rential equation in connection with the problem of lateral free vibrations of prismatic bars in Plasticity.

For simply supported bar at both ends the problem corresponds to find the solution u(x,t) of the followin system:

$$[f(u'')]'' = c\ddot{u},$$
$$u(o,t) = u(\pi,t) = u''(o,t) = u''(\pi,t) = o,$$
$$u(x,o) = \varphi_1(x),\ \dot{u}(x,o) = \varphi_2(x).$$

By taking properly the functions $\varphi_1(x)$, $\varphi_2(x)$, $f(u'')$, a solution of the type:

$$u(x,t) = \sum_{m=1}^{\infty} v_m(t)\cdot \sin mx \qquad \text{is given.}$$

The function $v_m(t)$ is determined with the help of an infinite system of nonlinear equations of degree r, which is solved by the method of successive approximations. The constructed solution fulfills, under certain limitations, all Hadamard's postulates, then it represents the reality.

ΑΣΤΡΟΝΟΜΙΑ.—**Study of the potential in the plane of symmetry of a stellar system,** *by* ***G. Contopoulos*** *. Ἀνεκοινώθη ὑπὸ τοῦ κ. Ἰωάνν. Ξανθάκη.

In his fundamental paper on the dynamics of stellar systems Chandrasekhar[1] has studied the general case of a two dimensional system having a center of symmetry and some special cases of three-dimentional systems with axial symmetry. However the general problem of a stellar system with an axis and a plane of symmetry has not yet been considered.

In this paper the results of S. Chandrasekhar on the two-dimensional problem are proved by a new, more concise method. Then, by extention of this method, we study the potential function on the plane of symmetry of a three-dimentional stellar system having both an axis and a plane of symmetry.

I

We assume the validity of the following three postulates, as given by S. Chandrasekhar[2].

---

* ΓΕΩΡΓ. ΚΟΝΤΟΠΟΥΛΟΥ, *Ἔρευνα τοῦ δυναμικοῦ εἰς τὸ ἐπίπεδον συμμετρίας ἀστρικοῦ συστήματος.*

[1] S. Chandrasekhar, The Dynamics of Stellar Systems, *Ap. J.* **90**, 1939, 1, and **92**, 1940, 441.

[2] S. Chandrasekhar, Principles of Stellar Dynamics, Chicago 1942, p. 89, E. von der Pahlen, Einführung in die Dynamik von Sternsystemen, Basel 1947, S. 127.

a) At any given point (x, y, z) corresponds a mean velocity (u_o, v_o, w_o), which is a continuous function of position and time.

b) The distribution of the different velocities in every point is given by a distribution function of the «generalized Schwarzschild type»:

$$\Psi(x, y, z;\ u, v, w;\ t) \equiv \Psi(Q + \sigma)$$

where:

$$Q = a(u-u_o)^2 + b(v-v_o)^2 + c(w-w_o)^2 + 2f(v-v_o)(w-w_o) + 2g(w-w_o)(u-u_o) + \\ + 2h(u-u_o)(v-v_o)$$

The coefficients a, b, c, f, g, h and σ are continuous functions of position and time.

c) The motions of the individual stars are governed by a potential function V(x, y, z,; t) per unit mass.

From this condition Liouville's theorem is deduced, which may be expressed:

$$\frac{\partial\Psi}{\partial t} + u\frac{\partial\Psi}{\partial x} + v\frac{\partial\Psi}{\partial y} + w\frac{\partial\Psi}{\partial z} - \frac{\partial V}{\partial x}\frac{\partial\Psi}{\partial u} - \frac{\partial V}{\partial y}\frac{\partial\Psi}{\partial v} - \frac{\partial V}{\partial z}\frac{\partial\Psi}{\partial w} = 0$$

The general problem is to find the form of V and the restrictions on the coefficients a, b etc. of the velocity ellipsoid, so that the above three postulates should be fulfilled.

S. Chandrasekhar has found that if we write $(Q + \sigma)$ in the form:

$$Q + \sigma = au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv - 2\Delta_1 u - 2\Delta_2 v - 2\Delta_3 w - \chi,$$

then the coefficients a, b, c, f, g, h, Δ_1, Δ_2 and Δ_3 are given by the formulae:

$$(1)\quad \begin{cases} a = -2(h_{20} + h_{21}z)y - h_{40}y^2 - (a_o + 2g_{30}z + g_{40}z^2) \\ b = -2(f_{20} + f_{21}x)z - f_{40}z^2 - (b_o + 2h_{30}x + h_{40}x^2) \\ c = -2(g_{20} + g_{21}y)x - g_{40}x^2 - (c_o + 2f_{30}y + f_{40}y^2) \end{cases}$$

$$(2)\quad \begin{cases} f = (f_{10} + f_{11}x - h_{21}x^2) + (f_{20} + f_{21}x)y + (f_{30} + g_{21}x)z + f_{40}yz \\ g = (g_{10} + g_{11}y - f_{21}y^2) + (g_{20} + g_{21}y)z + (g_{30} + h_{21}y)x + g_{40}zx \\ h = (h_{10} + h_{11}z - g_{21}z^2) + (h_{20} + h_{21}z)x + (h_{30} + f_{21}z)y + h_{40}xy \end{cases}$$

$$(3)\quad \begin{cases} \Delta_1 = \\ = y\left(y\frac{dh_{30}}{dt} - x\frac{dh_{20}}{dt}\right) - z\left(x\frac{dg_{30}}{dt} - z\frac{dg_{20}}{dt}\right) + \frac{df_{11}}{dt}yz - \frac{1}{2}\frac{da_o}{dt}x + \beta_3 y + \gamma_2 z + \delta_1 \\ \Delta_2 = \\ = z\left(z\frac{df_{30}}{dt} - y\frac{df_{20}}{dt}\right) - x\left(y\frac{dh_{30}}{dt} - x\frac{dh_{20}}{dt}\right) - \frac{dg_{11}}{dt}zx - \frac{1}{2}\frac{db_o}{dt}y + \beta_1 z + \gamma_3 x + \delta_2 \\ \Delta_3 = \\ = x\left(x\frac{dg_{30}}{dt} - z\frac{dg_{20}}{dt}\right) - y\left(z\frac{df_{30}}{dt} - y\frac{df_{20}}{dt}\right) - \frac{dh_{11}}{dt}xy - \frac{1}{2}\frac{dc_o}{dt}z + \beta_2 x + \gamma_1 y + \delta_3 \end{cases}$$

