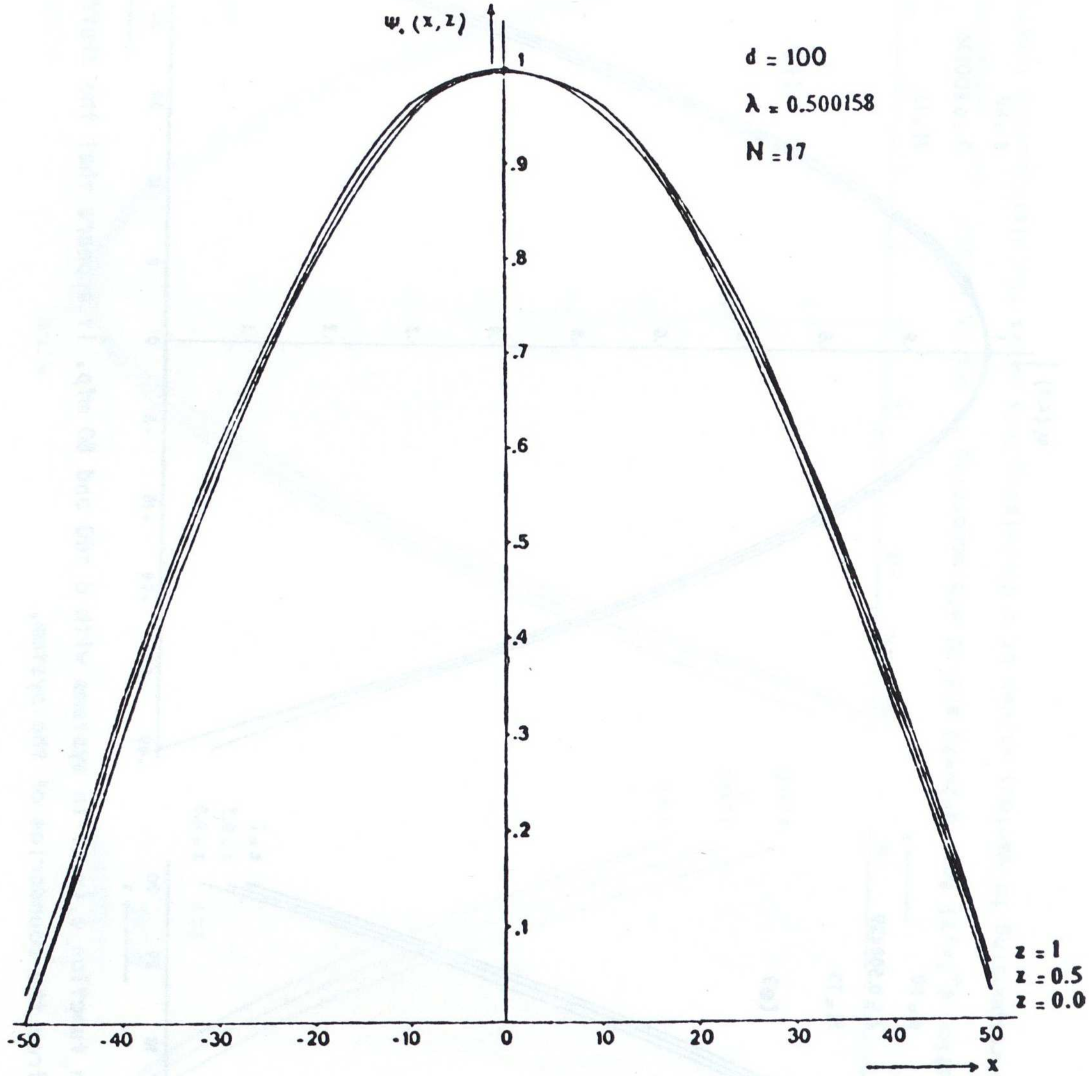


Fig. 4g. Angular distribution function $\psi_+(x, z)$ for $d = 100$ mfp. For increasing values of d increases the number, N, of terms required to obtain good convergence of the series representation of the distribution function. Inspite of this tact the number $N = 17$ is enough even for $d = 100$ mfp.

TABLE I

Components of the vectors q, p for $N = 13$. The rapidity of decrease of (p_n, p_n) with increasing $n \leqslant N$ insures the rapid convergence of the series.

n	0	1	2	3	4	5	6	7	8	9
q	1,0	1,25	—3,25.10^−2	—1,64.10^−2	—7,36.10^−3	—6,23.10^−3	—3,69.10^−3	2,0.10^−3	—9,8.10^−4	3,8. 10^−4
p	1,0	—1,25	—3,28.10^−2	1,62.10^−2	—7,49.10^−3	—6,31.10^−3	—3,73.10^−3	—2,08.10^−3	—9,94.10^−4	—3,83.10^−

n	10	11	12	13
q	—1,11.10^−4	2,19.10^−5	—2,17.10^−6	—2,93.10^−9
p	—1,12.10^−4	—2,22.10^−5	—2,23.10^−6	—2,93.10^−9

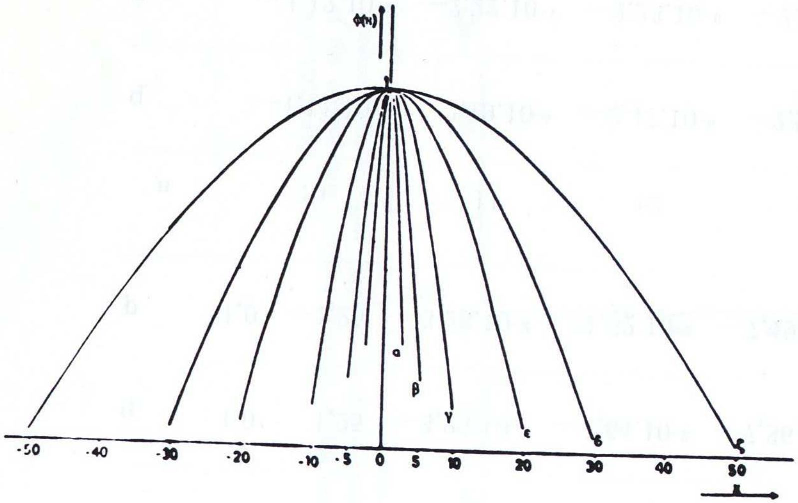


Fig.5. Graphs of Integrated distribution functions $\{\phi(x)\}$ for different values of the linear dimension, d, of the system. The value on the boundary decreases rapidly with increasing value of d. The graphs α, β γ, δ ε, ζ correspond the values d=5,20,40,60 and 100 mfp.

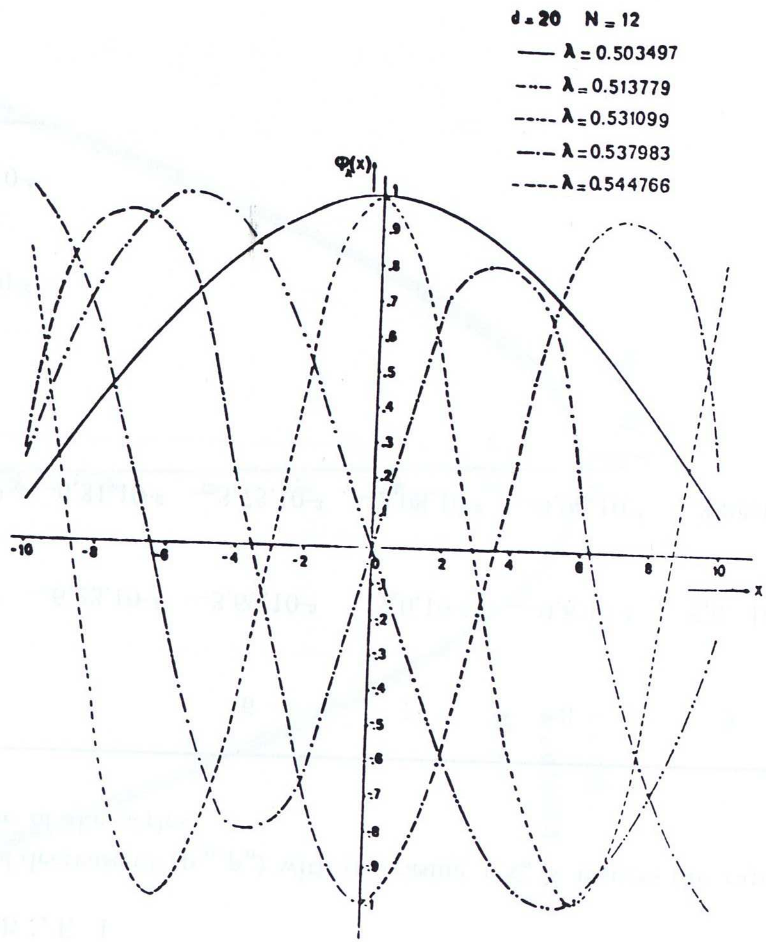


Fig.6. Fundamental and higher order eigenvalues and corresponding eigenfunctions $\{\phi_\lambda(x)\}$ for a system of d =20 mfp.

or explicitly for $\lambda = \frac{1}{2}$:

$$D^{(1)}(\lambda) = \left|\begin{matrix} (2l-1) & 0 & \frac{2\lambda^3}{3} & 0 & \frac{2\lambda^5}{5} & 0 & \cdots\cdots\cdots & \frac{\lambda^n}{n} \\ 0 & 0 & \lambda(2\lambda-1) & 0 & \frac{2\lambda^4}{3} & 0 & \frac{2\lambda^6}{5} \;\cdots & 0 \\ 0 & 0 & 0 & \lambda^2(2\lambda-1) & 0 & \frac{2\lambda^5}{3} & 0 \;\cdots & \frac{\lambda^n}{n-1} \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ 0 & 0 & 0 & 0 & \cdots\cdots\cdots & & 0 & \lambda^n(2\lambda-1) \end{matrix}\right| . \tag{3.60}$$

The rank of the n-th order determinant of this type equals $n-2$ and Eq. (3.23) should have at least one solution for each $n \geqslant o$. However, due to the particular inner structure of the determinant only the coefficients q_0 and q_1 remain arbitrary while $q_\nu = o$; $\nu \geqslant 2$. This proves 1°.

To prove 2° let us write $\omega_m(x)$ in the form

$$\omega_m(x) = \sum_{\nu=o}^{m} \omega_{\nu m} \frac{(x-a)^\nu}{\nu!} \tag{3.61}$$

and observe that $n = m$ for the Eq. (3.56) to have solution. There are two distinct cases: (i) $\omega_m(x)$ is proportional to $\varphi(x)$ and (ii) $\omega_m(x)$ is completely arbitrary.

If (i) is the case, there holds $\omega_{\nu m} = \lambda^\nu \kappa q_\nu$ and Eq. (3.59) becomes

$$D_{km}{}^{(1)}(\lambda) = \begin{cases} 0\,; & k > n, \\ \lambda^n \dfrac{(2\lambda-1+\kappa)}{n-k+1}\,; & k = n \\ \lambda^{n\pm1} \dfrac{1+(-)^{n-k}}{n-k+1}\,; & k > n. \end{cases} \tag{3.62}$$

In this case there is still one λ-value of Eq. (3.58) which is no longer $\lambda = \dfrac{1}{2}$ but rather

$$\lambda = \frac{1-\kappa}{2}\,. \tag{3.63}$$

The rank of the determinant remains the same $(n-2)$. Furthermore, there is a solution only for $n = m$ and for only $m = o$ or 1. Therefore,

$$\psi^1(x, z) = \begin{cases} q_0 \; ; & m = o \\ \psi_1^{1+}(x, z) + \psi_1^{1-}(x, z) \; ; & m = 1. \end{cases} \tag{3.64}$$

In case (ii) Eq. (3.23) is inhomogeneous while the determinant $D^{(1)}(\lambda)$ has still the form of Eq. (3.51). In this case there is a solution of Eq. (3.58) $\{\forall\lambda \mid D^{(1)}(\lambda) \neq 0\}$ and in particular for $\lambda \neq \frac{1}{2}$. Furthermore, the degree of $\psi^1(x, z)$ satisfies the relations

$$\deg_x \psi^1(x, z) = \deg_z \psi^1(x, z) = m\,. \tag{3.65}$$

Remark IX. Theorem IX assures the existence of polynomial solutions, of Eq. (3.58) satisfying inhomogeneous Dirichlet boundary conditions. However, these boundary conditions cannot be arbitrarily given, for $\{q_\nu \mid \nu = 0, 1, 2, \ldots m\}$ are the solutions of an algebraic system of equations. In case 1º the coefficients q_0 and q_1 are arbitrary.

Remark X. In a class of physical problems application of mixed boundary conditions is required. Mixed boundary conditions are defined as the linear sum of the boundary conditions of the first and second kind. This case was treated earlier (199).

Theorem X. Let $\psi(x, z)$ be a solution of Eq. (2.1) and let it satisfy the boundary conditions

$$\psi(a, z) = \psi_N(z) \quad ; \quad (\forall z \mid z \in B+)$$

and

$$\psi(b, z) = 0 \quad ; \quad (\forall z \mid z \in B-)\,.$$

Then:

1º. $\psi(x, z)$ has the form

$$\psi_+(x, z) = \sum_{n=o}^{N} \lambda^n q_n S_n(x-a, z) + \sum_{n=N+1}^{\infty} \lambda^n q_n \left[S_n(x-a, z) - (-z)^n e^{-\frac{x-a}{z}} \right] ; \quad (\forall z \mid z \in B_+)$$

2º. $$\psi_-(x, z) = \sum_{n=o}^{\infty} \lambda^n p_n \left[S_n(x-b, z) - (-z)^n e^{-\frac{x-b}{z}} \right] ; \quad (\forall z \mid z \in B_-).$$

3º. The vector $q \equiv \{\mathbf{q}_n \mid n = N+1, N+2, \ldots.\}$ is determined from the equation $D^{(2)}(\lambda; N)q = -D_0(\lambda; q_1, q_2 \ldots q_N)$.

4°. The matrix $D^{(2)}(\lambda; N)$ is derived from that given in Eq. (3.28) by omitting the columns o, ... N.

5°. $D_0(\lambda; q_0, q_1 \ldots q_N)$ is a one-column matrix defined by

$$(D_0)_k = \sum_{n=o}^{N} \lambda^n q_n V_{kn}(d/2).$$

The proof is given in another place(199).

