ΜΑΘΗΜΑΤΙΚΑ.— **On the properties of certain functions orthogonal over circular domains. Generalisation of Zernike polynomials** [1], *by Nicholas Chako* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ὀ. Πυλαρινοῦ.

In the theory of diffraction of light or electromagnetic waves by circular apertures and discs we meet certain types of integrals of the form

$$(1) \qquad I_{m,\beta,\sigma}(\varrho, z, a) = \int_0^\infty J_m(k\varrho t)\, J_\beta(kat)\, e^{\pm z\sqrt{t^2-k^2}}\, t^{2\sigma-\beta} \frac{dt}{\sqrt{t^2-k^2}}, \quad (z \lessgtr 0)$$

where ϱ, z are cylindrical coordinates, m, β and σ are real constant quantities and **a** is a fixed constant, the radius of the circular domain (aperture or disc). By choosing the parameters appropriately, the integral (1) is found to be a solution of the axial Helmholtz equation and in addition it will satisfy the Sommerfeld radiation condition and the Bouwkamp-Meixner edge condition and, either of the boundary conditions: $u=0$, or $\frac{\partial u}{\partial z}=0$ on the screen of the aperture or on the disc, if u is expressed, $u(x, y, z) = I_{m,\beta,\sigma} \exp(im\varphi)$. The purpose of this paper, however, is not the discussion of (1) which is done elsewhere [1], but to study the properties of certain functions and polynomials (hypergeometric) derived from it.

Let us consider the following function defined by the integral

$$(2) \qquad G_{m,\beta,\nu}(\varrho, a) = \int_0^\infty J_m(\varrho t)\, J_\beta(at)\, t^{m+2\nu+1-\beta}\, dt,$$

which is obtained by differentiating (1) with respect to $\pm z$ and letting z tend to zero. In (2) $k=1$ and $2\sigma = 2\nu + m + 1$. The function $G_{m,\beta,\nu}$

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1. After modifications, the contents of this paper are taken from paragragh 5. of Chapter VIII of my principal thesis for the «doctoral d'Etat ès sciences» (1966), Sorbonne.

is discontinuous on the rim of the aperture or the edge of the disc. Evaluating the integral, we obtain the following expressions :

$$(3) \qquad G_{m,\beta,}(\varrho, a) = \frac{1}{a}\left(\frac{2}{a}\right)^{m+2\nu+1-\beta}\left(\frac{\varrho}{a}\right)^{m} \frac{\Gamma(m+\nu+1)}{\Gamma(m+1)\,\Gamma(\beta-m-\nu)}$$

$$ {}_2F_1\left(m+\nu+1,\ m+\nu+1-\beta;\ m+1;\ \frac{\varrho^2}{a^2}\right) =$$

$$= \frac{1}{a}\left(\frac{2}{a}\right)^{m+2\nu+1-\beta}\left(\frac{\varrho}{a}\right)^{m} \frac{\Gamma(m+\nu+1)}{\Gamma(m+1)\,\Gamma(\beta-m-\nu)}\left(1-\frac{\varrho^2}{a^2}\right)^{\beta-m-2\nu-1}$$

$$ {}_2F_1\left(-\nu,\ \beta-\nu;\ m+1;\ \frac{\varrho^2}{a^2}\right)$$

valid for $0<\varrho<a$ and $\beta-m-1-2\nu>-1$.

On the other hand if $\varrho>a$, we have

$$(4) \qquad G_{m,\beta,\nu}(\varrho, a) = \frac{1}{2}\left(\frac{a}{2}\right)^{\beta-m-2(\nu+1)}\left(\frac{a}{\varrho}\right)^{m+2(\nu+1)} \frac{\Gamma(m+\nu+1)}{\Gamma(\beta+1)\,\Gamma(-\nu)}$$

$$ {}_2F_1\left(m+\nu+1,\ \nu+1;\ m+1;\ \frac{a^2}{\varrho^2}\right) =$$

$$= \frac{\cos\left(\nu+\frac{1}{2}\right)\pi}{2\pi}\ \frac{\Gamma(m+\nu+1)\,\Gamma(\nu+1)}{\Gamma(\beta+1)}$$