The paremeters $h_{20} \ldots \delta_3$ are functions of time, except f_{40}, g_{40}, h_{40}, f_{21}, g_{21} and h_{21}, which are constants.

We have also the following relations:

$$(4) \quad f_{11} + g_{11} + h_{11} = 0, \quad \beta_1 + \gamma_1 = \frac{df_{10}}{dt}, \quad \beta_2 + \gamma_2 = \frac{dg_{10}}{dt}, \quad \beta_3 + \gamma_3 = \frac{dh_{10}}{dt}$$

If the stellar system has an axis of symmetry, V is a function of $\tau = \frac{1}{2} w^2 = \frac{1}{2}(x^2 + y^2)$, z and t only. V is related to the function χ by the equations:

$$(5) \quad \begin{cases} (ax + hy)\dfrac{\partial V}{\partial \tau} + g\dfrac{\partial V}{\partial z} + \dfrac{\partial \Delta_1}{\partial t} = -\dfrac{1}{2}\dfrac{\partial \chi}{\partial x} \\ (hx + by)\dfrac{\partial V}{\partial \tau} + f\dfrac{\partial V}{\partial z} + \dfrac{\partial \Delta_2}{\partial t} = -\dfrac{1}{2}\dfrac{\partial \chi}{\partial y} \\ (gx + fy)\dfrac{\partial V}{\partial \tau} + c\dfrac{\partial V}{\partial z} + \dfrac{\partial \Delta_3}{\partial t} = -\dfrac{1}{2}\dfrac{\partial \chi}{\partial \chi} \\ (\Delta_1 x + \Delta_2 y)\dfrac{\partial V}{\partial r} + \Delta_3\dfrac{\partial V}{\partial z} = \dfrac{1}{2}\dfrac{\partial \chi}{\partial t} \end{cases}$$

Eliminating χ from these four equations we find six integrability conditions; the form of V is to be found by the solution of these conditions in every given case.

II

Is $z = o$ is the plane of symmetry of the stellar system, we evidently have:

$$\frac{\partial V(\tau, z, t)}{\partial z} = -\frac{\partial V(\tau, -z, t)}{\partial z}$$

for every value of τ, z and t. Therefore for $z = o$ it is:

$$\frac{\partial V}{\partial z} = o \text{ and } \frac{\partial^2 V}{\partial z \partial \tau} = o, \ \frac{\partial^2 V}{\partial z \partial t} = o.$$

Under these restrictions the six integrability conditions take the following form:

I) Differentiating the first of (5) for y and the second for x and substracting:

$$(6) \quad \left[h_{10}(y^2 - x^2) + (b_0 - a_0)xy + 2(h_{30}y - h_{20}x)\tau \right] \frac{\partial^2 V}{\partial \tau^2} + 3(h_{30}y - h_{20}x)\frac{\partial V}{\partial \tau} + \left[3(h''_{30}y - h''_{20}x) + \beta'_3 - \gamma'_3 \right] = 0$$

II) Differentiating the first of (5) for t and the fourth for x and adding :

$$
(7)\quad \left\{\begin{aligned}
& x\left[-\frac{1}{2}a'_0x^2-\frac{1}{2}b'_0y^2+h'_{10}xy+\delta_1x+\delta_2y\right]\frac{\partial^2V}{\partial\tau^2}+\\
&+\left[-a_0x+h_{10}y+(h_{30}y-h_{20}x)y\right]\frac{\partial^2V}{\partial t\partial\tau}+\\
&+\left[-2a'_0x+2h'_{10}y+(h'_{30}-h'_{20}x)y+\delta_1\right]\frac{\partial V}{\partial\tau}+\\
&+\left[(h'''_{30}y-h'''_{20}x)y-\frac{1}{2}\alpha'''_0x+\beta''_3y+\delta''_1\right]=0
\end{aligned}\right.
$$

III) Differentiating the second of (5) for t and the fourth for y and adding:

$$
(8)\quad \left\{\begin{aligned}
& y\left[-\frac{1}{2}a'_0x^2-\frac{1}{2}b'_0y^2+h'_{10}xy+\delta_1x+\delta_2y\right]\frac{\partial^2V}{\partial\tau^2}+\\
&+\left[h_{10}x-b_0y-(h_{30}y-h_{20}x)x\right]\frac{\partial^2V}{\partial t\partial\tau}+\\
&+\left[2h'_{10}x-2b'_0y-(h'_{30}y-h'_{20}x)y+\delta_2\right]\frac{\partial V}{\partial\tau}+\\
&+\left[-(h'''_{30}y-h'''_{20}x)x-\frac{1}{2}b'''_{30}y+\gamma''_3x+\delta''_2\right]=0
\end{aligned}\right.
$$