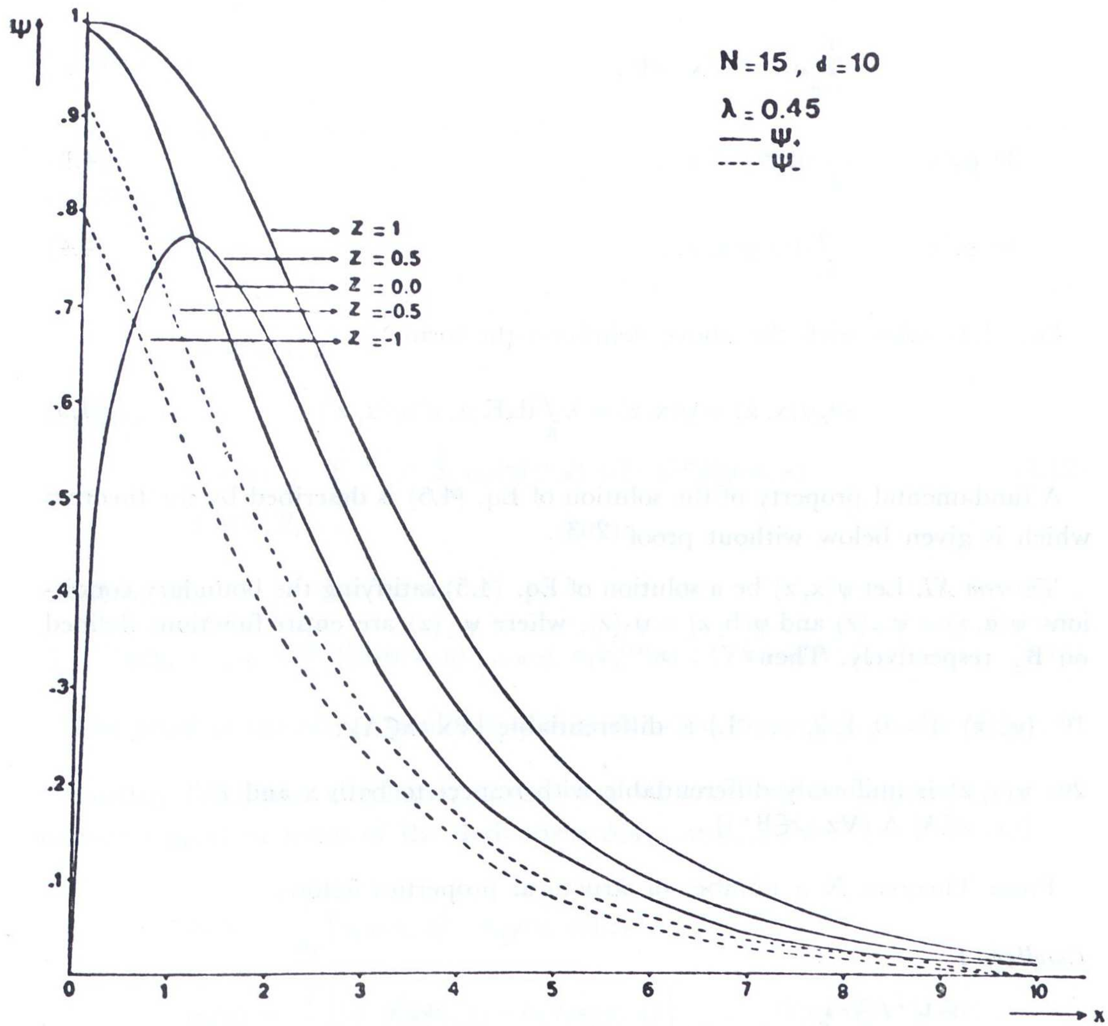


Fig. 7. Graphs of the angular distribution functions $\psi_+(x, z)$ and $\psi_-(x, z)$ for monodirectional surface of unit strength on the plane $x = 0$. The characteristic value of the system is $= 0.45$ and $d = 10$ mip. It is seen that the boundary conditions for both $z > 0$ and $z < 0$ are exactly satisfied.

4. STRUCTURAL PROPERTIES IN ANISOTROPIC SCATTERING

In this section there will be considered Eq. (1.1) with a degenerate kernel. We shall use the following

Definition V.

$$1^{o}\ K(z, z') = \sum_{l=0}^{L} \alpha_e\ (z \cdot z')^e\ ;\ L \geqslant 0\ , \tag{4.1}$$

$$2^{o}\ g(x, z) = \int_B dz' K(z, z') \psi(x, z') \tag{4.2}$$

$$= \sum_{l=0}^{L} \alpha_e .\ z^e \varphi_e(x) > 0\ ,$$

$$3^{o}\ \varphi_e(x) = \int_B dz z^e \psi(x, z)\ , \tag{4.3}$$

$$4^{o}\ \varphi_e(x) = \int_{B\pm} dz z^e \psi(x, z)\ . \tag{4.4}$$

Eq. (1.1) takes with the above definition the form

$$z\partial_x \psi(x, z) + \psi(x, z) = \lambda \int_B dz K(z, z') \psi(x, z')\ . \tag{4.5}$$

A fundamental property of the solution of Eq. (4.5) is described by the theorem which is given below without proof (203).

Theorem XI. Let $\psi(x, z)$ be a solution of Eq. (4.5) satisfying the boundary conditions $\psi(\alpha, z) = \psi_+(z)$ and $\psi(b, z) = \psi_-(z)$, where $\psi_\pm(z)$ are entire functions defined on $B_\pm$ respectively. Then

1o $\{\varphi_e(x) \mid l = 0, 1, 2, \ldots L\}$ is differentiable $\{\forall x \mid x \in A\}$.

2o $\psi(x, z)$ is uniformly differentiable with respect to both x and z $\{(x \mid x \in A) \wedge (\forall z \mid z \in B^{\cdot})\}$.

From Theorem X a number of structural properties follow:

Corollary I

$$\partial_x \varphi_{e'}(x) = \sum_{l+0}^{L} \alpha_l \{ -\int_a^x E_{l+l'}(x - x') \varphi_e(x') dx' + (-)^{l+l'} \int_x^b E_{l+l'}(x' - x) \varphi_e(x') dx'$$

$$+ \frac{1 - (-)^{l+l'}}{l+1}\ \varphi_e(x)\} + \partial_x \chi_{e'}(x)\ ;\ (l' = 0, 1, 2, \ldots L), \tag{4.6}$$

where $\qquad \chi_e(x) = \chi_{e+}(x) + \chi_{e-}(x)$ $\qquad$ and

$$\chi_{e+}(x) = \int_B \exp\left(-\frac{x-a}{z}\right) z^e \psi_+(z)dz\,, \tag{4.7}$$

$$\chi_{e-}(x) = \int_{B-} \exp\left(-\frac{x-b}{z}\right) z^e \psi_-(z)dz\,. \tag{4.8}$$

Corollary II.

$$\lim_{z\to+0} z\partial_x\psi(x, z) = 0\ ;\ (\forall x \mid x\in A^\cdot) \tag{4.9}$$

Corollary III.

$$\psi(x, 0) = \lambda\varphi_0(x)\ ;\ (\forall x \mid x\in A^\cdot) \tag{4.10}$$

Corollary IV.

$$\frac{1}{n!}\partial_x\psi(x, z)\Big|_{x=0} = \sum_{l+v=n} (-)^v \alpha_e\varphi_e^{(v)}(x)\ ;\ (\forall x \mid x\in A^\cdot) \tag{4.11}$$

$n = 0, 1, 2, \ldots$

Corollary V.

$$\psi(x, z) = \sum_{v=0}^{n-1} (-)^v \sum_{l=0}^{L} \alpha_e z^e \varphi^{(v)}(x) + (-z)^n \partial_n^x\psi(x, z) \tag{4.12}$$

$n \equiv 1, 2, \ldots$

Corollary VI.

$$\pm\partial_x^{n+1}\psi(x, z)\,|_{z=0} = \mp\,\partial_x^n\partial_z\psi(x, z)\,|_{x=0} \pm \alpha_1\varphi_1^{(v)}(x)\ ;\ (\forall x \mid x\in A^\cdot). \tag{4.13}$$

The proof of the above results is analogous to the respective results of Sec. 2.

Corollary VII. From Corollaries II - IV and VI one finds the expressions of the moments $\varphi_e(x)$ in terms of the derivatives $\partial_x^n\psi(x, z)\,|_{z=0}$, i.e.

$$\varphi_1(x) = \frac{1}{\alpha_1}\left[\partial_z\psi(x, z) - \partial_x\psi(x, z)\right]_{z=0}\,,$$

$$\varphi_2(x) = \frac{1}{\alpha_2}\left[\frac{1}{2!}\partial_z^2\psi(x, z) - \partial_x\partial_z\psi(x, z)\right]_{z=0}\,;\qquad (\forall x \mid x\in A^\cdot)\ \text{etc.}\,.$$

Before proceeding to the determination of the eigenvalue spectrum of Eq. (4.5) we shall establish an upper bound for the eigenvalue characterized by

Theorem XII. Let $\psi_i^2(x,z)$, $\psi_j^2(x,z)$ be solutions of Eq. (4.5) with the positive eigenvalues λ_i, λ_j satisfying the boundary conditions $\psi_s^2(a,z) = 0$; $(\forall z \in B_+)$ and $\psi_s^2(b,z) = 0$; $(\forall z \mid z \in B_-)$ with $s = i, j$, and let the quantities $\{ A_{ij}^{(l)}, B_{ij} \}$ be defined by

$$A_{ij}^{(l)} = \int_A dx \left(\int_B dz z^e \psi_i^2(x, z)\right) \left(\int_B dz z^e \psi_j^2(x, z)\right), \tag{4.14}$$

$$B_{ij} = \int_A dx \int_B \psi_i^2(x, z) \psi_j^2(x, z). \tag{4.15}$$

Let further the kernel $K(z, z')$ satisfy the positivity condition

$$\lambda g_s(x, z) \equiv (K(z, z'), \psi_s^2(x, z')) > 0. \tag{4.16}$$

Then:

1° $\psi_i^2(x, z)$, $\psi_j^2(x, z)$ are semi-positive functions on $A \otimes B$.

2° The eigenvalues λ_i, λ_j satisfy the spectral relations

$$\lambda_i \gtrless 1 \quad \text{for} \quad \sum_{l=0}^{L} \beta_e A_{ii}^{(l)} \gtrless B_{ii} \quad \text{respectively.}$$

3° The spectrum, Λ_L, of the eigenvalues λ_i, is a bounded set.

Proof. From Eq. (4.5) one deduces that

$$\psi_{s+}^2(x, z) = \lambda \int_a^x e^{-\frac{x-x'}{z}} g_s(x', z) \frac{dx'}{z} \quad ; \{ (\forall x \mid x \in A) \wedge (\forall z \mid z \in B_+) \} \tag{4.17}$$

and

$$\psi_{s-}^2(x, z) = \lambda \int_x^b e^{-\frac{x-x'}{z}} g_s(x', z) \frac{dx'}{z} \quad ; \{ (\forall x \mid x \in A) \wedge (\forall z \mid z \in B_-) \} \tag{4.18}$$

From Eqs. (4.16) - (4.18) it follows that $\psi_s^2(x, z)$ is a semi-positive function on $A \otimes B$· and satisfies the boundary conditions, Eqs. (4.14) – (4.15). It is also easily shown that $\psi_s^2(x, z)$ has the same property on $A \otimes B$. To this end it suffices to show that $(\forall x \mid x \in A)$

(i) $\psi_{s+}^2(x, 0) = \psi_{s-}^2(x, 0)$ and

(ii) $\psi_s^2(x, 0) \geqslant 0$.

From Eq. (4.17) it follows that

$$\lim_{z \to +0} \psi_{s+}^2(x, z) = \lambda \int_a^x \delta(x - x') \varphi_0(x') dx' = \lambda \varphi_0(x) \tag{4.19}$$

and

$$\lim_{z \to -0} \psi^2_{s-}(x, z) = \lambda \int_x^b \delta(x' - x)\varphi_0(x')dx' = \lambda\varphi_0(x).$$

Eqs. (4.17) - (4.18) prove assertion (i). The proof of (ii) follows immediately from the condition in Eq. (4.16) for $z = 0$. Therefore,

$$\psi^2_s(x, z) \geqslant 0 \; ; \; \{ (\forall x \mid x \in A) \wedge (\forall z \mid z \in B) \}.$$

Next we prove 2°. Let us write Eq. (4.5) for the eigenvalues λ_i, λ_j.

$$z\partial_x\psi_i(x, z) + \psi_i(x, z) = \lambda_i g_i(x, z), \tag{4.20}$$

$$z\partial_x\psi_j(x, z) + \psi_j(x, z) = \lambda_j g_j(x, z). \tag{4.21}$$

From Eqs. (4.20) - (4.21) and from the symmetry of the system it follows that

$$\int_A dx \{ \int_B dz \psi_i(x, z)\psi_j(x, z) - \frac{\lambda_i + \lambda_j}{2} \int_B dz \psi_j(x, z) g_i(x, z) \} \leqslant 0. \tag{4.22}$$

Since $g_i(x, z) = \sum_{l=0}^{L} \beta_e z^e \varphi_{ei}(x)$ with $\varphi_{ei}(x)$ given by Eq. (4.3), Eq. (4.22) can also be written in the form

$$\int_A dx \{ \int_B dz \psi_i(x, z)\psi_j(x, z) - \frac{\lambda_i + \lambda_j}{2} \sum_{l=0}^{L} \beta_e (\int_B dz z^e \psi_i(x, z)) (\int dz z^e \psi_j(x, z)) \} \geqslant 0. \tag{4.23}$$

Eq. (4.23) implies that

$$\lambda_i + \lambda_j \geqslant \frac{2B_{ij}}{\sum_{l=0}^{L} \beta_e \overset{(l)}{A_{ij}}} \tag{4.24}$$

and therefore for $i = j$ the assertion 2° follows.

We have still to prove 3°. From the positivity condition, Eq. (4.16), and from 1° follows that

$$\begin{aligned} \int_A dx \int_B dz g_i(x, z)\psi_j(x, z) &= \sum_{l=0}^{L} \beta_e \int_A dx \varphi_{ei}(x) \int_B dz z^e \psi_j(x, z) \\ &= \sum_{l=0}^{L} \beta_e \int_A dx \varphi_{ei}(x)\varphi_{ej}(x) \\ &= \sum_{l=0}^{L} \beta_e \overset{(l)}{A_{ij}} \\ &> 0 \; . \end{aligned} \tag{4.25}$$

Consequently, the denominator in Eq. (4.24) is strictly positive. On the other hand B_{ij} is a finite positive quantity. If we call I the index set and we define

$$\sup_{i \in I} \frac{B_{ii}}{\sum_{l=0}^{L} \beta_e A_{ii}^{(l)}} = \bar{\lambda} \tag{4.26}$$

and

$$\inf_{i \in I} \frac{B_{ii}}{\sum_{l=0}^{L} \beta_e A_{ii}^{(l)}} = \underline{\lambda} \tag{4.27}$$

then $\underline{\lambda} \leq \lambda_i \leq \bar{\lambda}$ for all $i \in I$. From this result 3° follows. (4.28)

Q.E.D.

The notation of Def. IV is in the case of anisotropic scattering generalized in the following

Definition VI.

$$1^o.\ \beta_{kn}^{ll'}(x-d) = \partial_x^k \int_{B+} \left[S_n(x-d,z) - \varepsilon_\eta\,(-z)^n e^{-\frac{x-d}{z}} \right] z_{e+e'} dz \tag{4.29}$$

$$2^o.\ \gamma_{kn}^{ll'}(x-b) = \partial_x^k \int_{B-} \left[S_n(x-b,z) - \varepsilon_\eta\,(-z)^n e^{-\frac{x-d}{z}} \right] z^{e+e'} dz \tag{4.30}$$

$$3^o.\ \zeta_{kn} = \begin{cases} (d/2)^{n-k} / (n-k)! \ ; & k \leqslant n \\ 0 & ; k > n \end{cases} \tag{4.31}$$

$$4^o.\ E_n = \begin{cases} 0 \ ; \ n = 1 \\ 1 \ ; \ n = 2\ . \end{cases} \tag{4.32}$$

Theorem XIII. Let $\psi^2(x,z) = \psi_+^2(x, z) + \psi_-^2(x,z)$ be a solution of Eq. (4.5) satisfying the boundary conditions ($\eta = 2$)

$$\psi_+^2(d, z) = 0\ ;\ (\forall z \mid z \in B_+) \tag{4.33}$$

and

$$\psi_-^2(b, z) = 0\ ;\ (\forall z \mid z \in B_-). \tag{4.34}$$

Let furthermore $\{ \psi_{l+}^2(x, z) \mid l = 0, 1, 2, \ldots L \}$ be a set of isotropic scattering solutions satisfying boundary conditions of the same kind. Then:

1°. The two superpositions

$$\psi^2_+(x, z) = \sum_{l=0}^{L} \alpha_e z^e \psi^2_{l+}(x, z) \ ; \{ (\forall x \mid x \in A) \wedge (\forall z \mid z \in B_+) \} \tag{4.35}$$

and

$$\psi^2_-(x, z) = \sum_{l=0}^{L} \alpha_e z^e \psi^2_{l-}(x, z) \ ; \{ \forall x \mid x \in A) \wedge (\forall z \mid z \in B_-) \} \tag{4.36}$$

are solutions of Eq. (4.5) provided at least one of the kernel coefficients $\{ \alpha_e \mid l = 0, 1, 2, \ldots L \}$ belongs to the set of eigenvalues of the supermatrix

$$\overset{ll'}{D}_{kn} = \left[\alpha_e \ \overset{ll'}{\beta}_{kn} + \sum_{\nu=k}^{n} \overset{ll'}{\gamma}_{k\nu}\zeta_{nk} \right] - 2^{n-k}\zeta_{nk}\delta_{ee'} \ . \tag{4.37}$$

2°. $\psi^2(x, z)$ satisfies homogeneous Dirichlet boundary conditions.