$$ {}_2F_1\left(m+\nu+1,\ \nu+1;\ m+1;\ \frac{a^2}{\varrho^2}\right)$$

provided ν is not a positive integer, otherwise it vanishes. Consequently, if $\nu=n$, $(n=0, 1, \ldots)$, we obtain

$$(5) \qquad G_{m,\beta,n}(\varrho, a) =$$

$$= \frac{1}{a}\left(\frac{2}{a}\right)^{m+2n+1-\beta}\left(\frac{\varrho}{a}\right)^{m} \frac{\Gamma(m+n+1)}{\Gamma(m+1)\,\Gamma(\beta-m-n)}\left(1-\frac{\varrho^2}{a^2}\right)^{\beta-m-2n-1}$$

$$ {}_2F_1\left(-n,\ \beta-n;\ m+1;\ \frac{\varrho^2}{a^2}\right) \qquad \text{if } \varrho<a$$

$$= 0 \quad \text{if } \varrho>a.$$

On the other hand if $\varrho \to a$ our function becomes

(6) $G_{m,\beta,\nu}(\varrho \to a, a) =$

$$= \frac{1}{a}\left(\frac{2}{a}\right)^{m+2\nu+1-\beta}\left(\frac{\varrho}{a}\right)^{m} \frac{\sin(\beta-m-\nu)\pi\, \Gamma(m+2\nu+1-\beta)}{\pi} \lim_{\varrho\to a}\left[1-\frac{\varrho^2}{a^2}\right]^{\beta-m-2\nu-1}.$$

Therefore, $G_{m,\beta,}(\varrho, a)$ becomes singular at the edge of the aperture for $\beta < m+2n+1$, whether ν is integer or not. If $\beta = m+2n+1$, $G_{m,\beta,n}$ vanishes as ϱ tends to **a** and thus, we obtain a function which is finite within the circular aperture or disc, and vanishes on the boundary, as well as, on the screen itself.

Suppose that ν is a positive integer n. In this case the hypergeometric function appearing in (3), namely, ${}_2F_1(-n, \beta-n; m+1; x^2)$, $x = \frac{\varrho}{a}$, reduces to a polynomial of degree 2n in x. Therefore we can express $G_{m,\beta,n}$ in terms of the Jacobi polynomials $G_n(\alpha, \gamma, x^2)$. Since the Jacobi polynomials are expressed by

$$(7) \qquad G_n(\alpha, \gamma; x) = {}_2F_1(-n, \alpha+n; \gamma; x), \qquad (0<x<1)$$

provided $\gamma \neq 0, -1, -2, \ldots, -n+1$, we obtain the following relation between the Jacobi polynomials and $G_{\alpha,\beta,\sigma}$:

(8) $G_n(\alpha, \gamma; x) =$

$$= a(x)^{\frac{1-\gamma}{2}}\left(\frac{2}{a}\right)^{\alpha-\gamma} \frac{\Gamma(\gamma)\,\Gamma(\alpha+n+1-\gamma)}{\Gamma(\gamma+n)}(1-x)^{\gamma-\alpha}\, G_{\gamma-1,\alpha+2n,\,n}(\sqrt{x}), \quad \left(x = \frac{\varrho}{a}\right).$$

To obtain $G_{m,\beta,n}$ in terms of G_n, the Jacobi polynomial, we must put $\alpha = \beta - 2n$, $\gamma = m+1$. The result is

(9) $G_{m,\beta,n}(\varrho, a) =$

$$= \frac{1}{a}\left(\frac{2}{a}\right)^{m+2n+1-\beta}\left(\frac{\varrho}{a}\right)^{m} \frac{\Gamma(m+n+1)}{\Gamma(m+1)\;\Gamma(\beta-m-n)}\left(1-\frac{\varrho^2}{a^2}\right)^{\beta-m-2n-1}$$
$$G_n\left(\beta-2n,\, m+1;\, \frac{\varrho^2}{a^2}\right).$$