IV) Differentiating the first of (5) for z and the third for x and substracting :

$$
(9)\quad \left\{\begin{aligned}
& \left[g_{10}-2h_{11}y+4g_{30}x+h_{21}xy-f_{21}y^2\right]\frac{\partial V}{\partial\tau}+\\
&+x\left[g_{10}x+f_{10}y-h_{11}xy+f_{20}y^2+g_{30}x^2\right]\frac{\partial^2V}{\partial\tau^2}-\\
&-\left[g_{10}+g_{11}y-f_{21}y^2+g_{30}x+h_{21}xy\right]\frac{\partial^2V}{\partial z^2}+\\
&+\left[3g''_{30}x-(h''_{11}-f''_{11})y+\beta'_2-\gamma'_2\right]=0
\end{aligned}\right.
$$

V) Differentiating the second of (5) for z and the third for y and substracting:

$$
(10)\quad \left\{\begin{aligned}
& \left[f_{10}-2h_{11}x+4f_{20}y+f_{21}xy-h_{21}x^2\right]\frac{\partial V}{\partial\tau}+\\
&+y\left[g_{10}x+f_{10}y-h_{11}xy+f_{20}y^2+g_{30}x^2\right]\frac{\partial^2V}{\partial\tau^2}-\\
&-\left[f_{10}+f_{11}x-h_{21}x^2+f_{20}y+f_{21}xy\right]\frac{\partial^2V}{\partial z^2}+\\
&+\left[3f''_{20}y-(h''_{11}-g''_{11})+\gamma'_1-\beta'_1\right]=0
\end{aligned}\right.
$$

VI) Finally, differentiating the third of (5) for t and the fourth for z and adding:

$$(11)\quad \begin{cases} \left[g'_{10}x + f'_{10}y + \gamma_2 x + \beta_1 y\right]\dfrac{\partial V}{\partial \tau} + \\ + \left[g_{10}x + f_{10}y - h_{11}xy + f_{20}y^2 + g_{30}x^2\right]\dfrac{\partial^2 V}{\partial t \partial \tau} + \\ + \left[g'_{30}x^2 + f'_{20}y^2 - h'_{11}xy + \beta_2 x + \gamma_1 y + \delta_3\right]\dfrac{\partial^2 V}{\partial z^2} + \\ + \left[g'''_{30}x^2 + f'''_{20}y^2 - h'''_{11}xy + \beta''_2 x + \gamma''_1 y + \delta''_3\right] = 0 \end{cases}$$

Equations (6), (7) and (8) are the same as equations (3.604), (3.605) and (3.606) of S. Chandrasekhar of the two-dimensional problem[1]. From the solution of these three equations this author deduces some results that will, be shown here in the following shorter way.

If we write $x = w\cos\vartheta$ and $y = w\sin\vartheta$ and insert these values in (6), (7) and (8) we will find three equations for ϑ that must be fulfilled for every value of ϑ. Therefore the coefficients of $\cos\vartheta$, $\sin\vartheta$, $\sin^2\vartheta$, $\cos\vartheta\sin\vartheta$ and the term independent of ϑ, must vanish separately. This requires that the coefficients of x, y, y^2, xy and the term which does not include x or y separately but only w or τ, must be reduced to zero. (It must be noted here that $x^2 = w^2 - y^2$, i.e. the coefficient of x^2 does not vanish separately).

Then from equation (6) we obtain the following relations:

$$(12)\ \text{(coefficient of x)}:\quad -2h_{20}\tau\frac{\partial^2 V}{\partial \tau^2} - 3h_{20}\frac{\partial V}{\partial \tau} - 3h''_{20} = 0$$

$$(13)\ (\quad » \quad » \ y):\quad 2h_{30}\tau\frac{\partial^2 V}{\partial \tau^2} + 3h_{30}\frac{\partial V}{\partial \tau} + 3h''_{30} = 0$$

$$(14)\ (\quad » \quad » \ xy):\quad (b_0 - a_0)\frac{\partial^2 V}{\partial \tau^2} = 0$$

$$(15)\ (\quad » \quad » \ y^2):\quad 2h_{10}w^2\frac{\partial^2 V}{\partial \tau^2} = 0$$

$$(16)\ \text{(independent term)}:\quad -h_{10}w^2\frac{\partial^2 V}{\partial \tau^2} + \beta'_3 - \gamma'_3 = 0$$

---

1 Principles of Stellar Dynamics, p. 106. The whole investigation of this problem by S. Chandrasekhar is carried out in polar coordinates. *See Ap. J.* 92, 1940, 459.

If we exclude the case $\frac{\partial^2 V}{\partial w^2} = k$, where k is a function of time only[1] we will find: $\frac{\partial^2 V}{\partial \tau^2} \neq 0$.

Thus from (14) and (15) we conclude:

$$b_0 = a_0 \quad \text{and} \quad h_{10} = 0$$

Therefore from (16): $\beta'_3 = \gamma'_3$, $\beta_3 = \gamma_3 + \text{const.}$ and from the last of conditions (4), $\beta_3 + \gamma_3 = \frac{dh_{10}}{dt} = 0$, thus: $\beta_3 = -\gamma_3 = \text{const.}$

Finally if (12) and (13) are fulfilled, we must have: either $h_{20} = h_{30} = 0$,

$$\text{or} \quad \frac{h''_{30}}{h_{30}} = \frac{h''_{20}}{h_{20}} \quad \text{and} \quad 2\tau\frac{\partial^2 V}{\partial \tau^2} + 3\frac{\partial V}{\partial \tau} + 3\frac{h''_{20}}{h_{20}} = 0 \qquad (17)$$

Equation (7) is now written:

$$x\left[-\frac{1}{2}a'_0 w^2 + \delta_1 x + \delta_2 y\right]\frac{\partial^2 V}{\partial \tau^2} + \left[-a_0 x + h_{30}y^2 - h_{20}xy\right]\frac{\partial^2 V}{\partial t \partial \tau} +$$

$$+\left[h'_{30}y^2 - h'_{20}xy - 2a'_0 x + \delta_1\right]\frac{\partial V}{\partial \tau} + \left[h'''_{30}y^2 - h'''_{20}xy - \frac{a'''_0}{2}x + \delta''_1\right] = 0$$

and from it the following relations are found:

(18) (coefficient of x): $\quad -\frac{1}{2}a_0' w^2 \frac{\partial^2 V}{\partial \tau^2} - a_0\frac{\partial^2 V}{\partial t \partial \tau} - 2a_0'\frac{\partial V}{\partial \tau} - \frac{a_0'''}{2} = 0$

(19) (» » xy): $\quad \delta_2\frac{\partial^2 V}{\partial \tau^2} - h_{20}\frac{\partial^2 V}{\partial t \partial \tau} - h_{20}'\frac{\partial V}{\partial \tau} - h_{20}''' = 0$

(20) (» » y^2): $\quad -\delta_1\frac{\partial^2 V}{\partial \tau^2} + h_{30}\frac{\partial^2 V}{\partial t \partial \tau} + h_{30}'\frac{\partial V}{\partial \tau} + h_{30}''' = 0$

(21) (independent term): $\quad \delta_1 w^2\frac{\partial^2 V}{\partial \tau^2} + \delta_1\frac{\partial V}{\partial \tau} + \delta_1'' = 0$

Equation (21) may be written:

(22) $$\delta_1\frac{\partial^2 V}{\partial w^2} = -\delta_1''$$

Therefore, as $\frac{\partial^2 V}{\partial w^2} \neq k$, we find: $\delta_1 = 0$

---

[1] In this case: $\frac{\partial V}{\partial w} = kw + \Lambda$, i.e. the force in this field is the sum of two terms, one proportional to the distance from the center and the other constant; but this latter term must be zero, because such a force does not exist in nature.

By a similar discussion of equation (8) we find: $\delta_2 = 0$.

From equations (19) and (20) we conclude (if h_{20} and h_{30} are not both zero):

$$(h_{30} h_{20}' - h_{30}' h_{20}) \frac{\partial V}{\partial \tau} + (h_{30} h_{20}''' - h_{30}''' h_{20}) = 0$$

and as $\frac{\partial V}{\partial \tau}$ is not a function of time only, it follows: $h_{30} h_{20}' - h_{30}' h_{20} = 0$,

i.e.
$$\frac{h_{20}}{h_{30}} = \text{const}.$$

If we write: $h_{20} = H_{20}\varphi$ and $h_{30} = H_{30}\varphi$, equation (17) becomes:

$$2\tau \frac{\partial^2 V}{\partial \tau^2} + 3 \frac{\partial V}{\partial \tau} + 3 \frac{\varphi''}{\varphi} = 0 \tag{23}$$

Equations (19) and (20) are reduced to:

$$\varphi \frac{\partial^2 V}{\partial t \partial \tau} + \varphi' \frac{\partial V}{\partial \tau} + \varphi''' = 0 \tag{24}$$

Equation (18) may be written:

$$\tau \frac{da_0}{dt} \frac{\partial^2 V}{\partial \tau^2} + a_0 \frac{\partial^2 V}{\partial t \partial \tau} + 2 \frac{da_0}{dt} \frac{\partial V}{\partial \tau} + \frac{1}{2} \frac{d^3 a_0}{dt^3} = 0 \tag{25}$$

The conditions (23) and (24) are valid only in the special case, when H_{20} and H_{30} are not both zero. In the general case, when $H_{20} = H_{30} = 0$, only (25) must be necessarily fulfilled.

We have thus found S. Chandrasekhar's results of the two-dimensional case. By further study of equations (23), (24) and (25) it is found [1]:

In the special case (H_{20} and/or $H_{30} \neq 0$)

$$\frac{\partial V}{\partial \tau} = -\frac{\varphi''}{\varphi} + \frac{q_0}{\varphi} \cdot \frac{1}{\tau^{3/2}} \tag{26}$$

where: $q_0 = \text{const.}$ and $a_0 = -\varphi^2$.

In the general case ($H_{20} = H_{30} = 0$)

$$\frac{\partial V}{\partial \tau} = -\frac{\varphi''}{\varphi} + \frac{1}{\varphi^4} W\left(\frac{\tau}{\varphi^2}\right) \tag{27}$$

III

We apply the above method for the solution of the other three equations (9), (10) and (11). Assuming that $\frac{\partial V}{\partial \tau}$ is given by equation (27), the

[1] Principles of Stellar Dynamics, p. 107-114.

problem now consists in defining the form of the unknown function W.

In equations (9), (10) and (11) we have terms containing $\cos^2\vartheta\sin\vartheta$, $\cos^3\vartheta$, $\cos\vartheta\sin^2\vartheta$ and $\sin^3\vartheta$ besides the terms with $\cos\vartheta$, $\sin\vartheta$, $\sin^2\vartheta$, $\sin\vartheta\cos\vartheta$. Among these new terms only $\cos^2\vartheta\sin\vartheta$ and $\cos\vartheta\sin^2\vartheta$ are linearly independent[1]. Therefore besides the coefficients of x, y, xy, y^2 and the independent term, the coefficients of x^2y and xy^2 must also vanish.