3°. The series representing $\psi^2(x, z)$ are uniformly convergent on $A \otimes B$ provided $\mid \partial^n_x \psi^2(x, z) \mid \leqslant C_0(z) n!$, where $0 \leqslant C_0(z) < \infty$; $(\forall z \mid z \in B)$. This theorem was proved in ref. 190.

For $\eta = 1$ Def. VI, 1°, 2° gives:

$$\psi^1_{N+}(x, z) = \sum_{l=0}^{L} \alpha_e \sum_{n=0}^{N} q_{en} z^e S_n(x - d, z) \ ; \ (\forall z \mid z \in B_+) \tag{4.38}$$

$$\psi^1_{N-}(x, z) = \sum_{l=0}^{L} \alpha_e \sum_{n=0}^{N} P_{en} z^e S_n(x - b, z); \ (\forall z \mid z \in B_-). \tag{4.39}$$

These functions satisfy the conditions of the following

Theorem XIV. Let $\psi^1_{N+}(x, z)$ be polynomials of degree N satisfying Eq. (4.5) with arbitrarily fixed kernel coefficients $\{ \alpha_e \mid l = 0, 1, 2, \ldots L\}$ and the boundary conditions

$$\psi^1_{N+}(\alpha, z) = \psi_+(z) \ ; \ (\forall z \mid z \in B_+) \tag{4.40}$$

and

$$\psi^1_{N-}(b, z) = \psi_-(z) \ ; \ (\forall z \mid z \in B_-), \tag{4.41}$$

where $\psi_\pm(z)$ are polynomials of degree N and of definite parity (even or odd). Then:

1°. There exist particular x-independent solutions of even or odd parity and of degree L or L - 1 (if L or L - 1 = even).

2°. There exist general solutions of degree $N = 0$ or 1 in x and $L = 1$, in z corresponding to the eigenvalue $\lambda_0^{-1} = 2\alpha^0$ and to

$$\lambda_{\pm} = [\alpha_0 + \alpha_2/5 \pm (\alpha_0^2 + \alpha_2^2/25 + 2/(45\alpha_0\alpha_2))^{1/2}]\,[16\alpha_0\alpha_2/45]^{-1}.$$

3°. There exist no general solutions of degree N equal to or higher than 2.

Proof: From Eqs. (4.5) and (4.38) - (4.39) it follows that

$$\sum_{n=k}^{N}\sum_{l=0}^{L}\left[\lambda\alpha_e\Big(\beta_{kn}^{ll'} + \sum_{v=k}^{n}\gamma_{kv}^{ll'}\zeta_{vn}\Big) - \delta_{ee'}\delta_{kn}\right] q_{en} = 0 \tag{4.42}$$

for $k = 0, 1, 2, \ldots N$ and $l' = 0, 1, 2, \ldots L$.

On the other hand it follows from Def. VI that

$$\beta_{kn}^{ll'} = \begin{cases} \dfrac{(-)^{n-k}}{n-k+e+e'+1} & ;\ k \leqslant n \\ 0 & ;\ k > n \end{cases} \tag{4.43a}$$

$$\sum_{v=k}^{n}\gamma_k^{ll'}(-d)\zeta_{vn}(d) = \begin{cases} \dfrac{(-)^{e+e'}}{n-k+e+e'+1} & \\ 0 & ;\ k > 0\,. \end{cases} \tag{4.43b}$$

Let us first prove 1°.

The system of Eqs. (4.42) takes for $N = 0$ and $L > 0$ the form $(\mathbf{\Delta}_L - \lambda^{-1}\mathbf{1})q = 0$ where $\mathbf{1}$ is the unit matrix of dimension L. Explicitly this equation reads:

$$\begin{Bmatrix} 2\alpha_0 - \sigma & 0 & \dfrac{2\alpha_2}{3} & 0 & \ldots .0 & \dfrac{2\alpha_L}{L+1} \\ 0 & \dfrac{2\alpha_1}{3} - \sigma & 0 & \dfrac{2\alpha_3}{3} & \ldots . & 0 \\ \dfrac{2\alpha_0}{3} & 0 & \dfrac{2\alpha_2}{3} - \sigma & 0 & \ldots . & \dfrac{2\alpha_L}{L+3} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \dfrac{2\alpha_0}{L+1} & 0 & \dfrac{2\alpha_2}{L+3} & 0 & \ldots & \dfrac{2\alpha_L}{2L+1} - \sigma \end{Bmatrix} \cdot \begin{Bmatrix} q_0 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_L \end{Bmatrix} = 0 \tag{4.44}$$

where $\sigma = \lambda^{-1}$.

It is seen that Eq. (4.44) breaks down into two uncoupled sets of homogeneous equations. The characteristic equations from which the eigenvalues σ^{-1} follow are:

$$\sigma^{L_s}+1-\sigma^{L_s}\sum_{i=0}^{L_s} P\begin{vmatrix} 1 & 0\ldots0 & \alpha_{1i} & 0\ldots0 \\ 0 & 1 & & \\ \vdots & \ddots & & \\ \vdots & & 1 & \\ \vdots & & \alpha_{ii} & \\ \vdots & & 1 & \\ \vdots & & & \ddots \\ 0 & & 1\ \alpha_{Li} & 0\ldots01 \end{vmatrix} + \sigma^{L_s}-1\sum_{i\neq j=0}^{L_s} P\begin{vmatrix} 1 & 0\ldots0 & \alpha_{1i} & 0\ldots0 & \alpha_{1j} & 0\ldots0 \\ 0 & 1 & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \alpha_{ii} & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & 1 & & \\ & & & & \alpha_{jj} & \\ & & & & 1 & \\ & & & & & \ddots\ 0 \\ 0 & \ldots0 & \alpha_{Li} & 0\ldots0 & \alpha_{Lj} & 0\ldots01 \end{vmatrix}$$

$$+\ \ldots\ +(-)^{L_s+1}\Delta_{L_s}=0\ ;\ (s=1,2) \tag{4.45}$$

with $\quad L_s = L_1,\ L_2,\ ;\ L_1+L_2 = L+1 \quad$ and

$$\alpha_{ee'} = \alpha_e\,\frac{1+(-)^{e+e'}}{e+e'+1}\ ;\ e,e'=0,1,2,\ldots L. \tag{4.46}$$

The sums in Eq. (4.45) extend over all possible values of column indices, while the "index ordering" operator P puts the columns $\alpha_{.i}$, $\alpha_{.j}$ etc. in their physical order without change of the determinant sign. Since the element sets of each of the determinants Δ_{L_1} and Δ_{L_2} differ in general, are different from each other too the sets of the roots Λ_{L_1} and Λ_{L_2} of them. Consequently, there are in general no equal eigenvalues and, therefore, the two sets of linear homogeneous equations are incompatible for $L \geqslant 2$. The incompatibility is resolved, if we assume that either $\{q_e \equiv 0 \mid e = \text{even}\}$

or $\{q_e \equiv 0 \mid e = \text{odd}]$, i.e. we are left with either

$$(\boldsymbol{\Delta}_{L_1} - \sigma\mathbf{1})q^{(o)} = 0 \tag{4.47}$$

or

$$(\boldsymbol{\Delta}_{L_2} - \sigma\mathbf{11})q^{(e)} = 0, \tag{4.48}$$

of which the solubility conditions are again given by Eq. (4.45). This proves assertion 1°.

To prove 2°, let us introduce the following notation

$$(D^{ee'}_{kn}) = \begin{Bmatrix} \mathbf{\Delta}_L(0,0) & \mathbf{\Delta}_L(0,1) & \dots & \mathbf{\Delta}_L(0,N) \\ 0 & \mathbf{\Delta}_L(1,1) & \dots & \mathbf{\Delta}_L(1,N) \\ . & 0 & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & & \mathbf{\Delta}_L(NN) \end{Bmatrix}, \tag{4.49}$$

where

$$\mathbf{\Delta}_L(k+\nu, n+\nu) = \mathbf{\Delta}_L(k, n),$$

$$(\mathbf{\Delta}_L(k, n)_{ee'} = \alpha_e \frac{(-)^{n-k} + (-)^{e+e'}}{n-k+e+e'+1} \tag{4.50}$$

and $\mathbf{\Delta}_L(0,0)_{ee'} = \mathbf{\Delta}_L$.

Next we consider the general case of Eq. (4.49): $D \cdot q = 0$, where

$$q \equiv (\overbrace{q_{00}, q_{10}, \dots q_{L0}}^{\bar{q}_0}, \overbrace{q_{01}, q_{01}, \dots q_{L1}}^{\bar{q}_1}, \dots \overbrace{q_{0N}, q_{1N}, \dots, q_{LN}}^{\bar{q}_N}).$$

Due to the definition given in Eq. (4.50) the diagonal elements of the supermatrix D are equal to each other and consequently the requirement $(q_{0N}, q_{1N}, \dots q_{LN}) \neq 0$ implies

$$\det \mathbf{\Delta}_L(0,0) = \det \mathbf{\Delta}_L(1,1) = \dots \det \mathbf{\Delta}_L(N,N) = 0 . \tag{4.51}$$

From Eqs. (4.49), (4.50), (4.51) it follows that

$$\mathbf{\Delta}_L(0,0)\bar{q}_0 = \mathbf{\Delta}_L(1,1)\bar{q}_1 = \dots = \mathbf{\Delta}_L(N,N)\bar{q}_N = 0 , \tag{4.52}$$

from which the vectors $\{ \bar{q}_n \mid n = 0, 1, \dots N \}$ are determined up to their last component, q_{Ln}. On the other hand the following additional conditions must be satisfied

$$\mathbf{\Delta}_L(N-1, N)\bar{q}_N = 0, \tag{a}$$

$$\mathbf{\Delta}_L(N-2, N-1)\bar{q}_{N-1} + \mathbf{\Delta}_L(N-2, N)\bar{q}_N = 0, \tag{ß}$$

$$\vdots$$

$$\mathbf{\Delta}_L(0,1)\bar{q} + \mathbf{\Delta}_L(0,2)\bar{q}_2 + \dots \mathbf{\Delta}_L(0,N)\bar{q}_N = 0 . \tag{4.53}$$

Now it is clear that Eq. (4.53a) is a condition on the elements of $\boldsymbol{\Delta}_L(N-1,N)$, which in general is not satisfied. It is observed however that the number of the conditions in Eq. (4.53) is an increasing function of N. For $N = 0$ we have the case 1° of the theorem. For $N = 1$ we have just one condition, Eq. (4.53a), which reads $\boldsymbol{\Delta}_L(0,1)\bar{q}_1 = 0$. In this case the sought vector is $q = \{\bar{q}_0, \bar{q}_1\} \equiv \{q_{00}, q_{10}, q_{01}, q_{11}\}$.

The condition $\boldsymbol{\Delta}_L(0,1)\bar{q}_1 = 0$ is satisfied if we put $L = 2$.

This is evident from the equations:

$$(2\alpha_0 - \sigma)q_{00} + \frac{2\alpha_2}{3} q_{20} - \frac{2\alpha_1}{3} q_{11} = 0\,,$$

$$\left(\frac{2\alpha_1}{3} - \sigma\right)q_{10} - \frac{2\alpha_0}{3} q_{20} - \frac{2\alpha_2}{5} q_{21} = 0\,,$$

$$\frac{2\alpha_0}{3} q_0^0 + \left(\frac{2\alpha_2}{3} - \sigma\right)q_{20} + \frac{2\alpha_1}{5} q_{11} = 0\,, \tag{4.54}$$

$$(2\alpha_0 - \sigma)q_{00} + \frac{2\alpha_2}{3} q_{11} = 0\,,$$

$$\frac{2\alpha_0}{3} q_{01} + \left(\frac{2\alpha_2}{5} - \sigma\right)q_{21} = 0\,.$$

The solution is

$$L = 2,\ N = 1 : q_{00} = \frac{2\alpha_0}{(32\alpha_0 - \sigma)} q_{21}\,.$$

$$q_{10} = \left[\frac{4\alpha_0\alpha_2}{3(2\alpha_0 - 3\sigma)\,(2\alpha_0 - \sigma)} + \frac{2\alpha_2}{5}\right] q_{21}\,,$$

$$q_{01} = \frac{2\alpha_2}{3(2\alpha_0 - \sigma)} q_{21}\,. \tag{4.55}$$

The eigenvalue corresponding to this solution is given by the expression

$$\lambda_\pm = \frac{\left[\alpha_1 + \frac{\alpha_2}{5} \pm \alpha_0^2 + \frac{\alpha_2^2}{25} + \frac{2}{45}\alpha_0\alpha_2\right]^{1/2}}{\frac{16}{45}\alpha_0\alpha_2}\,. \tag{4.56}$$

This proves 2°.

The proof of 3° follows from the impossibility to satisfy Eqs. (4.52) - (4.53).

Q.E.D.

Remark XI. Theorem XIV states implicitly the conditions on which solutions of degree $N \geqslant 2$ do exist. We have therefore

Corollary I. Corresponding to each degree of the kernel, $K(z, z')$, there exist particular sets of values of $\{ \alpha_e \mid e = 0, 1, \dots L \}$ for which there exist polynomials, $\psi_N(x, z)$, of degree $L = N$ in x and in z which satisfy Eq. (4.5.). These particular sets $\{ \alpha_e \mid e = 0, 1, \dots L \}$ are the roots of the equations resulting from Eqs. (4.53) after elimination of the vectors $\{ \bar{q}_1, \bar{q}_2 \dots \bar{q}_N \}$ with the help of Eqs. (4.52).

Remark XII. Theorem XIV, Corollary I explain why polynomial approximation in transport theory yield surprisingly satisfactory results. This happens when the physical values of $\{ a_e \mid e = 0, 1, \dots L \}$ are near the roots of Eqs. (4.53).

Example: In case $L = N = 2$ the condition for the existence of the solution $\psi_2(x, z)$ is given by

$$\alpha_1 = - \frac{3.5.\sigma}{4} \left(\frac{3}{7} - \frac{1}{5} \right) , \tag{4.57}$$

where $\sigma = \lambda_{\pm}^{-1}$.

5. SYSTEMS WITH STEP-WISE CHANGING PROPERTIES

In this section an application is given of the simplest case represented by Eq. (2.1). The properties of the system characterizing A are determined by the kernel K and the total mean free-path.

Definition VII.

1° $\lambda^i = K^i =$ constant, is the kernel on $A' \otimes B \otimes B$.

2° σ_t^i is the total cross-section on A^i.

3° a^i, b^i are the limit points of the set A^i.

4° $b^i = \alpha^{i+1}$; $\sigma_t^i \cdot b^i \neq \sigma_t^{i+1} \cdot \alpha^{i+1}$; $\sigma_t^i \alpha^i \equiv \alpha^i$; $\sigma_t^i b^i \equiv \beta^i$.

5° $\psi^i(x, z)$ is the solution on $A^i \otimes B$.

6° $\varphi^i(x) = \int_A \psi^i(x, z) dz$.