From the orthogonal properties of Jacobi polynomials, we derive the following orthogonal relations between our functions, namely

$$(10)\quad \int_0^a \left[G_{\gamma-1,\,\alpha+2n,\gamma+n-\frac{1}{2}}(\varrho,a)\, G_{\gamma-1,\alpha+2p,\gamma+p-\frac{1}{2}}(\varrho,a)\right]\left(1-\frac{\varrho^2}{a^2}\right)^{\gamma-\alpha}\varrho\, d\varrho = $$

$$= 0 \quad \text{if} \quad p \neq n$$

$$= N^2 \quad \text{if} \quad p = n$$

where N^2 is equal to

$$(11)\qquad N^2 = \frac{1}{a^2}\left(\frac{4}{a^2}\right)^{\gamma-\alpha} \frac{\Gamma(\gamma+n)\,\Gamma(n+1)}{\Gamma(\alpha+n)\,\Gamma(\alpha+n+1-\gamma)\,(\alpha+2n)}\,.$$

Replacing α by $\beta-2n$ and γ by $m+1$, we find the following orthogonality relation for $G_{m,\beta,n}(\varrho,a)$ (*)

$$(12)\quad \int_0^a G_{m,\beta,n}(\varrho,a)\, G_{m,\beta,p}(\varrho,a)\left[1-\frac{\varrho^2}{a^2}\right]^{m+n+p+1-\beta}\varrho\, d\varrho = 0 \quad \text{if} \quad p \neq n$$

$$= N^2 \quad \text{if} \quad p = n$$

with

$$(13)\qquad N^2 = \frac{1}{a^2}\left(\frac{\alpha^2}{4}\right)^{\beta-m-2n-1} \frac{\Gamma(m+n+1)\,\Gamma(n+1)}{\Gamma(\beta-m-n)\,\Gamma(\beta-n)}\,\frac{1}{\beta}\,.$$

From (12) we conclude that the functions $G_{m,\beta,n}$ form an orthogonal set over a circular domain of any radius **a**, with the weight function $w(\varrho)$,

$$(A)\qquad w(\varrho) = \varrho\left[1-\frac{\varrho^2}{a^2}\right]^{m+2n+1-\beta}\,.$$

We now define a function $K_{m,\beta,n}$ to be

$$(14)\qquad K_{m,\beta,n}(\varrho,a) = \frac{1}{N}\,G_{m,\beta,n}(\varrho,a)\left[1-\frac{\varrho^2}{a^2}\right]^{\frac{m+2n+1-\beta}{2}}\sqrt{\varrho}$$

and in general

$$(15)\quad K_{\alpha,\gamma;n}(\varrho,a) = K(\alpha,\gamma;\varrho) = \frac{1}{N}\,G_{\gamma-1,\alpha+2n,\gamma+n-\frac{1}{2}}(\varrho,a)\left[1-\frac{\varrho^2}{a^2}\right]^{\frac{\gamma-\alpha}{2}}\sqrt{\varrho}$$

(*) We would like to remark that the notation used here for the function $G_{m,\beta,n}$ corresponds to $G_{m,\beta,m+n+1/2}$ of the thesis. The last index of our function is equal to half the exponent σ appearing in the integrand of (2).

These functions form an orthonormal set over a circular two dimensional domain. We have

$$(16) \qquad \int_0^a K_{\alpha,\gamma;n}(\varrho, a)\, K_{\alpha,\gamma;p}(\varrho, a)\ d\varrho = 0 \quad \text{if} \quad p \neq n$$
$$= 1 \quad \text{if} \quad p = n\,.$$

So far we have let the parameter β to be an arbitrary real quantity. In the following discussion we shall let β to assume certain values for which the function $G_{m,\beta,n}$ plays an important role in the theory of diffraction, as mentioned in the beginning of this article.