We find thus the following relations:

(28) (coefficient of x) : $$4g_{30}\frac{\partial V}{\partial \tau} + g_{30}w^2\frac{\partial^2 V}{\partial \tau^2} - g_{30}\frac{\partial^2 V}{\partial z^2} + 3g''_{30} = 0$$

(29) (» » y) : $$-2h_{11}\frac{\partial V}{\partial \tau} - g_{11}\frac{\partial^2 V}{\partial z^2} - (h''_{11} - f''_{11}) = 0$$

(30) (» » xy) : $$h_{21}\frac{\partial V}{\partial \tau} + f_{10}\frac{\partial^2 V}{\partial \tau^2} - h_{21}\frac{\partial^2 V}{\partial z^2} = 0$$

(31) (» » y^2) : $$-f_{21}\frac{\partial V}{\partial \tau} - g_{10}\frac{\partial^2 V}{\partial \tau^2} + f_{21}\frac{\partial^2 V}{\partial z^2} = 0$$

(32) (» » x^2y) : $$-h_{11}\frac{\partial^2 V}{\partial \tau^2} = 0$$

(33) (» » xy^2) : $$(f_{20} - g_{30})\frac{\partial^2 V}{\partial \tau^2} = 0$$

(34) (independent term): $$g_{10}\frac{\partial V}{\partial \tau} + g_{10}w^2\frac{\partial^2 V}{\partial \tau^2} - g_{10}\frac{\partial^2 V}{\partial z^2} + (\beta'_2 - \gamma'_2) = 0$$

From (32) and (33) we conclude that:

(35) $$h_{11} = 0 \quad \text{and} \quad f_{20} = g_{30}.$$

Therefore the first of (4) bocomes: $f_{11} + g_{11} = 0$ and thus (29) may be written:

(36) $$g_{11}\frac{\partial^2 V}{\partial z^2} + g''_{11} = 0$$

From equation (10) we find:

(29′) (coefficient of x) : $$-2h_{11}\frac{\partial V}{\partial \tau} - f_{11}\frac{\partial^2 V}{\partial z^2} - (h''_{11} - g''_{11}) = 0$$

(28′) (» » y) : $$4f_{20}\frac{\partial V}{\partial \tau} + f_{20}w^2\frac{\partial^2 V}{\partial \tau^2} - f_{20}\frac{\partial^2 V}{\partial z^2} + 3f''_{20} = 0$$

---

1 $\cos^3\vartheta$ and $\sin^3\vartheta$ can be expressed linearly by means of the others:
$$\cos^3\vartheta = \cos\vartheta - \cos\vartheta\sin^2\vartheta, \qquad \sin^3\vartheta = \sin\vartheta - \cos^2\vartheta\sin\vartheta.$$

(31′) (condition of xy) : $f_{21}\frac{\partial V}{\partial \tau} + g_{10}\frac{\partial^2 V}{\partial \tau^2} - f_{21}\frac{\partial^2 V}{\partial z^2} = 0$

(30′) (» » x^2) : $-h_{21}\frac{\partial V}{\partial \tau} - f_{10}\frac{\partial^2 V}{\partial \tau^2} + h_{21}\frac{\partial^2 V}{\partial z^2} = 0$

(33′) (» » x^2y) : $(g_{30} - f_{20})\frac{\partial^2 V}{\partial \tau^2} = 0$

(32′) (» » xy^2) : $-h_{11}\frac{\partial^2 V}{\partial \tau^2} = 0$

(37) (independent term): $f_{10}\frac{\partial V}{\partial \tau} + f_{10}w^2\frac{\partial^2 V}{\partial \tau^2} - f_{10}\frac{\partial^2 V}{\partial z^2} + (\gamma'_1 - \beta'_1) = 0$

The equations (32′), (33′), (30′) and (31′) are the same as (32), (33), (30) and (31). The (28′) and (29′) are reduced to (28) and (29) by means of (35).

Thus the only new condition besides (28) - (34) is (37).

Finally from (11) we derive the following relations if we take into account the conditions (4) and (35):

(38) (coefficient of x) : $(2g'_{10} - \beta_2)\frac{\partial V}{\partial \tau} + g_{10}\frac{\partial^2 V}{\partial t \partial \tau} + \beta_2\frac{\partial^2 V}{\partial z^2} + \beta''_2 = 0$

(39) (» » y) : $(2f'_{10} - \gamma_1)\frac{\partial V}{\partial \tau} + f_{10}\frac{\partial^2 V}{\partial t \partial \tau} + \gamma_1\frac{\partial^2 V}{\partial z^2} + \gamma''_1 = 0$

(40) (independent term) : $g_{30}w^2\frac{\partial^2 V}{\partial t \partial \tau} + (g'_{30}w^2 + \delta_3)\frac{\partial^2 V}{\partial z^2} + (g'''_{30}w^2 + \delta''_3) = 0$

We shall now prove that $g_{10} = 0$. Because if $g_{10} \neq 0$, (38) is written:

(41) $\frac{\partial}{\partial t}\left(g_{10}^2\frac{\partial V}{\partial \tau}\right) - \beta_2 g_{10}\left(\frac{\partial V}{\partial \tau} - \frac{\partial^2 V}{\partial z^2}\right) + \beta''_2 g_{10} = 0$

From (31) and (34) on the other hand we find:

(42) $(g_{10} - f_{21}w^2)\left(\frac{\partial V}{\partial \tau} - \frac{\partial^2 V}{\partial z^2}\right) + (\beta'_2 - \gamma'_2) = 0$

and

(43) $g_{10}(f_{21}w^2 - g_{10})\frac{\partial^2 V}{\partial \tau^2} + f_{21}(\beta'_2 - \gamma'_2) = 0$

From this last equation we derive:

$$\frac{\partial V}{\partial \tau} = -\frac{(\beta'_2 - \gamma'_2)}{2g_{10}}\log(2f_{21}\tau - g_{10}) + k(t)$$

If the above values of $\frac{\partial V}{\partial \tau}$ and $\left(\frac{\partial V}{\partial \tau} - \frac{\partial^2 V}{\partial z^2}\right)$ are inserted into (41) we find:

$$-\frac{\partial}{\partial t}\left[\frac{(\beta'_2-\gamma'_2)}{2}g_{10}\right]\log(2f_{21}\tau-g_{10})+\frac{(\beta'_2-\gamma'_2)g_{10}}{2}\cdot\frac{g'_{10}}{(2f_{21}\tau-g_{10})}+$$

$$+\frac{\partial}{\partial t}[g_{10}^2k(t)]-\beta'_2g_{10}\cdot\frac{(\beta'_2-\gamma'_2)}{(2f_{21}\tau-g_{10})}+\beta''_2g_{10}=0$$

This equation must be fulfilled for every value of τ, therefore the coefficient of $\frac{1}{(2f_{21}\tau-g_{10})}$ must vanish. (It must be noted that equation (31) gives $f_{21}\neq 0$ if $g_{10}\neq 0$, otherwise $\frac{\partial^2 V}{\partial\tau^2}$ should be zero).