7° $x \rightarrow \bar{x} = \sigma_t \cdot x$.

The solution pertaining to each A^i of Def. VII is that given previously by Theorem V (199), where now the boundary conditions are represented by non-vanishing functions at the points α^i and β^i.

For simplicity it will be assumed that the boundary conditions at the first $a^i(i = 1)$ and at the last $b^i(i = I)$ are of the homogeneous Dirichlet type

$$\psi^1(\alpha^1, z) = 0; \ (\forall z \mid z \in B_+) \ , \tag{5.1}$$

$$\psi^I(\beta^I, z) = 0; \ (\forall z \mid z \in B_-) \ . \tag{5.2}$$

In all intermediate surfaces we shall have:

$$\psi^i(\beta^i, z) = \psi^{i+1}(\alpha^{i+1}, z); \{ (i = a, 2, \dots I - 1) \wedge (z \in B) \}. \tag{5.3}$$

Due to the linearity of Eq. (2.1) each inhomogeneous boundary condition implies the existence of two distinct contributions to the solution, i.e., the solutions on $A^i \otimes B$; ($i = 1, I$ (first and last A^i)) consist of three distinct parts.

Each part of the solution contains a set of infinitely many coefficients $\{q^i_n \mid n = 0, 1, 2, \dots]$. Hence the solution on $A^i \otimes B$ is identified by the following sets:

$$\psi^1_+(x, z) \langle \Longleftrightarrow \rangle \{ q^1_n \mid n = 0, 1, 2, \dots \}. \tag{5.4}$$

$$\psi^1_-(x, z) \langle \Longleftrightarrow \rangle \{ p^1_n, \ \bar{p}^1_n \mid n = 0, 1, 2, \dots \} \tag{5.5}$$

The arrow "$\langle \Longleftrightarrow \rangle$" expresses the equivalence relation, provided the kind of boundary conditions at $x = a^1, \beta^1$ is given.

On $A^I \otimes B$ the solution is identified by

$$\psi^I_+(x, z) \langle \Longleftrightarrow \rangle \{q^I_n, \ \bar{q}^I_n \mid n = 0, 1, 2, \dots \} \tag{5.6}$$

and

$$\psi^I_-(x, z) \langle \Longleftrightarrow \rangle \{p^I_n, \ \bar{p}^I_n \mid n = 0, 1, 2, \dots \} . \tag{5.7}$$

Every set $A^i \otimes B$ being neither the first ($i = 1$) nor the last ($i = I$) has the following solution:

$$\psi^i_+(x, z) \langle \Longleftrightarrow \rangle \{q^i_n, \bar{q}^i \mid n = 0, 1, 2, \dots \} \tag{5.8}$$

and

$$\psi^i_-(x, z) \langle \Longleftrightarrow \rangle \{p^i_n, \ \bar{p}^i_n \mid n = 0, 1, 2, \dots \} . \tag{5.9}$$

To simplify the presentation we shall make use of the notation

$\mathscr{B}_{\pm}\{q^i, \bar{q}^i, q^i, \bar{q}^i\} = 0$ to represent the Boltzmann equation on $A^i \otimes B_{\pm}$ for $i \neq 1$, I. If $i = 1$, then we have owing to Eqs. (5.4) - (5.7), $B_{\pm}\{q^1, q^1. \bar{p}^1\} = 0$ and for $i = I$ $\mathscr{B}_{\pm}\{q^I\ q^I.\ p^I\} = 0$.

The boundary conditions at $x = b^i = a^{i+1}$ and $z \in B$ will be represented by $\mathscr{T}_{\pm}\{X^i, X^i, \bar{X}^{i+1}, \bar{X}^{i+1}\}^* = 0$. If $i = 1$, I, then we shall have from Eqs. (5.5) - (5.7) $\mathscr{T}_{+}\{q^1, q^2, \bar{q}^2\} = 0$, $C_{+}\{q^{I-1}, \bar{q}^{i-1}, q^I, \bar{p}^I\} = 0$ for $z \in B_{+}$ and $\mathscr{T}_{-}\{p^1, \bar{p}^1, p^2, \bar{p}^2\} = 0$, $\mathscr{T}_{-}\{p^{I-1}, p^{I-1}, p^I =\} 0$ for $z \in B_{-}$.
From Eqs. (5.4) - (5.9) it follows that we have $4 \times (I - 2) + 2 \times 3 = 4 \times I - 2$ sets of unknown coefficients. The number of equations for their determination is as follows:

I	equations	of the	type		B_{+}
I	»	»	»		B_{-}
I - 1	»	»	»	»	$\mathscr{T}_{+}$
I - 1	»	»	»	»	$\mathscr{T}_{-}$

This gives again $4 \times I - 2$ equations. To obtain $4 \times I - 2$ sets of equations (each with an infinite number of equations) we equate coefficients of equal powers of x or equivalently (since $\varphi^i(x)$ has finite derivatives at all points of $A^{\cdot}$) for the $\mathscr{B}_{\pm}\{\ \}$-type equation we put

$$\partial_x^k \mathscr{B}_{\pm}\{q^i, \bar{q}^i, p^i, \bar{p}^i\}_{x=x^i} = 0\ ;\ k = 0, 1, 2, \ldots;\ x^i = \frac{\alpha^i + \beta^i}{2}. \tag{5.10}$$

For the $\mathscr{T}_{\pm}\{\ \}$-type equations we have similarly

$$\partial_z^k \mathscr{T}_{\pm}\{X^i, \bar{X}^i, \bar{X}^{i+1}, \bar{X}^{-i+1}\}_{z=o} = 0\ ;\ k = 0, 1, 2, \ldots \tag{5.11}$$

Eqs. (5.10) and (5.11) represent a linear homogeneous set of [$(4 \times I - 2)$x infinite] equations from which the unknown coefficients $\{q^i, \bar{q}^i, p^i, \bar{p}^i \mid i = 1, 2, \ldots I, n = 0, 1, 2, \ldots\}$ can be determined. If we write $D(\lambda;1;I)$ for the matrix of the above system of equations, then

$$\det \mathbf{D}(\lambda;1;I) = 0 \tag{5.12}$$

is the solubility condition and the spectral equation of the problem. This is the general straightforward solution of the problem.

We wish next to give a more elegant and simpler solution of the same problem.

* $X \equiv q$ for $z \in B_{+}$ and $X \equiv p$ for $z \in B_{-}$

In this second procedure explicit use will be made of the structural properties of the functions $\psi^i_+(x, z)$. To this end we write first $\psi^i_+(x, z)$ for each $A^i \otimes B_\pm$.

$$\psi^1_+(x, z) = \sum_{n=0}^{\infty} (\lambda^1)^n q^1_n \left[S_n(x-\alpha^1, z) - (-z)^n e^{-\frac{x-\alpha^1}{z}} \right], \tag{5.13}$$

$$\psi^1_-(x, z) = \sum_{n=0}^{\infty} (\lambda !)^n p^1_n \left[S_n(x-\beta^1, z) - (-z)^n e^{-\frac{x-\beta^1}{z}} \right]$$

$$+ \psi^2_-(\alpha^2, z) e^{-\frac{x-\beta^1}{z}}, \tag{5.14}$$

$$\psi^2_+(x, z) = \sum_{n=0}^{\infty} (\lambda^2)^n q^2_n \left[S_n(x-\alpha^2, z) - (-z)^n e^{-\frac{x-\alpha^2}{z}} \right]$$

$$+ \psi^1_+(\beta^1, z) e^{-\frac{x-\alpha^2}{z}}, \tag{5.15}$$

$$\psi^2_-(x, z) = \sum_{n=0}^{\infty} (\lambda^2)^n p^2_n \left[S^n(x-\beta^2, z) - (-z)^n e^{-\frac{x-\beta^2}{z}} \right]$$

$$+ \psi^3_-(\alpha^3, z) e^{-\frac{x-\beta^2}{z}}, \tag{5.16}$$

$$\vdots$$

$$\psi^I_+(x, z) = \sum_{n=0}^{\infty} (\lambda I)^n q^2_n \left[S_n(x-\alpha^I, z) - (-z)^n e^{-\frac{x-\alpha^I}{z}} \right]$$

$$+ \psi^{I-1}_+(\beta^{-1}, z) e^{-\frac{x-\alpha^I}{z}}, \tag{5.17}$$

$$\psi^I_-(x, a) = \sum_{n=0}^{\infty} (\lambda^I)^n p^I_n \left[S_n(x-\beta^I, z) - (-z)^n e^{-\frac{x-\beta^I}{z}} \right]. \tag{5.18}$$

Next we observe that A^i are ordered on the x-axis in the natural order of the indices $\{i\}$. Moreover, the factor $\psi^2_-(a^2, z)$ in Eq. (5.14) is given by the first sum in Eq. (5.16) at $x = \alpha^2$.

Similar relations exist among the other equations. This parametrization of the solutions makes disappear several of coefficients $\{\bar{q}, \bar{p}\}$ and reduces the number of equations for the determination of the remaining ones.

Let us see more precisely how this runs.

The boundary condition satisfied by $\psi^1_+(x, z)$ at $x = \alpha^1$ is an identity provided the convention for taking limits is applied as given by Eq. (1.7). The boundary condition $C_+\{q^1, \bar{q}^2, q^{-2}\}$ becomes now $\mathscr{T}_+\{q^1, q^2, q^1\} \equiv 0$. In general, we have

$$C_+\{q^i, q^{i-1}, q^{i+1}, q^i\} \equiv 0\,, \tag{5.19}$$

$$C_-\{p^i, p^{i+1}, p^{i-1}, p^i\} \equiv 0\,. \tag{5.20}$$

Hence the number of equations reduces to $2 \times I$ since all equations of the C-type become identities. At the same time all variables $\{\bar{q}^i, \bar{p}^i\}$ are expressed by $\{q^i, p^i.\}$ Moreover, owing to Eq. (3.19) which now becomes

$$p^i_k = \sum_{n=k}^{\infty} q^i(\lambda^i)^{n-k} \frac{(\beta^i - \alpha^i)^{n-k}}{(n-k)!}\ ;\ k = 0, 1, 2, \ldots \tag{5.21}$$

I variables are eliminated. Consequently, we are left with I variables and I equations of the $\mathscr{B}_+$-type which is considerably simpler than Eqs. (5.10) - (5.11). It is obvious from Eqs. (5.13) - (5.18) that the equation $\mathscr{B}_+\{\ \}$becomes now

$$\partial^k_x \mathscr{B}_+\{\, q^{i-1}, q^i, q^{i+1}\} = 0; \{(k = 0, 1, 2, \ldots) \wedge (i = 1, 2, \ldots I)\}\,. \tag{5.22}$$

In the case of $i = 1$ we have either $\{q^0\} \equiv 0$, if homogeneous boundary conditions are used or $\{q^0\}$ is given in advance by the left boundary conditions, $\psi_+(z)$. For $i = I$ we have analogously either $\{p^{I+1}\} = 0$, or $\{p^{I+1}\}$ is given in advance by the right boundary condition, $\psi_-(z)$.

To complete Sec. 5 we give next the elements of the matrix $\mathbf{D}(1;1, I)$.

We consider first Eq. (5.22).

$$\sum_{n=k}^{\infty} (\lambda^i)^{n-k} q^i_n \frac{(\delta^i/2)^{n-k}}{(n-k)!} = \lambda^i \sum_{n=0}^{\infty} (\lambda^i)^{n-k} \Bigg\{\Bigg[\, \beta_{kn}(\delta^i/2) + \sum_{\nu=0}^{\infty} \gamma_{k\nu}(-\delta^i/2)\zeta^i{}_{\nu n} \Bigg] q^i{}_n +$$

$$\Gamma_{kn}(\delta^{i-1}, \delta^i) q^{i-1}_n + \Delta_{kn}(\delta^{i+1}, \delta^i)^{i+1}_n \Bigg\}\ ;\ \left(\begin{matrix} k = 0, 1, 2, \ldots \\ i = 1, 2, 3, \ldots I], \end{matrix}\right) \tag{5.23}$$

where β_{kn}, γ_{kn} are defined in Eqs. (3.13) - (3.14).

In Eq. (5.23) we have defined Γ_{kn} and Δ_{kn} by

$$\Gamma_{kn}(\delta^{i-1}, \delta^i) = (\lambda^{i+1})^{n-k} \partial^k_x \int_{B+} \Bigg[S_n(\delta^{i-1}, z) - (-z)^n e^{\frac{\delta^{i-1}}{z}} (-z)^{-k} e^{-\frac{\delta^i}{2z}} \Bigg] dz \tag{5.24}$$

$$\Delta_{kn}(\delta^{i+1}, \delta^{i}) = (\lambda^{i+1})^{n-k}\partial_x^k \int_{B-} \left[S_n(-\delta^{i+1}, z) - (-z)^n e^{-\frac{\delta^{i+1}}{z}} (-z)^{-k} e^{\frac{\delta^{i}}{2z}} \right] dz \quad (5.25)$$

and $\zeta^{i}_{\nu n}$ is as in Eq. (3.24) with d replaced by δ^i given by

$$\begin{aligned}\delta^i &= \beta^i - \alpha^i \\ &= \sigma^i_t(b^i - \alpha^i).\end{aligned} \quad (5.26)$$

Hence the matrix $\mathbf{D}(\lambda;1;I)$ has three types of elements:

(i) The first-kind element, D'_{kn}, is identical to that given in Eq. (3.28).

(ii) $\quad D''_{kn} = \Gamma_{kn}(\delta^{i-1}, \delta^{i})$ (5.27)

and

(iii) $\quad D'''_{kn} = \Delta_{kn}(\delta^{i+1}, \delta^{i})$. (5.28)

It is recalled that

$$\begin{aligned} q^{o}_{n} &= \frac{(-)^n}{n!} (\lambda^1)^{-n} \left. \frac{d^n\psi_{+}(z)}{dz^n} \right|_{z=o}, \\ p^{I+1}_{n} &= \frac{(-)^n}{n!} (\lambda^I)^{-n} \left. \frac{d^n\psi_{-}(z)}{dz^n} \right|_{z=o}, \end{aligned} \quad (5.29)$$

which implies that $q^o_n = p^{I+1}_n \equiv 0$ for all $n \geqq 0$, if homogeneous boundary conditions are applied at $x = \alpha^1, \beta^I$.

It is easy to see that there exist in general I different countable sets $\{\lambda^j_n \mid n = 1, 2, \ldots]$ of eigenvalues. This conclusion follows from the fact that the values of the matrix elements given in Eqs. (3.28), (5.27), (5.28) depend on the combinations $\{\sigma^i_t, \lambda^i\}$ with the sets A^j; i, j = 1, 2, ... I. In order that Eq. (5.12) be satisfied we must have at least one free λ^j in the case of $\psi_{\pm}(z) \equiv 0$. The other λ^i-values can be fixed in advance.

The number of the permutations

$$\begin{pmatrix} \lambda^{i1}\ \lambda^{i2} \ldots \lambda^{ij-i}\ \lambda^{ij} \ldots \lambda^{ij+1} \\ A^1\ A^2 \ldots\ldots\ldots\ldots A^I \end{pmatrix} \quad (5.30)$$

is equal to I! and the set of values $\{\lambda^j\}$ taken on by λ^j in one permutation are in general different from the values taken on by $\lambda^{j'}$ in another permutation, because at least six columns of $D(\lambda;1;I)$ for a given permutation (5.30) are in general different from the corresponding columns of $D(\lambda;1;I)$ for another permutation provided $d^1 \neq d^2 \neq \dots \neq d^I$. Hence, we have $(I!)$ $\{\lambda_n^j \mid n = 1, 2, \dots\}$ sets of eigenvalues.

This conclusion is of importance from the physical point of view in optimisation considerations.