Before we take up the special cases, let us write (3) in another form by replacing the hypergeometric function by its series, which reduces to a finite form when ν is a positive integer. We have

$$(17) \quad G_{m,\beta,n}(\varrho,a) = G_{m,\beta,\,m+n+\frac{1}{2}}(\varrho,a) =$$
$$= \frac{1}{a}\left(\frac{2}{a}\right)^{m+2n+1-\beta} \frac{\Gamma(m+n+1)\ \Gamma(n+1)}{\Gamma(\beta-n)\ \Gamma(\beta-m-n)} \left(\frac{\varrho}{a}\right)^m$$
$$\left[1-\frac{\varrho^2}{a^2}\right]^{\beta-m-2n-1} \sum_{r=o}^{r=n} (-1)^r_{r!}\ \frac{\Gamma(r+\beta-n)}{\Gamma(r+m+1)\ \Gamma(n+1-r)} \left(\frac{\varrho}{a}\right)^{2r}$$

It is easy to show from (17) that as ϱ tends to a, we obtain formula (6). Moreover, if $\beta = m+2n+1$, (17) reduces to a polynomial of degree $m+2n$. In this case, $G_{m,\beta,n}$ reduces to an orthogonal polynomial with the weight function ϱ, that is:

$$(18) \qquad \int_0^a G_{m,\beta,n}(\varrho, a)\ G_{m,\beta,p}(\varrho, a)\ \varrho\, d\varrho = 0 \qquad \text{if} \quad p \neq n$$
$$= \frac{1}{a^2}\ \frac{1}{m+2n+1} \quad \text{if} \quad p = n$$

C a s e I. Let $\beta = m + 2n + \frac{3}{2}$.

For this value of β, $G_{m,\beta,n}$ reduces to a hypergeometric polynomial multiplied by $\sqrt{1-x^2}$, $\left(x = \frac{\varrho}{a}\right)$

$$(19) \quad G_{m,\,m+2n+\frac{3}{2}\,n}(\varrho, a) =$$
$$= \frac{1}{2a}\ \frac{\Gamma(m+n+1)}{\Gamma(m+1)\ \Gamma\left(n+\frac{3}{2}\right)} \left(\frac{\varrho}{a}\right)^m \sqrt{1-x^2}\ {}_2F_1\left(-n,\ m+n+\frac{3}{2};\ m+1;\ x^2\right)$$

Thus $G_{m, m+2n+\frac{3}{2}, n}$ vanishes as ϱ tends to **a** and its weight function according to (12) or (A) is equal to $\varrho\,(1-x^2)^{-\frac{1}{2}}$. The normalisation constant N is found to be as follows

$$N^2 = \frac{1}{2a} \frac{\Gamma(m+n+1)\,\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(m+n+\frac{3}{2}\right)} \frac{1}{m+2n+\frac{3}{2}}. \tag{20}$$

Case II. Let $\beta = m + 2n + \frac{1}{2}$.

For this value of β, $G_{m,\beta,n}$ is

(21) $G_{m, m+2n+\frac{1}{2}, n}(\varrho, a) =$

$$= \frac{2}{a^3} \frac{\Gamma(m+n+1)}{\Gamma(m+1)\,\Gamma\left(n+\frac{1}{2}\right)} \left(\frac{\varrho}{2}\right)^m \left[1-\frac{\varrho^2}{a^2}\right]^{-\frac{1}{2}} {}_2F_1\left(-n, m+n+\frac{1}{2}; m+1; \frac{\varrho^2}{a^2}\right).$$

The function $G_{m, m+2n+\frac{1}{2}, n}$ becomes singular as $\varrho \to a$. Its weight function is $\varrho\sqrt{1-x^2}$ and the normalisation constant is

$$N^2 = \frac{2}{a^3} \frac{\Gamma(m+n+1)\,\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+n+\frac{1}{2}\right)} \frac{1}{m+2n+\frac{1}{2}}.$$

Case III.

A special but interesting case is when we put $a = 1$ and $m+2n = p$. In this particular case our function becomes a Zernike polynomial.