This condition gives:

either that: $\beta'_2-\gamma'_2=0$

or: $\frac{1}{2}g'_{10}=\beta_2$ and from the relation: $\beta_2+\gamma_2=g'_{10}$ it is found: $\beta_2=\gamma_2$, therefore: $\beta'_2-\gamma'_2=0$ again.

Thus if $g_{10}\neq 0$, then $\frac{\partial V}{\partial\tau}=k(t)$, which case we have excluded. We conclude therefore that $g_{10}=0$.

Similarly from (39), (30) and (37) it is concluded that $f_{10}=0$.

Then from (34) and (37) we find:

$\beta'_2-\gamma'_2=0$; but: $\beta_2+\gamma_2=g'_{10}=0$; therefore: $\beta_2=-\gamma_2=\text{const.}$

and similarly: $\beta_1=-\gamma_1=\text{const.}$

From (30), (31), (38), and (39) we conclude now easily that, either:

$$f_{21}=h_{21}=\beta_2=\gamma_2=\beta_1=\gamma_1=0 \quad \text{(case a')}$$

or:

$$\frac{\partial V}{\partial\tau}=\frac{\partial^2 V}{\partial z^2} \quad \text{(case b').}$$

We notice now that if $g_{30}=0$, then (28) is fulfilled and (40) is written:

(44)
$$\delta_3\frac{\partial^2 V}{\partial z^2}+\delta''_3=0$$

If on the other hand $g_{30}\neq 0$, then (28) is written:

(45)
$$\frac{\partial^2 V}{\partial z^2}=4\frac{\partial V}{\partial\tau}+2\tau\frac{\partial^2 V}{\partial\tau^2}+3\frac{g''_{30}}{g_{30}}$$

If we insert this value of $\frac{\partial^2 V}{\partial z^2}$ into (40) we find:

(46)
$$2\tau g_{30}\frac{\partial^2 V}{\partial t\partial\tau}+(2\tau g'_{30}+\delta_3)\left(4\frac{\partial V}{\partial\tau}+2\tau\frac{\partial^2 V}{\partial\tau^2}+3\frac{g''_{30}}{g_{30}}\right)+(2g'''_{30}\tau+\delta''_3)=0$$

Finally if we put the values of $\frac{\partial V}{\partial \tau}$, $\frac{\partial^2 V}{\partial t \partial \tau}$ and $\frac{\partial^2 V}{\partial \tau^2}$ given by (27) into (46) we derive:

$$\frac{4\tau}{\varphi^5}(g'_{30}\varphi - \varphi' g_{30})\left(2W + \frac{\tau}{\varphi^2}W'\right) + \frac{2\delta_3}{\varphi^4}\left(2W + \frac{\tau}{\varphi^2}W'\right) + \tag{47}$$

$$+ 2\tau\left[-g_{30}\frac{d}{dt}\left(\frac{\varphi''}{\varphi}\right) + g'_{30}\left(-4\frac{\varphi'}{\varphi} + 3\frac{g''_{30}}{g_{30}}\right) + g'''_{30}\right] + \left[\delta''_3 - 4\delta_3\frac{\varphi'}{\varphi} + 3\delta_3\frac{g''_{30}}{g_{30}}\right] = 0$$

Equation (36) gives that:

either: $g_{11} = 0$

or: $\frac{\partial^2 V}{\partial z^2} = -\frac{g''_{11}}{g_{11}} = Q(t)$

The solution $\frac{\partial^2 V}{\partial z^2} = Q(t)$ is consistent with case a′ only, because the case $\frac{\partial V}{\partial \tau} = Q(t)$ has been excluded.

Therefore if $\frac{\partial^2 V}{\partial z^2} = Q(t)$, we distinguish two subcases:

1) If $g_{30} = 0$, then from (44) it follows: $\frac{\delta''_3}{\delta_3} = \frac{g''_{11}}{g_{11}} = -Q(t)$

2) If $g_{30} \neq 0$, then from (45) we have:

$$2\frac{\partial}{\partial \tau}\left(\tau^2\frac{\partial V}{\partial \tau}\right) = \left[Q(t) - 3\frac{g''_{30}}{g_{30}}\right]\tau \tag{48}$$

and from (46) ≡ (40):

$$2\tau g_{30}\frac{\partial^2 V}{\partial t \partial \tau} + (2\tau g'_{30} + \delta_3)\,Q(t) + (2\tau g'''_{30} + \delta''_3) = 0 \tag{49}$$

From (48) we deduce:

$$\frac{\partial V}{\partial \tau} = \frac{1}{4}\left[Q(t) - 3\frac{g''_{30}}{g_{30}}\right] + \frac{q(t)}{\tau^2} \tag{50}$$

The comparison of equations (50) and (27) gives:

$$\frac{1}{4}\left[Q(t) - 3\frac{g''_{30}}{g_{30}}\right] = -\frac{\varphi''}{\varphi} \quad \text{and} \quad \frac{q(t)}{\tau^2} = \frac{4}{\varphi^4}W\left(\frac{\tau}{\varphi^2}\right)$$

therefore: $Q(t) = 3\frac{g''_{30}}{g_{30}} - 4\frac{\varphi''}{\varphi}$ and $q(t) = q = \text{const.}$