We have therefore established the following

Theorem XV. Let $A^i; i = 1, 2, \dots I$ be closed sub-sets of R^1 with limit points $0 < d^i < b^i$ such that $d^{i+1} = b^i = 1, 2, \dots I-1$. Let $K^i(z, z') = \lambda^i$, σ_t^i be the values of the constant kernel on $A^i \otimes B$ and of the constant total cross-section on A^i respectively. Let further $\psi_{\lambda j}(x, z)$ be the solution of Eq. (2.1) valid on $(\bigcup_{i=1}^{I} A^i) \otimes B$. Then:

1° $\psi_{\lambda j}(x, z)$ is given by Eqs. (5.13) - (5.14) and is uniformly continuous everywhere on $(\bigcup_{i=1}^{I} A^i)^{\cdot} \otimes B$, where $(\bigcup_{i=1}^{I} A^i)^{\cdot}$ is given by $\bigcup_{i=1}^{I} A^i - \{d^i b^i \mid i = 1, 2, \dots I\}$.

2° $\psi_{\lambda j}(x, z)$ has finite partial derivatives of any order everywhere on $(\bigcup^{I} A^i)^{\cdot} \otimes B$ and on $(\bigcup_{i=1}^{I} A^i) \otimes B^{\cdot}$.

3° The elements of $\psi_{\lambda j}(x, z)$, $\{\psi^i_{\pm}(x, z) \mid i = 1, 2, \dots I\}$ satisfy exactly homogeneous or inhomogeneous Dirichlet conditions.

4° There are $(I!)$ countable sets of eigenvalues which are different from each other, if $\{\lambda^i, \sigma_t^i\} \neq \{\lambda^j, \sigma_t^j\}$ and $A^i \neq A^j$; $i, j = 1, 2, \dots I\}$.

6. DEGENERATE υ-, X-DEPENDENT KERNEL

The present section is concerned with an application of the theory to the velocity and angle-dependent kernel. This somewhat more general case with respect to kernels considered hitherto does not in fact present any new aspects from the methodological point of view.

It will be supposed that the kernel has the form

$$K(\upsilon, \upsilon'; z, z') = \sigma_t(\upsilon) \sum_{e=0}^{L} f_e(\upsilon, z) h_e(\upsilon', z'); \{\forall \upsilon', \upsilon, z, z' \mid (\upsilon, \upsilon' \in U \otimes U) \wedge (z, z' \in B \otimes B)\}, \tag{6.1}$$

where U is the velocity space.

A special case of Eq. (6.1) is the first order degenerate kernel $K(\upsilon, \upsilon') = M(\upsilon') \Sigma_S(\upsilon) \Sigma_S(\upsilon) / \bar{\Sigma}_S$ where $M(\upsilon)$ is the Maxwell distribution and $\bar{\Sigma}_S$ is the Maxwellian mean of the scattering cross-section $\Sigma_s(\upsilon)$. No restrictions will be imposed on the functions $\{f_e(\upsilon, z), h_e(\upsilon, z) \mid l = 1, 2, \ldots L\}$ except the conditions for normalisation

$$\int_U d\upsilon \int_B f_e(\upsilon, z) h_{e'}(\upsilon, z) g_n(x, \upsilon, z)\, dz < \infty \tag{6.2}$$

for all $1 \leqslant l, l' \leqslant L$ all $n \geqslant 0$ and all $x \in A$, here

$$g_n(x, \upsilon, z) = \begin{cases} S_n\left(x - a, \dfrac{z}{\sigma_t(\upsilon)}\right) - \left(\dfrac{-z}{\sigma_t(\upsilon)}\right)^n e^{-\dfrac{x-a}{z/\sigma_t(\upsilon)}} \;;\; \{\forall x, \upsilon, z \mid (x \in A) \wedge (\upsilon \in U) \wedge (z \in B_+)\} \\[2ex] S_n\left(x - b, \dfrac{z}{\sigma_t(\upsilon)}\right) - \left(\dfrac{-z}{\sigma_t(\upsilon)}\right)^n e^{-\dfrac{x-b}{z/\sigma_t(\upsilon)}} \;;\; \{\forall x, \upsilon, z \mid (x \in A) \wedge (\upsilon \in U) \wedge (z \in B_-)\}. \end{cases} \tag{6.3}$$

This implies that δ-functions may well be present in $K(\upsilon, \upsilon'; z, z')$ because it is not required that Eq. (6.1) be square integrable. According to Sec. 4 we write the solution in the form

$$\psi_+(x, \upsilon, z) = \sum_{e=1}^{L} f_e(\upsilon, z)\psi_{e+}(x, \upsilon, z); \{\forall x, \upsilon, z \mid (x \in A) \wedge (\upsilon \in U) \wedge (z \in B_-)\} \tag{6.4}$$

and

$$\psi_-(x, \upsilon, z) = \sum_{e=1}^{L} f_e(\upsilon, z)\psi_{e-}(x, \upsilon, z) \;;\; \{\forall x, \upsilon, z \mid (x \in A) \wedge (\upsilon \in U) \wedge (z \in B_-)\} \tag{6.5}$$

and impose for definiteness the boundary conditions

$$\psi_+(a, \upsilon, z) = 0 \tag{6.6}$$

and

$$\psi_-(b, \upsilon, z) = 0\,. \tag{6.7}$$

Now we wish to establish the conditions for which Eqs. (6.4) - (6.5) satisfy Eqs. (1.1) and (6.6 - 6.7), where the functions

$\{\psi_{e\pm}(x, \upsilon, z) \mid l = 1, 2, \ldots I\}$ are given by the expressions

$$\psi_{e+}(x, \upsilon, z) = \sum_{n=0}^{\infty} (\lambda)^n . \left[S_n(x - a, \zeta) - (-\zeta)^n e^{-\frac{x-a}{\zeta}} \right] . q_{en} \tag{6.8}$$

$$\psi_{e-}(x, \upsilon, z) = \sum_{n=0}^{\infty} (\lambda)^n . \left[S_n(x - b, \zeta) - (-\zeta)^n e^{-\frac{x-b}{\zeta}} \right] . p_{en} \tag{6.9}$$

with $\zeta = z / \sigma_t(\upsilon)$ and for $l = 1, 2, \ldots L$.

We shall use the following

Definition VIII.

$$1^o \ \bar{\beta}^{ee'}_{kn}(x-a) = \partial^k_x \int_U d\upsilon \int_{B+} dz \left[S_n(x-a, \zeta) - (-\zeta)^n e^{-\frac{x-a}{\zeta}} \right] f_{e'}(\upsilon, z) h_e(\upsilon, z)$$

$$2^o \ \bar{\gamma}^{ee'}_{kn}(x-b) = \partial^k_x \int_U d\upsilon \int_{B-} dz \left[S_n(x-b, \zeta) - (-\zeta)^n e^{-\frac{x-b}{\zeta}} \right] f_{e'}(\upsilon, z) h_e(\upsilon, z)$$

for $k = 0, 1, 2, \ldots$

$3^o \ \varphi_{e\pm}(x) = \int d\upsilon \int dz h_e(\upsilon, z) \psi_\pm(x, \upsilon, z)$

$4^o \ \varphi_e(x) = \varphi_{e+}(x) + \varphi_{e-}(x)$.

Next we observe that owing to the positivity of $\sigma_t(\upsilon)$; $\upsilon \in U$, the exponentials in Eqs. (6.8 - 6.9) are finite and unique everywhere on $A^\bullet \otimes B \otimes U$. Hence, the quantities $\bar{\beta}^{ee'}_{kn}(x-d)$, $\bar{\gamma}^{ee'}_{kn}(x-b)$ are well defined on $A^\bullet$.

From Eqs. (1.1), (6.4 - 6.9) and from Def. VIII it follows upon multiplying by ∂^k_x that

$$\sum_{e'=1}^{L} f_{e'}(\upsilon, z) \sum_{n=k}^{\infty} \lambda^{n-k} q_{e'n} \frac{(x-a)^{n-k}}{(n-k):} - \lambda \sum_{e=1}^{L} f_e(\upsilon, z) \varphi_e(x) = 0. \tag{6.10}$$

Eq. (6.10) must be satisfied everywhere on $A \otimes B \otimes U$.

Equating coefficients of $f_{e'}(\upsilon, z)$, $f_{e'}(\upsilon, z)$ for $l' = 1$ and defining the new quantities

$$\tilde{\beta}^e_{kn}(x-)\alpha q_{en} = \sum_{e'=1}^{L} \bar{\beta}^{ee'}_{kn}(x-\alpha) q_{e'n},$$

$$\tilde{\gamma}^e_{kn}(x-b) p_{en} = \sum_{e'=1}^{L} \bar{\gamma}^{ee'}_{kn}(x-b) p_{e'n}$$

we can bring Eq. (6.10) in the form

$$\sum_{n=0}^{\infty} \lambda^n \frac{(x-d)^n}{n!} q_{en} = \lambda \sum_{n=0}^{\infty} \lambda^n \left[\tilde{\beta}^e_n(x-d) q_{en} + \sum_{v=0}^{n} \tilde{\gamma}^e_n(x-b) p_{en} \right]. \tag{6.11}$$

Eq. (6.11) is formally identical to the equation valid for the case of isotropic and velocity-independent kernels.

Again putting equal to zero the coefficients of $f_e(\upsilon, z)$; $(l = 1, 2, \ldots L)$ in Eq.

(6.10) and evaluating the derivatives ∂_x^k at $x = \dfrac{\alpha + b}{2}$ we get the following set of equations for the determination of $\{q_{en} \mid e = 1, 2, \ldots L; n = 0, 1, 2, \ldots\}$:

$$\sum_{n=1}^{\infty} \lambda^n \sum_{e=1}^{L} \lambda \left\{ \lambda \left[\bar{\beta}_{kn}^{ee'}(d/2) + \sum_{\nu=0}^{n} \bar{\gamma}_{k\nu}^{ee'}(-d/2)\zeta_{\nu n} \right] - \alpha_{kn}\delta_{ee'} \right\} q_{en} = 0 \tag{6.12}$$

for all $l' = 0, 1, 2, \ldots L$ and $k = 0, 1, 2, \ldots$ where $\zeta_{\nu n}$, α_{kn} are given by Eqs. (3.24) and (3.28) respectively.

It should be observed that the functions $\varphi_{e\pm}(x)$ given in Def. VIII are not identical with the angle independent flux $\varphi(x, \upsilon)$. The latter being given by

$$\varphi(x, \upsilon) = \sum_{n=0}^{\infty} \lambda^n \sum_{e=0}^{L} \left\{ \int_{B+} dz f_e(\upsilon, z) \left[S_n(x-\alpha, \zeta) - (-\zeta)e^{n-\frac{x-\alpha}{\zeta}} \right] q_{en} + \right.$$

$$\left. + \int_{B-} dz f_e(\upsilon, z) \left[S_n(x-b, \zeta) - (-\zeta)^n e^{-\frac{x-b}{\zeta}} \right] p_{en} \right\}. \tag{6.13}$$

In the case of isotropic scattering ($K(\upsilon, \upsilon'; z, z')$ is constant on $B \otimes B$) Eq. (6.13) simplifies to the following expression:

$$\varphi_{\text{isotr.}}(x, \upsilon) = \sum_{n=1}^{\infty} \left[\frac{\lambda}{\sigma_t(\upsilon)} \right]^n \sum_{e=1}^{L} f_e(\upsilon) \left\{ \sum_{\nu=0}^{L} \frac{[\sigma_t(\upsilon)]^{n-\nu}}{(n-\nu)!(1+\nu)} [(b-x)^{n-\nu} p_{en} + (-)^\nu (x-\alpha)^{n-\nu} q_{en}] \right.$$

$$\left. - (-)^n E_{n+2}[\sigma_t(\upsilon)(x-\alpha)] q^-_{en} \, E_{n+2}[\sigma_t(\upsilon)(b-x)] p_{en} \right\}. \tag{6.14}$$

$\varphi_{\text{isotr}}(x, \upsilon)$ with its first-order derivative is continuous everywhere on A including the boundary points a, b, of A.

We have therefore established the following

Theorem XVI.

Let the kernel $K(\upsilon, \upsilon'; z, z')$ be represented on $U \otimes U \otimes B \otimes B$ by the sum

$$K(\upsilon, \upsilon'; z, z') = \sigma_t(\upsilon) \sum_{i=1}^{L} f_e(\upsilon, z) h_e(\upsilon', z'),$$

which may or may not be square-integrable.

Let the functions $\{f_e, h_e \mid l = 1, 2, \ldots L\}$, which may contain terms with delta-functions* be such that the super-matrix

(*) or other distributions.

$$D(\lambda;1;L) = (\bar{\beta}^{ee'}_{kn}(d/2) + \sum_{n=o}^{n} \bar{\gamma}^{ee'}_{k\nu}(-d/2)\zeta_{\nu n}) \tag{6.15}$$

has a finite determinant

$$|\det D\ (\lambda;1;L)\ | < \infty \tag{6.16}$$

for a non-empty set Λ_L of λ-neighbourhoods.

Then:

1° The general solution of Eq. (2.1) is represented by superpositions of constant-kernel solutions as given by Eqs. (6.8 - 6.9).

2° The solution satisfies homogeneous or inhomogeneous Dirichlet boundary conditions.

3° The sets of coefficients $\{q_{en}, p_{en} \mid l = 1, 2, \dots L\}$ are constants on $A \otimes B \otimes U$.

4° The equation

$$\det D(l;1;L) = 0 \tag{6.17}$$

is the spectral equation and the eigenvalues are constants on U.

Remark XIII.

From Eqs. (6.8 - 6.9) it follows that the case $z = 0$ is equivalent to the case $\sigma_t(\upsilon) = \infty$ and vice-versa.

Moreover, we have the continuity relations

$$\psi_{e+}(x, \upsilon, o) = \sum_{n=0}^{\infty} (\lambda)^n q_{en} \frac{(x-a)^n}{n!} = \varphi_e(x)$$

and also

$$\psi_{e-}(x, \upsilon, o) = \sum_{n=0}^{\infty} (\lambda)^n p_{en} \frac{(x-b)^n}{n!} = \varphi_e(x)$$

Inspite of the above results $\psi_\pm(x, \upsilon, o)$ remains energy-dependent owing to the factors $f_e(\upsilon, 0)$ in Eqs. (6.4 - 6.5).

Remark XIV.

The development of the present theory tacitly makes use of the theory of distributions. This becomes clear from the definition of the coefficients

$$q_n = \partial^n_x \varphi(x) \mid_{x=x_0} ;\ x_0 \in A^{\bullet}.$$

This can be seen as follows:

The coefficients in question are defined on a sub-set of A which is a compact sub-set of R^1. The linear form in the sense of distribution theory is the operator

$$T_n = z^n \frac{\partial^n}{\partial x^n} \; ; \; n = 0, 1, 2, \ldots,$$

which according to Eq. (2.35) satisfies the criterion for a linear form to be a distribution:

$$| < T_n, \varphi(x) > | \leq \frac{M}{r^n},$$

where $M_n = \sup_{x \in A} \sup_{z \in B_+} \psi\,(x, z) < \infty$ and r is a constant.