It is not difficult to deduce the following properties of $G_{m,\beta,n}$:

$$\int_0^a G_{m,\beta,n}(u, a)\, J_m(\varrho u)\, u\, du = \varrho^{m+2n-\beta}\, J_\beta(a\varrho) \tag{22}$$

and as special cases, for $\beta = m+2n$, we get

$$\int_0^a G_{m,\beta,n}(u, a)\, J_m(\varrho u)\, u\, du = J_{m+2n}(\varrho a) \tag{23}$$

and for $\beta = m+2n+1$,

$$\int_0^a G_{m, m+2n+1, n}(u, a)\, J_m(\varrho u)\, u\, du = \frac{J_{m+2n+1}(a\varrho)}{\varrho}. \tag{24}$$

Now if we put $m+2n=p$, $a=1$, formula (24) becomes

$$(25) \qquad \int_0^{a=1} G_{m,p+1,\frac{p-m}{2}}(u)\, J_m(\varrho u)\, u\, du = \frac{J_{p+1}(\varrho)}{\varrho}$$

or, in the notation employed in the thesis,

$$(25') \qquad \int_0^{a} G_{m,p+1,\frac{m+p+1}{2}}(u)\, J_m(\varrho u)\, u\, du = \frac{J_{p+1}(\varrho)}{\varrho}.$$

This formula is equivalent to that obtained by Nijboer [3], which reads

$$(26) \qquad \int_0^{1} Z_n^m(u)\, J_m(\varrho u)\, u\, du = (-1)^{\frac{n-m}{2}} \frac{J_{n+1}(\varrho)}{\varrho}.$$

Therefore the Zernike polynomials can be expressed as follows *

$$(27) \qquad Z_n^m(\varrho) = (-1)^{\frac{n-m}{2}} G_{m,n+1,\frac{m+n+1}{2}}(\varrho) \qquad (0<\varrho<1),$$

and $n-m=2p$, $(p=0,1,2,\ldots)$.

From (22) we can derive a number of interesting formulas. Since $G_{m,\beta,m+n+\frac{1}{2}} = G_{m,\beta,n}$ is expressed by the integral

$$(28) \qquad G_{m,\beta,m+n+\frac{1}{2}}(\varrho, a) = \int_0^{\infty} J_m(\varrho t)\, J_\beta(at)\, t^{m+2n+1-\beta}\, dt,$$

we can replace $J_\beta(at)\, t^{m+2n+1-\beta}$ by the equivalent expression (22) and obtain **

$$(29) \qquad G_{m,\beta,m+n+\frac{1}{2}}(\varrho, a) = \int_0^{\infty} J_m(\varrho t) t dt \int_0^{a} G_{m,\beta,m+n+\frac{1}{2}}(u, a)\, J_m(tu)\, u\, du.$$

This relation resembles the Fourier-Bessel (transformation) integral. In fact, if our function satisfies the Dirichlet condition, the integral

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* Using the integral representation of $G_{m,n+1,\frac{m+n+1}{2}}(\varrho, a=1)$ we may express the Zernike polynomial in an integral form

$$Z_n^m(\varrho) = A \int_0^{\infty} J_m(\varrho t)\, J_{n+1}(t)\, dt, \qquad A=(-1)^{\frac{n-m}{2}}.$$

This is the integral representation of the Zernike polynomials.

** The interchange of integrals is allowed because the function is absolutely integrable in the interval $[0, a]$, since $\beta-m-2n-1>-1$.

expression in (28) is equal to the average of the function evaluated at $\varrho+0$ and $\varrho-0$ if $0<\varrho<a$; it is equal to half the value of the function at $\varrho=a-0$, if $\varrho=a$ and half the value at $\varrho=0+$, if $\varrho=0$. We have seen that for $\varrho>a$, the function vanishes.

We remark here that for the special values of $\beta=m+2n+\frac{3}{2}$ and $m+2n+\frac{1}{2}$, the $G_{m,\beta,n}$ have also been studied by Nomura and his collaborators [4] and myself [1].

From the precedeing analysis it is not difficult to construct polynomials for annular circular domains.