Now (49) is reduced to the relation:

$$-2\tau g_{30}\frac{d}{dt}\left(\frac{\varphi''}{\varphi}\right) + (2\tau g'_{30} + \delta_3)\left(3\frac{g''_{30}}{g_{30}} - 4\frac{\varphi''}{\varphi}\right) + (2\tau g'''_{30} + \delta''_3) = 0$$

therefore:

(51) $$-g_{30}\frac{d}{dt}\left(\frac{\varphi''}{\varphi}\right)+3\frac{g'_{30}g''_{30}}{g_{30}}-4\frac{g'_{30}\varphi''}{\varphi}+g'''_{30}=0$$

and

(52) $$\delta_3 Q(t)+\delta''_3=0$$

From (52) we conclude: $Q(t)=-\frac{\delta''_3}{\delta_3}$

and (51) may be written:

(53) $$\frac{d}{dt}(g^3_{30}g''_{30})=\frac{d}{dt}\left(g^4_{30}\frac{\varphi''}{\varphi}\right)$$

therefore: $$g^3_{30}g''_{30}+c_1=g^4_{30}\frac{\varphi''}{\varphi}$$

and finally:

(54) $$\frac{\varphi''}{\varphi}=\frac{c_1}{g^4_{30}}+\frac{g''_{30}}{g_{30}}$$

In conclusion we have the following cases:

Case a'.

If A) $$\frac{\partial^2 V}{\partial z^2}=Q(t)$$

then: $$\frac{\delta''_3}{\delta_3}=\frac{g''_{11}}{g_{11}}=-Q(t)$$

When $g_{30}=0$, $\frac{\partial V}{\partial \tau}$ is given by (27) (or (26) if H_{20} and|or $H_{30}\neq 0$).

When $g_{30}\neq 0$, we find from (50):

(55) $$\frac{\partial V}{\partial \tau}=-\frac{\varphi''}{\varphi}+\frac{q}{\tau^2}$$

Where φ and g_{30} are connected by the condition (54).

If B) $$\frac{\partial^2 V}{\partial z^2}\neq Q(t)$$

then: $$g_{11}=0$$

When $g_{30}=0$, it follows that $\delta_3=0$ and $\frac{\partial V}{\partial \tau}$ is given by (27) or (26).

When $g_{30}\neq 0$, $\frac{\partial^2 V}{\partial z^2}$ is given by (45) and $\frac{\partial V}{\partial \tau}$ by (27) or (26) with the further restriction (47).

Equation (47) may be written:

(56) $$(2W+wW')(Aw+B)=Cw+D$$

where: $w = \frac{\tau}{\varphi^2}$ and A, B, C and D are functions of t through φ, g_{30} and δ_3. $(2W + wW')$ is a function of w only, therefore some relations are existing between φ, g_{30} and δ_3 (three in general) such that $\frac{Cw + D}{Aw + B}$ should be independent of t.

It is easily shown that the solution of equation (56) assumes the forms:

If $A \neq 0$,

$$W = C_1 + \frac{C_2}{w} + \frac{C_3}{w^2} \log (w + C_4) + \frac{C_5}{w^2} \tag{57}$$

where C_1, C_2, C_3, C_4 and C_5 are constants.

If $A = 0 \left(\text{i. e. if } \frac{g_{30}}{\varphi} = \text{const.} \right)$

$$W = C_1 w + C_2 + \frac{C_3}{w^2} \tag{58}$$

In the special case (formula (26)) we have: $W = \frac{q_0 \varphi^3}{\tau^{3/2}}$ and (47) may be written:

$$\frac{2}{\varphi^2}(g'_{30}\varphi - \varphi' g_{30}) \frac{q_0}{\tau^{1/2}} + \frac{\delta_3}{\varphi} \frac{q_0}{\tau^{3/2}} + 2\tau \left[- g_{30} \frac{d}{dt}\left(\frac{\varphi''}{\varphi}\right) + \right.$$

$$\left. + g'_{30}\left(- 4 \frac{\varphi''}{\varphi} + 3 \frac{g''_{30}}{g_{30}} \right) + g'''_{30} \right] + \left[\delta''_3 - 4\delta_3 \frac{\varphi''}{\varphi} + 3\delta_3 \frac{g''_{30}}{g_{30}} \right] = 0$$

therefore: $g'_{30}\varphi - \varphi' g_{30} = 0$, hence $\frac{g_{30}}{\varphi} = \text{const.}$

and $\delta_3 = 0$

Case b'.

Then $g_{11} = 0$

If now $g_{30} = 0$, it is: $\delta_3 = 0$, and $\frac{\partial V}{\partial \tau}$ is given by equation (27) or (26).

If $g_{30} \neq 0$, equation (45) gives:

$$3 \frac{\partial V}{\partial \tau} + 2\tau \frac{\partial^2 V}{\partial \tau^2} + 3 \frac{g''_{30}}{g_{30}} = 0 \tag{59}$$

This equation may be written:

$$\frac{\partial}{\partial \tau}\left(\tau^{3/2} \frac{\partial V}{\partial \tau} \right) = - \frac{3}{2} \tau^{1/2} \frac{g''_{30}}{g_{30}}$$

therefore: $$\frac{\partial V}{\partial \tau} = - \frac{g''_{30}}{g_{30}} + \frac{F(t)}{\tau^{3/2}}$$

Taking into account equation (27) we find: $\frac{\varphi''}{\varphi} = \frac{g''_{30}}{g_{30}}$ and $F(t) = \frac{q_0}{\varphi}$, and $\frac{\partial V}{\partial \tau}$ is given by (26). Then from (47) we find that $\frac{g_{30}}{\varphi} = \text{const.}$ and $\delta_3 = 0$.