On the other hand, due to Theorem I, Corollary II, there holds

$$\psi(x, 0) = \int_B \psi(x, z) dz = \varphi(x).$$

Therefore, there holds also

$$| < T_n, \varphi(x) > | \leq M'_n < \infty$$

and consequently the coefficients $\{q_n \mid n = 0, 1, 2, \ldots\}$ constitute a set of values for the distributions $\{T_n\}$, whereby $\varphi(x)$ is a test function. It is not difficult to show that the distribution function is the set of values of another distribution, $\widehat{\psi}(x, z)$. Theorem II, Corollary II states the conditions for which a distribution T_n can be calculated when the distributions $\{T_{n'} \mid n' = 0, 1, 2, \ldots n-1\}$ are given. With the help of them the distribution $\widehat{\psi}(x, z)$ can be expressed. To see this let us expand the distribution function $\psi(x, z)$ in a Taylor series in the right neighborhood of the point $a \in A$:

$$\psi(x, z) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \psi^{(n)}\,(a, z).$$

From Eq. (2.22) it follows that

$$\partial_x^n \psi(x, z) = \psi(x, z) - \sum_{v=0}^{n-1} (-z)^{v-n} \partial_x^v \, \varphi(x).$$

One obtains then immediately

$$\psi(x, z) = \psi(a, z) e^{-\frac{x-a}{z}} - \sum_{n=0}^{\infty} \sum_{v=0}^{n-1} (-z)^{v-n} \frac{(x-a)^n}{n!} \partial_x^v \, \varphi(x) \mid x = a \, .$$

Upon rearranging terms conveniently we find

$$\psi(x, z) = \psi(a, z)e^{-\frac{x-a}{z}} + \left[\frac{(x-a)}{z.1!} T_0 - \frac{(x-a)^2}{z^2 2!}(T_0-T_1) + \frac{(x-a)^3}{z^3 3!}(T_0-T_1+T_2) - \frac{(x-a)^4}{z^4 4!}(T_0-T_1+T_2-T_3) + \ldots\right] \varphi(x) \mid_{x=a} .$$

Upon summing-up terms having as a common factor T_n for $n = 0, 1, 2, \ldots$ we get

$$\psi(x, z) = \psi(a, z)e^{-\frac{x-a}{z}} + \sum_{n=0}^{\infty} \left(S_n(x-a, z) - (-z)^n e^{-\frac{x-a}{z}}\right) T_n\varphi(x) \mid_{x=a} ,$$

where S_n are the polynomials given previously. Obviously the distribution $\widehat{\psi}(x, z)$ is given by

$$\widehat{\psi}(x, z) = \psi(a, z)e^{-\frac{x-a}{z}} + \sum_{n=0}^{\infty} \left(S_n(x-a, z) - (-z)^n e^{-\frac{x-a}{z}}\right) T_n .$$

PART C

7. MANY-DIMENSIONAL SYSTEMS

The first part of the present work was concerned exclusively with transport problems in one-dimensional spaces, R^1. The purpose of the present part C is to give an account of the generalisations obtained sofar and to examine possibilities for further generalisations.

To be more specific we shall consider finite, many-dimensional systems bounded by convex surfaces of arbitrary shapes.

As basic equation will again be considered the Boltzmann equation

$$\left\{\partial_t + \vec{\upsilon} \cdot \nabla + \upsilon\Sigma_{tot}(\upsilon)\right\}\psi(x, \upsilon) = \lambda \int d\upsilon'^p \; K(\vec{\upsilon}, \vec{\upsilon}')\psi(\vec{x}, \vec{\upsilon}') \,, \tag{7.1}$$

where λ is the eigenvalue parameter.

We shall make use of the following

Definition IX.

1°. R^p is a sub-set of the p-dimensional Euclidean space E^p, $R^p \subseteq E^p$, containing all the points of a given physical system, such that $\vec{x} \in R^p$ is a point of the system.

2°. S is the convex surface of the physical system R^p, such that the normal on S at $\vec{x}'$ does not, if prolonged, meet S again for all $\vec{x}'$ on S.

3°. Σ is a sub-set of S, $\Sigma \subseteq S$, such that $\vec{x}' \in \Sigma$ is a source point on Σ.

4°. $\vec{n}$ is the outwards pointing normal to Σ at $\vec{x}'$.

5°. $\vec{\upsilon}$ is a vector independent of $\{\vec{x}, \vec{x}', \vec{n}\}$ belonging to the p-dimensional vector space U^p. U^p is the velocity space of the particles in a cell around the point $\vec{x} \in R^p$ of the system and $\vec{\Omega}$ is a unit vector $\vec{\Omega} = \vec{\upsilon}/\upsilon$ $(\upsilon = \vec{\upsilon})$.

6°. The operator polynomials S_n are given by:

$$S_n(A, B) = \sum_{\nu=0}^{\infty} \frac{(-)^\nu}{(n-\nu)!} A^{n-\nu} B^\nu , \tag{7.2}$$

where A, B are any operators acting on functions defined on $R^p \otimes U^p$.

7o. Further useful operator functions are given by:

$$H_n = \exp\left(-\frac{(\vec{x}-\vec{x}') \cdot \vec{n}}{\vec{\Omega} \cdot \vec{n}}\right)(-B)^n \tag{7.3}$$

and

$$G_n = \exp\left(-(\vec{x}-\vec{x}') \cdot \vec{\Omega}\right)(-B)^n . \tag{7.4}$$

8o. The reduced streaming operator is given by $\vec{\Omega} \cdot \nabla + 1$.

Remark XX.

From Def. XI, 6o it is seen that

$$S_n(-A, -B) = (-)^n S_n(A, B) . \tag{7.5}$$

We shall use the above definitions to prove the following

Theorem XVII.

Let A and B be given by $A \equiv (\vec{x}-\vec{x}') \cdot \nabla'$ and $B = \vec{\Omega} \cdot \nabla'$, where $\nabla' \equiv \partial_{\vec{x}'}$.
Then, if $\nabla \equiv \partial_{\vec{x}}$, there holds for every positive integer n

1o. $\nabla S_n(A, B) = -S_{n-1}(A, B)$ (7.6)

2o. $(B+1)S_n(A, B) = \dfrac{A^n}{n!}$. (7.7)

Proof: Assertion 1o is proved by construction.
The proof of 2o is based on the proof of 1o using the definition of the operator polynomials.

Remark XXI.

From Eq. (7.6) it is seen that the action of the streaming operator, B + 1, on the polynomial $S_n(A, B)$ renders it independent of B for all n. This property simplifies the construction of the distribution function satisfying Eq. (7.1).

Theorem XVIII.

The streaming operator annihilates the operator function H_n and G_n.
Proof: The proof is immediate.

Remark XXII.

The general solution of Eq. (7.1) may contain linear superpositions of the functions H_n and G_n with arbitrary constant coefficients.

In the present section we generalize some results obtained sofar to the case of any number, p, of dimensions using a degenerate velocity dependent scattering kernel. Apart from being degenerate this kernel will be otherwise arbitrary and in particular it will not need to be a Hilbert-Schmidt kernel.

More specifically we shall consider kernels of the form:

$$K(\vec{\upsilon}, \vec{\upsilon}') = \sum_{e=1} f_e(\vec{\upsilon})\, h_1(\vec{\upsilon}');\ \{\forall \vec{\upsilon} \text{ and } \forall \vec{\upsilon}' \mid \vec{\upsilon} \in U \text{ and } \vec{\upsilon}' \in U\}. \tag{7.8}$$

f_1', h_1 will be summable functions on U^p. The equation to be investigated here is the Boltzmann linear equation of the form:

$$(\vec{\upsilon} \cdot \nabla + \upsilon\sigma(\upsilon))\, \psi(\vec{x}, \vec{\upsilon}) = \lambda \int_U d\upsilon' P\, K(\vec{\upsilon}, \vec{\upsilon}')\psi)\vec{x}, \vec{\upsilon}') \tag{7.8'}$$

and $\upsilon\sigma(\upsilon)$ is either equal to $\upsilon\Sigma_{tot}(\upsilon)$, if the problem is stationary, or it equals $\mu + \upsilon\Sigma_{tot}(\upsilon)$ in the time-dependent case. In the second case the distribution function is given by:

$$\psi(\vec{x}, \vec{\upsilon}) = \mathscr{L}_t\{\psi(\vec{x}, \vec{u}, t)\},$$

where $\mathscr{L}_t\{\psi\}$ is the Laplace transform of the time-dependent distribution function and μ is the corresponding spectral parameter. The system for which Eq. (7.1) will be investigated is bounded by the convex surface S, such that V^p is the set of all the points $\vec{x} \in R^p$ inside S.

$\vec{x}'$ is a point belonging to the boundary surface, such that $\vec{x}' \in \Sigma$, where Σ is a subset of S such that the ingoing flux for $(\vec{n} \cdot \vec{\upsilon} < 0)$ $\psi_-(\vec{x}, \vec{x}', \vec{\upsilon})$ satisfies the boundary condition:

$$\psi_-(\vec{x}, \vec{x}', \upsilon) = \begin{cases} \psi_\Sigma(\vec{x}, \vec{\upsilon});\ \vec{x} = \vec{x}' \in \Sigma \\ 0\ ;\ \text{otherwise} \end{cases} \tag{7.9}$$

$\psi_\Sigma(\vec{x}', \vec{\upsilon})$ is the flux entering the surface S at $\vec{x}'$ and is an arbitrary given function.

In work published previously the solutions of Eq. (7.1) were expressed as a superposition of operator polynomials $S_n(A, B)$ and exponential functions as given by Eqs. (7.3) - (7.4).

These operator functions are a generalization of algebraic polynomials deduced in ref. (199) by using the structural properties of the distribution function on R^1 and by conveniently rearranging a converging series.

The operator functions in Eqs. (7.3), (7.4) can be used to construct the solution to Eq. (7.8) for every system bounded by the convex surfaces S with finite diameter $|\vec{\delta}|$ and satisfying homogeneous or inhomogeneous Dirichlet boundary conditions.

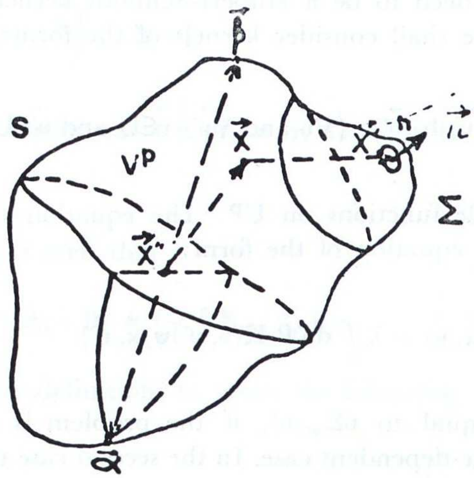


Fig. 8. On $\Sigma \subseteq S$ the surface-source function $\psi_\Sigma(\vec{x}', \vec{\upsilon})$ determines the boundary condition on S for $\vec{n}\cdot\vec{\upsilon} < 0$. For $\vec{n}\cdot\vec{\upsilon} > 0$ and $\vec{x}' \in S\text{-}\Sigma$ per definition the equation $\psi_\Sigma(\vec{x}', \vec{\upsilon}) = 0$ holds. The vector diameter, $\vec{\delta} = \max(\vec{\beta}\text{-}\vec{\alpha} \mid \vec{\alpha}, \vec{\beta} \in V_P)$, of V_p is finite. $\vec{x}''$ is a reference point $\vec{x}'' \in V_P$ and $\vec{x}$ is the observation point. $\psi_-(\vec{x}, \vec{x}', \vec{\upsilon})$ is the distribution function at $\vec{x}$ due to the surface source $\psi_\Sigma(\vec{x}', \vec{\upsilon})$ at $\vec{x}'$ for $\vec{n}.\vec{\upsilon} < 0$.

To make clear the method we proceed in a way close to the experimental observation methods. As a matter of fact in order to measure experimentally the distribution function at a given point $\vec{x}$ of the system (see Fig. 8) we introduce from the outside of S an appropriate detector along a given geometric line $\vec{x} - \vec{x}'$. At the point $\vec{x}'$ a surface source $\psi_\Sigma(\vec{x}', \vec{\upsilon})$ may $(\vec{x}' \in \Sigma)$ or not $\vec{x}' \in \Sigma - \Sigma)$ exist.

Next we choose a series representation of the ingoing part of the distribution function $\psi(\vec{x}, \vec{x}', \vec{\upsilon})$ along the line $\vec{x} - \vec{x}'$ such, that:

(i) $\psi_-(\vec{x}, \vec{x}', \vec{\upsilon}) = \psi_\Sigma(\vec{x}', \vec{\upsilon})$ for $\vec{x} = \vec{x}'$.

(ii) The coefficients of the series are determined such, that Eq. (7.1) is satisfied.

(iii) The outgoing part $\psi_+(\vec{x}, \vec{x}', \vec{\upsilon})$ is developed in a series along $\vec{x} - \vec{x}''$ and the coefficients are determined in such a way, that Eq. (7.1) is again satisfied.

The point $\vec{x}''$ belongs to the set V^p, $\vec{x}'' \in V^p - S$.

(iv) The distribution function $\psi(\vec{x}, \vec{x}', \vec{\upsilon})$ satisfies the condition

$$\int_{U^p} d\upsilon^p \int_{V^p} dx^p \int_{\Sigma} dx'^{p-1}\, \psi(x, x', \upsilon) = C, \tag{7.10}$$

where $\psi(\vec{x}, \vec{x}', \vec{\upsilon}) = \psi_-(\vec{x}, \vec{x}', \vec{\upsilon}) + \psi_+(\vec{x}, \vec{x}', \vec{\upsilon})$, and C is a positive finite constant.

Due to the complexity of the subject we decompose in the sequel the presentation into a number of sections of progressing generality. In addition we shall omit the arguments $\vec{x}'$, $\vec{x}''$ for simplicity and we shall write the distribution function in the form $\psi(\vec{x}, \vec{\upsilon})$.

8. DISTRIBUTION FUNCTIONS FOR CONSTANT KERNEL

Next we consider the simple case of a constant scattering kernel and we observe that in this case the equation

$$(\vec{\Omega}.\nabla + 1)\exp\left(- \frac{(\vec{x} - \vec{x}') \cdot \vec{n}}{\vec{\Omega}.\vec{n}} \right) (\vec{\Omega} \cdot \nabla')^n\, \varphi(\vec{x}') = 0 \;;\; \vec{\Omega} = \frac{\vec{\upsilon}}{\upsilon} \tag{8.1}$$

is satisfied according to Theorem XVIII. Theorem XVIII suggests that we can construct two kinds of solutions to Eq. (7.1) $\psi^i_\pm$; $(i = 1, 2)$ for constant kernel with fundamentally different analytical behaviours.

First kind

$$\psi^1_- = \sum_{n=0}^{\infty} \left(S_n((\vec{x} - \vec{x}') \cdot \nabla', \vec{\Omega} \cdot \nabla') - \exp\left(\frac{(\vec{x} - \vec{x}') \cdot \vec{n})}{\vec{\Omega} \cdot \vec{n}}\right) (- \vec{\Omega} \cdot \nabla')^n \right) \varphi(\vec{x}')$$

$$+ \psi_\Sigma(\vec{x}', \vec{\Omega}) \exp\left(- \frac{(\vec{x} - \vec{x}') \cdot \vec{n}}{\vec{\Omega} \cdot \vec{n}} \right) \quad \vec{n} \cdot \vec{\Omega} < 0 \tag{8.2}$$

and

$$\psi^1_+ = \sum_{n=0}^{\infty} \left(S_n(\vec{x} - \vec{x}'') \cdot \nabla'', \vec{\Omega} \cdot \nabla'' \right) \varphi(\vec{x}''), \quad \vec{n} \cdot \vec{\Omega} > 0. \tag{8.3}$$

Second kind

$$\psi^2_- = \sum_{n=0}^{\infty} \left(S_n((\vec{x} - \vec{x}')\nabla', \vec{\Omega} \cdot \nabla') - \exp\left(- (\vec{x} - \vec{x}) \cdot \vec{\Omega} \right) (- \vec{\Omega} \cdot \nabla') \right) \varphi(\vec{x}') +$$

$$+\psi_\Sigma(\vec{x},\vec{\Omega})\exp\left(-\vec{x}-\vec{x}'.\vec{\Omega}\right) ; \vec{n}.\vec{\Omega}<0 \tag{8.4}$$

and

$$\psi^2{}_+ = \sum_{n=0}^{\infty} S_n\left((\vec{x}-\vec{x}'').\nabla'',\vec{\Omega}.\nabla''\right)\varphi(\vec{x}'') ; \vec{n}.\vec{\Omega}>0, \tag{8.5}$$

where the polynomials S_n have been given in Def. IX, 6º.