Finally, one can express $G_{m,\beta,n}$ for $\beta=m+2n+\frac{1}{2}$ and $\beta=m+2n+\frac{3}{2}$, etc., in terms of associate Legendre polynomials. For example:

$$(30)\quad G_{m,\,m+2n+\frac{3}{2},\,m+n+\frac{1}{2}}(\varrho,a)=$$

$$=\frac{\Gamma(n+1)}{2^{m-\frac{1}{2}}a^{\frac{3}{2}}\,\Gamma\left(m+n+\frac{3}{2}\right)}P^{m}_{m+2n+1}\left(\sqrt{1-x^2}\right),\ \text{for } 0<\varrho<a$$

$$=0\ \text{if } \varrho>a\quad\left(x=\frac{\varrho}{a}\right).$$

To obtain formula (30) one utilises the properties of hypergeometric functions. One could also deduce (30) by following Whittaker [5], for a solution of Laplace equation in cylindrical coordinates (ϱ, φ, z) is given by an integral of type (1), where in place of the exp $\left(\pm z\sqrt{t^2-k^2}\right)$ one substitutes $\pm$ zt and drops out the denominator $\sqrt{t^2-k^2}$, and after evaluating the integral let z tend to zero. This limiting value is the function $G_{m,\beta,n}$. On the other hand the Laplace equation is separable in prolate spheroidal coordinates (ξ, η, φ), so a solution is also given in terms of spheroidal coordinates of the form $F^m_n(\xi)\,H^m_n(\eta)\,e^{im\varphi}$ (Meixner and Schäfke [6], Nielsen [7]). Then $G_{m,\beta,n}$ would correspond to the limiting value of the product $F^m_n(\xi)\,H^m_n(\eta)$ as $\xi\rightarrow 0$, since letting $\xi=0$ is equivalent to letting $z=0$, for $\varrho=a\sqrt{(1+\xi^2)(1-\eta^2)}$, $z=a\xi\eta$. The limiting value of $F^m_n(\xi)\,H^m_n(\eta)$ as $\xi\rightarrow 0$ is the right hand side of (30).

The polynomials given by (25) play an important role in the theory of diffraction of electromagnetic waves by circular aperture or discs

[2, 4, 8] and in many problems of mathematical physics and on analytic theory of numbers.

It should be noted that these functions are not only generalisations of Zernike and Jacobi polynomials but also of a number of functions employed in physics, such as Lommel, Struve and Llambda functions, etc. In fact, one can use this functions in treat the diffraction theory of aberrations where the use of Zernike polynomials fails as we have shown in the thesis and in reference [8].

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SUMMARY

In this article we have given some of the most important properties of certain functions which are orthogonal over a two dimensional circular domain and the polynomials associated with these. These

functions are of some interest because they occur in many problems of mathematical physics and especially in diffraction problems. It is shown that the Jacobi and Zernike polynomials are special forms of these functions.

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Ὁ Ἀκαδημαϊκὸς κ. **Ὄθων Πυλαρινὸς** *κατὰ τὴν ἀνακοίνωσιν τῆς ὡς ἄνω ἐργασίας εἶπε τὰ κάτωθι :*

Ὁ *συγγραφεὺς* εἰς τὴν ἐργασίαν ταύτην μελετᾷ συστήματα συναρτήσεων ὡρισμένης μορφῆς τῆς ἀποστάσεως ἀπὸ ἑνὸς σημείου ἑνὸς διδιαστάτου εὐκλειδείου χώρου, αἱ ὁποῖαι εἶναι ὀρθογώνιοι ἐντὸς κυκλικῶν πεδίων τοῦ χώρου τούτου ἐχόντων ὡς κέντρον τὸ ἐν λόγῳ σημεῖον, καὶ δίδει τὰς σημαντικωτέρας τῶν ἰδιοτήτων τῶν συναρτήσεων τούτων· ἀποδεικνύει δὲ πρὸς τούτοις ὅτι τὰ πολυώνυμα τῶν JACOBI καὶ ZERNIKE εἶναι εἰδικαὶ περιπτώσεις τοιούτων συναρτήσεων. Ἡ μελέτη τῶν συναρτήσεων τούτων παρουσιάζει ἐνδιαφέρον, διότι αὗται ἐμφανίζονται εἰς πολλὰ προβλήματα τῆς Μαθηματικῆς Φυσικῆς καὶ δὴ εἰς προβλήματα περιθλάσεως.