IV

The force per unit mass on the plane of symmetry of a stellar system having both a plane and an axis of symmetry is given by the formula:

$$\frac{\partial V}{\partial w} = -\frac{\varphi''}{\varphi} w + \frac{w}{\varphi^4} W \left(\frac{w^2}{2\varphi^2}\right) \tag{60}$$

which is a direct consequence of equation (27).

The function W in some cases assumes special forms, as the forms (57), (58), $W = \frac{4q\varphi^4}{w^4}$ (equation (55)), or $W = \frac{2^{3/2} q_0 \varphi^3}{w^3}$ (as in the special case, when H_{20} and/or $H_{30} \neq 0$).

In this last case: $\frac{\partial V}{\partial w} = -\frac{\varphi''}{\varphi} w + \frac{2^{3/2} q_0}{\varphi} \cdot \frac{1}{w^2}$

i.e. the force is the sum of a newtonian force due to a central mass and a force proportional to the distance, due to an homogeneous ellipsoid, containing the attracted points.

The coefficients of the velocity ellipsoid (equations (1)-(3)) in the cases discussed assume simpler forms. E. g. in case a′ we have:

$$a = \varphi^2 - 2H_{20}\varphi y - h_{40} y^2$$
$$b = \varphi^2 - 2H_{30}\varphi x - h_{40} x^2$$
$$f = -g_{11} x + g_{30} y$$
$$g = g_{11} y + g_{30} x$$
$$h = H_{20}\varphi x + H_{30}\varphi y + h_{40} xy \quad \text{etc.}$$

If $g_{11} = g_{30} = 0$, then $f = g = 0$, therefore two of the principal axes of the velocity ellipsoid lie on the plane of symmetry of the system. But in general there exists a deviation of the vertex, i. e. the principal axis of the velocity ellipsoid is not directed towards the center. If e is the angle between the direction of the principal axis and the axis of x, then[7]:

$$\tan 2e = \frac{2h}{a-b}$$

and the deviation of the vertex is given by the difference $(e-\vartheta)$, where:

$\tan\vartheta = y/x$. Therefore the deviation of the vertex depends upon the values of H_{20}, H_{30}, h_{40} and φ, as well as on the position of the point (x, y), except if $H_{20} = H_{30} = 0$, when the deviation of the vertex is always zero[1].

ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἐργασία βασίζεται εἰς τὴν θεωρίαν περὶ τοῦ ἐλλειψοειδοῦς ταχυτήτων τοῦ S. Chandrasekhar. Κατὰ πρῶτον εὑρίσκομεν διὰ μιᾶς συντομωτέρας μεθόδου τὰ συμπεράσματα τοῦ S. Chandrasekhar ἐπὶ τοῦ προβλήματος τῶν δύο διαστάσεων. Δι' ἐφαρμογῆς τῆς μεθόδου ταύτης εὑρίσκομεν τὴν μορφὴν τοῦ δυναμικοῦ πεδίου εἰς τὸ ἐπίπεδον συμμετρίας τριαξονικοῦ ἀστρικοῦ συστήματος, συμμετρικοῦ ὡς πρὸς ἄξονα καὶ ἐπίπεδον. Ἡ δύναμις ἀνὰ μονάδα μάζης δίδεται ὑπὸ τὴν μορφήν:

$$\frac{\partial V}{\partial w} = -\frac{\varphi''}{\varphi} w + \frac{w}{\varphi^4} W\left(\frac{w^2}{2\varphi^2}\right)$$

ὅπου $w = \sqrt{x^2 + y^2}$, φ αὐθαίρετος συνάρτησις τοῦ χρόνου καὶ W συνάρτησις τοῦ $\frac{w^2}{2\varphi^2}$, ἡ ὁποία ἀναλόγως τῆς περιπτώσεως εἴτε παραμένει αὐθαίρετος εἴτε λαμβάνει εἰδικὰς μορφάς, ὅπως τὰς (57), (58), $W = \dfrac{4q\varphi^4}{w^4}$ ἢ $W = \dfrac{2^{3/2} q_0 \varphi^8}{w^3}$.—

ΦΩΤΟΓΡΑΜΜΕΤΡΙΑ. — **Le procédé le plus favorable d'orientation relative d'après le prof. A. Brandenberger appliqué à l'autographe Wild A6,** *par* ***C. Cladas****. Ἀνεκοινώθη ὑπὸ τοῦ κ. Βασ. Αἰγινήτου[2].

Je donne dans ce qui suit une application à l' autographe Wild A6 d'une nouvelle méthode d'orientation relative d'après le Dr. A. Brandenberger.

A) Détermination des corrections, des éléments d'orientation relative.

Les relations analytiques pour la parallaxe verticale sont (*M. Zeller*, Traité de Photogrammétrie, page 220)

$$pv_i' = +F\,\psi'\,(dY'' - dY') \tag{1α}$$

$$pv_i'' = -F\,\psi''\,(dY'' - dY') \tag{1β}$$

Pour des levers nadiraux on a

$$\psi' = \psi'' = \frac{1}{Z}$$

donc $pv' = -pv'' = F\,\frac{1}{Z}\,(dY'' - dY')$ (2)

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[1] E. von der Pahlen, p. 179.

* Κ. ΚΛΑΔΑ, Ὁ εὐνοϊκώτερος τρόπος σχετικοῦ προσανατολισμοῦ κατὰ τὸν καθηγητὴν A. Brandenberger ἐφαρμοζόμενος εἰς τὸν αὐτοχαρτογράφον Wild A6.

[2] Ἀνεκοινώθη κατὰ τὴν συνεδρίαν τῆς 19 Μαΐου 1955.