Obviously Eqs. (8.2), (8.3) give the value of the distribution function on the segments $\vec{x}-\vec{x}'$. Moreover, considering the boundary condition

$$\psi_-(\vec{x},\vec{\upsilon}) = \begin{cases} \psi_\Sigma(\vec{x},\vec{\upsilon}) ; \left(\forall\vec{x}\,\middle|\,\vec{x}=\vec{x}'\right)\Lambda\,\vec{x}'\in\Sigma \\ 0 \qquad ; \left(\forall\vec{x}\,\middle|\,\vec{x}=\vec{x}'\right)\Lambda\vec{x}'\in S-\Sigma \end{cases} \tag{8.6}$$

we see that the behavior of the distribution function changes according to whether $\vec{x}'$ belongs to Σ or to $S-\Sigma$.

Regarding the difference between $\psi^1{}_-$ and $\psi^2{}_-$ it is observed that, for $\vec{\upsilon}_0 \perp \vec{n}$, $\psi^1{}_-$ is irregular becoming non-uniformly continuous at $\vec{\upsilon}_0$, $\vec{x}=\vec{x}'$. On the contrary, $\psi^2{}_-$ is regular everywhere on $R^p \otimes U^p$. In addition, it is pointed out that the distribution functions, Eqs. (8.2), (8.4), are in fact general, since they are represented by all positive powers of the components of the vectors $\{\vec{x},\vec{\upsilon}\}$ and satisfy any boundary condition represented by given functions. This is an expression of the completeness of the operator functions S_n and H_n, with respect to the possible boundary functions $\psi_\Sigma(\vec{x}',\vec{\upsilon})$. Even in the general case, i.e., the case of non-constant kernel, it would be admitted that, due to the arbitrary choice of functions $\{\varphi_e\}$ the polynomials $\{S_n(A, B)\varphi_e\}$ form a complete set. Moreover, since S_n and $(-\vec{\Omega}.\nabla')^n$ are polynomials of ∇', the resulting equations for the unknown coefficients $D^k\varphi_e(\vec{x}')$ (See ref. 206), obtained by inserting $\psi^i{}_\pm$ in Eq. (7.1) and equating to zero the co efficients of all powers $\partial^{k_1}_{x_1}\,\partial^{k_2}_{x_2}\ldots\partial^{k_p}_{x_{k_p}}$, are algebraic. From these equations the coefficients $D^k\varphi_e(\vec{x}')$ and the corresponding eigenvalues λ_ν, $\nu=1, 2, \ldots$, can be determined.

To conclude this Section we make one more remark concerning the multiplicity of the coefficients in Eqs. (7.8), (7.10). These coefficients are used for the superposition of the polynomials S_n and exponentials H_n and follow from the action of ∇' on $\varphi_e(\vec{x}')$ or of ∇'' on $\varphi_e(\vec{x}'')$. Obviously, the operator αD^k acting on $\varphi_e(\vec{x}')$ generates as many unknown coefficients as the number of its different terms is. These coefficients are in general different. On the other hand it is well known that $\nabla' \varphi_e(\vec{x}')$ is a vector giving the direction of maximal variation of $\varphi_e(\vec{x}')$. For systems in which $-\nabla'\varphi_e(\vec{x}')$ is parallel to the outer normal $\vec{n}$ on S at $\vec{x}'$ for all $\vec{x}' \in S$, the number of coefficients reduces considerably and the corresponding equations for their determination are simpler.

In fact, if we replace $\vec{\Omega} \, . \, \nabla'\varphi(\vec{x}')$ by $\vec{\Omega} \, . \, \vec{n}q$, where q is any real number, we have instead of p coefficients only one coefficient. This property suggests a considerable simplification of the distribution function, when the surface S is spherically symmetric. The symmetric case is considered in the next Section.

Remark XXIII.

In the many dimensional space, E^p, the compact sub-set is represented by R^p. Due to the vector character of $\vec{x} - \vec{x}'$ and of ∇' the polynomials $S_n(A, B)$ are now themselves distributions. The test function in the present case is again

$$\varphi(\vec{x}) = \int \psi(\vec{x}, \vec{\upsilon}) d\upsilon^3.$$

The distribution giving rise to the distribution function $\psi(\vec{x}, \vec{\upsilon})$ is now a linear form of the polynomials $S_n(A, B)$. For example,

$$\hat{\psi}^1_+(\vec{x}, \vec{\upsilon}) = \sum S_n (A, B) \; ; \; \vec{n} \, . \, \vec{\Omega} > 0, \text{ etc.}$$

It appears, therefore, that the theory of distributions is the natural way of expressing the distribution functions in the present theory.

9. DISTRIBUTION FUNCTION FOR SYMMETRICAL SYSTEMS

We shall now consider in more details systems characterized by the relation

$$-\nabla'\varphi(\vec{x}') \,\Big\|\, \vec{n} \tag{9.1}$$

at every point $\vec{x}'$ of the system's surface, S.

Relation (9.1) is a consequence of the geometric symmetry and of the homogeneity of the physical system. From this it follows that (see Fig. 9) we have also ..

$$-\lim_{\vec{x} \leftarrow \vec{x}''} \nabla'' \varphi(\vec{x}'') \,\|\, \vec{n}\,.$$

Therefore, from Eqs. (8.2) - (8.5) we get the following simplified distribution function, provided $\vec{n}$ is taken at the point $\vec{x}'$, where the line $\vec{x} - \vec{x}'' + t \cdot (\vec{x} - \vec{x}'') \,/\, \left|\vec{x} - \vec{x}''\right|$, $(t > 0)$, intersects the surface S, of the system.

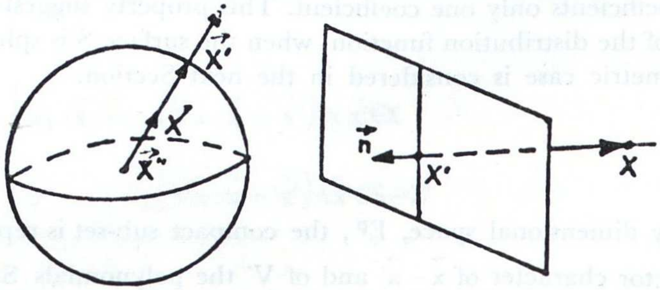


Fig. 9. The normal $\vec{n}$ at any point $\vec{x}'$ of the surface of a hypersphere or of an infinitely extended plane is an axis of rotational symmetry. This is true for any parallel sphere or plane respectively. The property $-\nabla\, \varphi(\vec{x}') \,||\, \vec{n}$ for all $\vec{x}' \in S$ reduces considerably the multiplicity of the superposition coefficients in Eqs. (3.5) - (3.8).

To make this clear we give the new scalar variables in

Definition X.

1o. $\xi = (\vec{x} - \vec{x}') \,.\, \vec{n}$, (9.2)

2o. $\xi' = (\vec{x} - \vec{x}'') \,.\, \vec{n}$, (9.3)

3o. $\zeta = \vec{\Omega} \,.\, \vec{n}$. (9.4)

Using the above variables we can write the distribution functions in question as follows:

First kind

$$\psi^3_- = \sum_{n=3}^{\infty} q_n \left[S_n(\xi, \zeta) - (-\zeta)^n \exp\left(-\frac{\xi}{\zeta} \right) \right] + \psi_\Sigma(\vec{x}, \vec{\Omega}) \exp\left[-\frac{\xi}{\zeta} \right]; \zeta < 0 \quad (9.5)$$

and

$$\psi_+^3 = \sum_{n=0}^{\infty} p_n S_n (\xi', \zeta) \; ; \zeta > 0 \tag{9.6}$$

Second kind

$$\psi_+^4 = \sum_{n=0}^{\infty} q_n \left[S_n(\xi, \zeta) - (-\zeta)^n \exp\left(-(\vec{x}-\vec{x}').\vec{\Omega}\right)\right] +$$

$$+ \psi_\Sigma (\vec{x}', \vec{\Omega}) \exp\left(- (\vec{x} - \vec{x}') . \vec{\Omega}\right) ; \zeta < 0 \tag{9.7}$$

and

$$\psi_+^4 = \sum_{n=0}^{\infty} p_n S_n (\xi', \zeta) \; ; \zeta > 0 \, . \tag{9.8}$$

In the above particular case the coefficients $\{ D^{k'} \varphi_e (\vec{x}'') \}$, becoming simply $[q_n, p_n]$, satisfy an algebraic linear system of equations.

We shall in particular give here without derivation the simplified form of the second kind solution of Eq. (7.1) with the kernel as given by Eq. (7.8):

$$\psi_{L-} = \sum_{n=0}^{L} f_1(\vec{\upsilon}) \sum_{n=0}^{\infty} q_{ne} \left[S_n \left(\xi, \frac{\zeta}{\sigma(\upsilon)}\right) - \left(- \frac{\zeta}{\sigma(\upsilon)}\right)^n \right.$$

$$\left. \cdot \exp\left[- \sigma(\upsilon) \; (\vec{x} - \vec{x}') \, . \, \vec{\Omega} \; \right]\right] + \psi_\Sigma (\vec{x}', \vec{\Omega}) \exp\left[- (\vec{x} - \vec{x}') \, . \, \vec{\Omega}\right] ; \zeta < 0 \tag{9.9}$$

and

$$\psi_{L+} = \sum_{l=1}^{\infty} f_1(\upsilon) \sum_{n=0}^{\infty} p_{ne} S_n \left(\xi', \frac{\zeta}{\sigma(\upsilon)}\right) ; \zeta > 0 \tag{9.10}$$

In Eqs. (9.9), (9.10) the coefficients $\{q_{ne}, p_{ne} \,|\, 1 < l < L, n > 0\}$ are to be determined and depend on the boundary function, whilst $f_e(\vec{\upsilon})$ is defined by the relation

$$f_e(\vec{\upsilon}) = [\upsilon\sigma(\upsilon)]^{-1} f_e(\vec{\upsilon}) \, . \tag{9.11}$$

The comparison of Eqs. (8.2) - (8.5) with Eqs. (9.9.), (9.10) shows that on the one hand the latter contain no operator functions and on the other hand they possess a much smaller multiplicity of coefficients.

For the simplicity of the presentation we shall conduct here our argument only for the distribution functions applicable to systems with rotational symmetry.

Before verifying that ψ_{L+} satisfy indeed Eq. (7.8') it will be shown that ψ_{L-} satisfies the boundary condition, Eq. (7.9). If $\vec{x} = \vec{x}'$, ψ_{L-} becomes equal to $\psi_\Sigma (\vec{x}', \vec{\upsilon})$. From the definition of the polynomials, S_n, it follows that

$$S_n(0, \zeta/\sigma(\upsilon)) = (-\zeta/\sigma(\upsilon))^n \; ; n = 0, 1, 2, \ldots \tag{9.12}$$

and consequently the bracket in Eq. (9.9) vanishes identically at $\vec{x} = \vec{x}'$ and there results

$$\psi_{L-}(\vec{x}, \vec{x}', \vec{\upsilon}) \Big|_{\vec{x} = \vec{x}'} = \psi_\Sigma(\vec{x}', \vec{\upsilon}) \; ; \zeta < 0 \,. \tag{9.13}$$

Next we observe that there is a linear relation between the coefficients $[q_{en}]$ and $[p_{en}]$. This relation can be found by comparing Eqs. (9.9) and (9.10) at a particular direction of $\vec{\upsilon} = \vec{\upsilon}_0$. We choose $\vec{\upsilon}_0$ to lie in the plane tangent to S at $\vec{x}'$, such that $\zeta = 0$.

Consequently,

$$\psi_{L-} - \psi_{L+} = \psi_\Sigma(\vec{x}, \vec{\upsilon}_0) \exp[-\sigma(\upsilon_0)(\vec{x} - \vec{x}') \,.\, \vec{\Omega}_0] + \\ + \sum_{e=1}^{\infty} f_1(\upsilon_0) \sum_{n=3}^{\infty} \left(\xi^n q_{en} - \xi'^n p_{en}\right) / n! \,, \tag{9.14}$$

where it is supposed that

$$\lim_{\zeta \to +0} f_e(\vec{\upsilon}) = \lim_{\zeta \to -0} f_e(\vec{\upsilon}) \; ; l = 1, 2, \ldots, L \,. \tag{9.15}$$

Given ψ_{L-} in advance it is always possible to represent ψ_{L+} by Eq. (9.14) for constant $\vec{\upsilon}_0$, because the set of powers $[\xi^n \mid n = 0, 1, 2, \ldots]$ is a complete set of functions.

Now the continuity of ψ_{L+} on U^p implies that $\psi_{L-} - \psi_{L+} = 0$ $(\zeta = 0)$, and therefore

$$\sum_{e=1}^{\infty} f_e(\vec{\upsilon}_0) \sum_{n=0}^{\infty} \left(\xi^n q_{en} - \xi'^n p_{en}\right) / n! = \\ = -\psi_\Sigma(\vec{x}', \vec{\upsilon}_0) \exp[-(\vec{x} - \vec{x}') \,.\, \vec{\Omega}_0] \; ; (\zeta = 0). \tag{9.16}$$

Upon taking the k-th derivative of both sides of Eq. (9.16) with respect to $\vec{x}$ and putting $\vec{x} = \vec{x}''$ we get

$$\sum_{e=1}^{L} f_e(\vec{\upsilon}_0)\, p_{ek} = \left[-\sigma(\upsilon_0)\, \vec{n} \,.\, \vec{\Omega}_0\right]^k \exp[-(\vec{x}'' - \vec{x}') \,.\, \vec{\Omega}_0] \,.\, \psi_\Sigma(\vec{x}', \vec{\upsilon}_0) +$$

$$+ \sum_{e=1}^{L} f_e(\vec{\upsilon}_0) \sum_{n=k}^{\infty} q_{en} \left[(\vec{x} - \vec{x}') \cdot \vec{n}\right]^{n-k} / (n-k)! \tag{9.17}$$

for $k = 0, 1, 2, \ldots$

Equation (9.16) simplifies considerably if the homogeneous Dirichlet boundary condition is imposed, ie. $\psi_\Sigma(\vec{x}', \vec{\upsilon}_0) = 0$. In this case if follows from Eq. (9.17) that

$$p_{ek} = \sum_{n=k}^{\infty} \frac{\left[(\vec{x}'' - \vec{x}') \cdot n\right]^{n-k}}{(n-k)!} q_{en}, \quad 1 < 1 < L,\ n, k > 0 . \tag{9.18}$$

Eqs. (9.17) and (9.18) are necessary for the complete determination of the constant coefficients $[q_{en}]$ and $[p_{en}]$. Next we inquire into the conditions for which Eqs. (9.9) and (9.10) satisfy Eq. (7.8′). To keep simple the expression we shall assume that $\psi_\Sigma(\vec{x}, \vec{\upsilon}) = 0$ and introduce the following

Definition XI.

1º. $$\beta_{kn}^{ee''}(\xi, \mu) = (\vec{n} \cdot \nabla)^k \int_{\vec{n}\cdot\vec{\upsilon}<o} d\upsilon^p f_e(\vec{\upsilon})\, h_e'(\vec{\upsilon}) \cdot$$

$$\cdot \left[S_n(\xi, \zeta / \sigma(\upsilon)) - \left(-\zeta/\sigma(\upsilon)\right)^n \exp\left[-\sigma(\upsilon)\,(\vec{x} - \vec{x}') \cdot \vec{\Omega}\right]\right] . \tag{9.19}$$

2º. $$\gamma_{kn}^{ee'}(\xi, \mu) = (\vec{n} \cdot \nabla)^k \int_{\vec{n}\cdot\vec{\upsilon}>o} d\upsilon^p f_e(\vec{\upsilon}) h_e'(\vec{\upsilon}) S_n\left(\xi', \zeta/\sigma(\upsilon)\right) . \tag{9.20}$$

Using the above definitions we obtain from Eqs. (7.8'), (9.9) and (9.10) the equation

$$\sum_{e'=1}^{L} f_e'(\vec{\upsilon}) \sum_{u=0}^{\infty} \frac{q_{e'n}}{n!} \xi^n = \lambda \sum_{e=1}^{L} f_e(\vec{\upsilon}) \sum_{e'=1}^{L} \sum_{n=0}^{\infty} \cdot$$

$$\cdot \left[\beta_{0n}^{ee'}(\xi, \mu) q_{e'n} + \gamma_{0n}^{ee'}(\xi', \mu) p_{e'n}\right] . \tag{9.21}$$

Comparison of coefficients of $f_e(\vec{\upsilon})$, $1 = 1, 2, \ldots, L$, in both members of Eq. (9.12) leads to the set of equations

$$\sum_{n=0}^{\infty} \frac{q_{en}}{n!} \xi^n = \lambda \sum_{n=0}^{\infty} \sum_{l'=1}^{L} \left(\beta_{0u}^{ee'} (\xi, \mu) + \sum_{\nu=0}^{n'} \gamma_{0\nu}^{ee'} (\xi, \mu) \zeta_{\nu n} \right) q_{en} , \tag{9.22}$$

where $l = 1, 2, \ldots, L$,

$$\zeta_{\nu n} = \begin{cases} \left(-\frac{1}{2} \vec{d} \cdot \vec{n} \right)^{n-\nu} , & \nu < n, \\ 0 , & \nu > 0, \end{cases} \tag{9.23}$$

and $\vec{d} = \vec{x}' - \vec{x}''$.

From Eq. (9.22) the coefficients $[q_{en} \mid l < l < L ; n > 0]$ can be determined by equating to zero the coefficients of all powers of $\vec{x}$. This is done by multiplying both sides of Eq. (9.22) by $(\vec{n} \cdot \nabla)^k$ and putting $\vec{x} = \vec{\alpha}$, where $\vec{\alpha}$ is any fixed point of V^p. It is convenient, however, to take $\vec{\alpha} = \vec{x}'$ or $\vec{\alpha} = \vec{x}''$, so that we have the set of linear algebraic equations

$$\sum_{e=1}^{L} \sum_{n=0}^{\infty} \left\{ \left[\beta_{kn}^{ee'} (0, \mu) + \sum_{\nu=0}^{n} \gamma_{kn}^{ee''} \left(\frac{\vec{d} \cdot \vec{n}}{2} \right) \zeta_{\nu n} \right] - \delta_{ee'} \, \delta_{kn} \cdot \kappa \right\} q_{en} = 0 \tag{9.24}$$

where $\kappa = \lambda^e$, $l' = 1, 2, \ldots, L$ and $k = 0, 1, 2, \ldots$

The multiplication by $(\vec{n} \cdot \nabla)^k$ is allowed since all expressions are uniformly continuous on $R^p \otimes U^p$. Since Eq. (9.24) is homogeneous the condition for the existence of nontrivial solutions $\{q_{en}\}$ is identical with the secular equation

$$\det \mid D(\mu \, ; p \, ; L) - \kappa I \mid = 0 , \tag{9.25}$$

where I is the unit matrix.

The elements of the supermatrix $D(\mu \, ; p \, ; L)$ are given by the equation

$$D_{kn}^{ee'} = \beta_{kn}^{ee'} (0, \mu) + \sum_{\nu=0}^{n} \gamma_{kn}^{ee'} \left(\frac{\vec{d} \cdot \vec{n}}{2} \right) \zeta_{\nu n} . \tag{9.26}$$

It is obvious that in the present spherical system $\vec{x}''$ may be identified with the centre of the sphere and therefore $\left| \vec{n} \cdot \vec{d} \right| = \left| \vec{d} \right|$ equals the radius of the system.

Next as an example we specialize to the case $p = 3$. The following results are easily established using the Def. XI, 1° and 2°.

$$\beta_{kn}^{ee'}(0,\mu) = \int_{\vec{n}\,.\,\vec{\upsilon}} d\upsilon^3 \, \frac{f_e(\vec{\upsilon})h_{e'}(\vec{\upsilon})}{[\upsilon\sigma(\upsilon)]^{n-k}} \left\{ \begin{array}{ll} \left[(-)^{n-k} - (-)^{n} (\vec{n}\,.\,\vec{\Omega})^{2k}\right] (\vec{n}\,.\,\vec{\Omega}) & ;\ k < n \\ (-)^{n+k+1} (\vec{n}\,.\,\vec{\Omega})^{n+k} & ;\ k > n \end{array} \right. \tag{9.27}$$

$$\gamma_{kn}^{ee'}\left(\frac{\vec{d}\,.\,\vec{n}}{2}\right) = \sum_{\nu=0}^{n} (-)^{\nu} \frac{\left((\vec{d}\,.\,\vec{n})/2\right)^{n-\nu-k}}{(n-n-k)!} \,.$$

$$\cdot \int_{\vec{n}\,.\,\vec{\upsilon} > o} d\upsilon^3 \, \frac{f_e(\vec{\upsilon})h_{e'}(\vec{\upsilon})}{[\upsilon\sigma(\upsilon)]^{\nu}} \left\{ \begin{array}{ll} (\vec{n}\,.\,\vec{\Omega})^{\nu} & ;\ k < n \\ 0 & ;\ k > n \end{array} \right. \tag{9.28}$$

Introducing Eqs. (9.27) and (9.28) into Eq. (9.22) one obtains the system of equations from which $[q_{en}]$ can be determined. The coefficients $[p_{en}]$ are then calculated from Eq. (9.18). The eigenvalue spectrum follows from the determinental equation, Eq. (9.25).

10. THE COMPLETENESS IN THE GENERAL CASE

From the results described in Sect. 9 it becomes clear what kind of generalizations are required in order to be able to treat the case of a system with an asymmetrical convex surface S. Obviously this is done if we abandon the restriction imposed by condition Eq. (9.1). If we do so, then the quantities changing are:

(i) the multiplicity of the coefficients $[p_{en}, q_{en} \mid l = 1, 2, \ldots L;\ n = 0, 1, 2, \ldots]$

(ii) the structure of the elements $[\beta_{kn}^{ee'}, \gamma_{kn}^{ee'}]$ of the supermatrix D.

By using the distribution function given by the expressions

$$\psi_{L-}(\vec{x}, \vec{x}', \vec{\upsilon}) = \sum_{e=1}^{L} f_e(\vec{\upsilon}) \sum_{n=0}^{\infty} \Big\{ S_n\left((\vec{x}-\vec{x}')\,.\,\nabla', \frac{\vec{\upsilon}\,.\,\nabla'}{\upsilon\sigma(\upsilon)}\right) -$$

$$- \exp\left[-\sigma(\upsilon)\,(\vec{x}-\vec{x}')\,.\,\vec{\Omega}\right] \cdot \left[-\frac{\vec{\upsilon}\,.\,\nabla'}{\upsilon\sigma(\upsilon)}\right]^{n} \Big\}\, \varphi_{en}(\vec{x}') \tag{10.1}$$

$$\psi_{L+}(\vec{x}, \vec{x}'', \vec{\upsilon}) = \sum_{e=1}^{L} f_e(\vec{\upsilon}) \sum_{n=0}^{\infty} S_n\left((\vec{x}-\vec{x}'')\,.\,\nabla'', \frac{\vec{\upsilon}\,.\,\nabla''}{\upsilon\sigma(\upsilon)}\right) \varphi_{en}(\vec{x}'') \tag{10.2}$$

the elements $[\beta_{kn}^{ee'}, \gamma_{kn}^{ee'}]$ become operators and can be evaluated by operating with them on $\varphi_{en}(\vec{x})$ and $\varphi_{en}(\vec{x}'')$ respectively. More precisely, we get from Def. XI the following expressions in which we again use for conciseness of the formulas the boundary condition $\psi_{\Sigma}(\vec{x}', \vec{\upsilon}) = 0$.

$$\beta_{kn}^{ee'}(\vec{x} - \vec{x}', \nabla', \mu)\varphi(\vec{x}') = (\vec{n} \cdot \vec{\nabla})^k \int_{\vec{n} \cdot \vec{\upsilon} < 0} d\upsilon^p f_e(\upsilon) \cdot h_e'(\upsilon) \cdot \left\{ S_n\left((\vec{x} - \vec{x}')\nabla', \frac{\vec{\upsilon} \cdot \nabla'}{\upsilon\sigma(\upsilon)} \right) - \right.$$

$$\left. - \exp\left[- \frac{\sigma(\upsilon)\,(\vec{x} - \vec{x}') \cdot \vec{n}}{\vec{\Omega} \cdot \vec{n}} \right] \left(- \frac{\vec{\upsilon} \cdot \nabla'}{\upsilon\sigma(\upsilon)} \right)^n \right\} \varphi(\vec{x}') \tag{10.3}$$

$$\gamma_{nk}^{ee'}(\vec{x} - \vec{x}'', \nabla'', \mu)\varphi(\vec{x}'') = (\vec{n} \cdot \vec{\nabla})^k \int_{\vec{n} \cdot \vec{\upsilon} > 0} d\upsilon^p\, f_e(\upsilon)\, h_e(\upsilon) \cdot$$

$$\cdot\, S_n\left((\vec{x} - \vec{x}'')\nabla'', \frac{\vec{\upsilon} \cdot \nabla''}{\upsilon\sigma(\upsilon)} \right) \varphi(\vec{x}'') . \tag{10.4}$$

We wish now to see how Eq. (9.24) should be modified as a consequence of the asymmetry of the surface S. To do this we use again the continuity condition

$$\lim_{\vec{n} \cdot \vec{\upsilon} \to -0} \psi_{L-} = \lim_{\vec{n} \cdot \vec{\upsilon} \to +0} \psi_{L+} \tag{10.5}$$

to establish the relationship between $(\vec{n} \cdot \nabla')^k \varphi(\vec{x}')$ and $(\vec{n} \cdot \nabla'')^k \varphi(\vec{x}'')$. Since Eq. (10.5) holds at $\vec{\upsilon} = \vec{\upsilon}_0$ for all $x \in V^p$, upon inserting Eqs. (10.1) and (10.2) into Eq. (10.5) and putting thereby $\vec{x} = \vec{x}''$, we get the equation

$$\sum_{n=k}^{\infty} \sum_{\nu=0}^{n} \left\{ \frac{\left[(\vec{x}'' - \vec{x}')\nabla' \right]^{n-\nu-k} (\vec{n} \cdot \nabla')^k \left(- \vec{\upsilon}_0 \cdot \nabla' / \upsilon\sigma(\upsilon) \right)^\nu}{(n - \nu - k)!} \left(- \sigma(\upsilon) \right)^k \right.$$

$$\left. \cdot \exp\left(- \frac{\sigma(\upsilon)(\vec{x}'' - \vec{x}') \cdot \vec{n}}{\vec{\Omega}_0 \cdot \vec{n}} \right) \left(- \frac{\vec{\upsilon}_0 \cdot \nabla'}{\upsilon\sigma(\upsilon)} \right)^n \right\} \cdot \varphi(\vec{x}', \vec{\upsilon}_0) =$$

$$= \sum_{n=k}^{\infty} \left[- \frac{\vec{\upsilon}_0 \cdot \nabla''}{\upsilon\sigma(\upsilon)} \right]^{n-k} \varphi(\vec{x}'', \vec{\upsilon}_0) \quad , \tag{10.6}$$

where

$$\varphi(\vec{x}', \vec{\upsilon}_0) \sum_{e=1}^{L} f_e(\vec{\upsilon}_0) \int_{U^p} d^p \upsilon' h_e(\vec{\upsilon}')\psi_1 = \sum_{e=1}^{L} f_e(\vec{\upsilon}_0)\varphi_e(\vec{x}') \,. \tag{10.7}$$

Eq. (10.6) allows us to eliminate the constant coefficients $D^k\varphi_e(\vec{x}'')$ from ψ_{L+} occurring in Eq. (7.8') after the integration over U^p.

After some simple calculations one again obtains an equation for the determination of the coefficients $D^k\varphi_e(\vec{x}')$. This equation is analogous to the simpler Eq. (9.24).

The condition for the existence of the determinant det $D(\mu;p;L)$ which is of infinite order have been investigated and applied in Ref. 207 for the case of R^1.

To complete this Section we still indicate that the functions given in Eqs. (13.1) and (10.2) are the most general ones. It can be seen that this is indeed the case if we only show that the operator polynomials $[S_n(A, B) \mid n = 0, 1, \ldots]$ form a complete set. To do this, we observe that A and B are operators having as range E_1, the set of functions

$$[\varphi_e(\vec{x}) \mid l = 1, 2, \ldots, L)] = E_1 \,,$$

where $\varphi_e(\vec{x})$ are infinitely many times differentiable functions, and as domain the Banach space E_2 of the functions

$$[S_n(A, B)\varphi_e(x) \mid l = 1, 2, \ldots, L;\ n = 1, 2, \ldots]$$

defined on $R^p \otimes U^p$. The completeness under the above conditions is shown in Ref. 208.

We can orthogonalize $S_n(A, B)\varphi_e$ on $R^p \otimes U^p$ and this can be done by the procedure described in Ref. 209.

It is easily shown that indeed the functions

$$[S_n\varphi_e(A, B)\ (\vec{x}_0);\ \vec{x}_0 = \vec{x}', \vec{x}'' \mid l = 1, 2, \ldots, L;\ n = 0, 1, 2, \ldots]$$

satisfy the axioms defining the Banach space.

In particular one verifies at once that

$$\lim_{m,n\to\infty} \left\| \Big(S_m(A, B) - S_n(A, B)\Big)\, \varphi_1(\vec{x}_0) \right\| = 0 \,, \tag{10.8}$$

where $\| S_n \|$ is the norm.

We conclude, therefore, according to the Weierstrass theorem that Eqs. (10.1) and (10.2) can represent any function defined on $R^p \otimes U^p$ and satisfying Eq. (7.8′), Ref. 2.10.

We collect the above results in the following

Theorem XIX.

Let V^p be a subset of R^p with finite diameter, $\vec{\delta}$, having a convex boundary surface, S, and let U^p be the velocity space. Let further the kernel $K(\vec{\upsilon}, \vec{\upsilon}')$ have the form given in Eq. (7.8) with $[f_e(\vec{\upsilon}), h_e(\vec{\upsilon}) \mid l = 1, 2, \ldots, L]$ such that the integrals

$$[\beta_{kn}^{ee'}, \gamma_{kn}^{ee'} \mid l, l' = 1, 2, \ldots L;\ k, n > 0]$$

exist and satisfy the condition $\mid \det D \mid < \infty$.

Then, there exist solutions of Eq. (7.8′) regular everywhere on V^p given by Eqs. (10.1) and (10.2) and satisfying the boundary condition given by Eq. (9.3).

One also can easily show the following results based in particular on Section 10.

Corollary I.

The regularity behaviour of $\psi_{L\pm}$ on U^p is determined by the behaviour of $f_e(\vec{\upsilon})$ on U^p.

Corollary II.

The solution of Eq. (7.8′) is a superposition of constant-kernel solutions with coefficients following from the scattering kernel.

Corollary III.

The point spectrum of the Boltzmann operator is given by the secular equation, Eq. (9.25).

Remark XXIV

It is not required for the kernel to be square integrable.

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