ΜΑΘΗΜΑΤΙΚΑ.— **Certain relations between generalized topology and Universal Algebra**, *by Peter B. Krikelis.** Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κ. Παπαϊωάννου.

I. Backgrounds.

The notions of sef and homomorphism of sefs, introduced by S. P. Zervos and studied by him in N. S. 1 - N. S. 5, (**4**), constitute the Universal Algebra background used below (we effectively use only N. S. 1).

The notions of generalized topology and neighborhood spaces, as exposed in Z. P. Mamuzic's book «Introduction to General Topology» (**2**), constitute the topological backgrounds of the present paper. General reference: Djuro Kurepa (**1**). Terminology, abbreviations and notations. Iff = if and only if. Resp. = respectively. | denotes the end of a proof. P(A) = the set of all subsets of the set A. Function = Mapping = single valued mapping.

Ia. Definition of the W-neighborhood of $a \in E$.

Let E, K and M be three non-empty sets, g a mapping $E \times K \rightarrow M$ and σ a mapping $E \rightarrow P(P(M))$. For every $a \in E$, $\sigma(a)$ may always be written in the form of a family $(X_{l_a})_{l_a \in L_a}$, where L_a is an index set, depending on a.

Notation. For $(x, y) \in E \times K$, we write, as usual, $g(x, y)$ in place of $g((x, y))$.

We consider the relation in the «unknown» x, $g(a, x) \in X_{l_a}$ (α) and make the supposition 1) for every (a, l_a), there exists at least one $x \in K$ for which this relation is true, and 2) if $a \in E \cap K$, $g(a, a) \in X_{l_a}$ is true for all $l_a \in L_a$. We shall call the set of all solutions of (α), i. e. the set $\{x \mid x \in K \text{ and } g(a, x) \in X_{l_a}\}$, W-neighborhood of $a \in E$ and shall denote it by $W_{l_a}(a)$. The family $(W_{l_a}(a))_{l_a \in L_a}$ will, accordingly, be called the family of W-neighborhoods of $a \in E$. We shall refer to this definition as definition Ia.

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* ΠΕΤΡΟΥ Β. ΚΡΙΚΕΛΗ, Σχέσεις τινὲς μεταξὺ τῆς γενικευμένης τοπολογίας καὶ τῆς καθολικῆς ἀλγέβρας.

Obvious consequences. For all $a \in E$ and $l_a \in L_a$, $W_{l_a}(a) \neq \varnothing$ (β) and $g^{-1}(X_{l_a}) = \{(a, x) \mid x \in W_{l_a}(a)\} = \{a\} \times W_{l_a}(a)$ (γ).

The special case $K = E$ (then supposition 1 becomes a consequence of supposition 2) was considered by Z. P. Mamuzic in his book (**2**), p. 121, where he defined in this way essentially, a neighborhood space. He used this general setting for describing and studying generalized topological spaces as well as topological spaces.

Ib. Example.

Let us consider a Newtonian field due to a single attracting material point placed at the origin 0 of the axes. We shall take : $E = \{0\}$ (hence, 0 is the only element a satisfying $a \in E$; hence we shall write L_0 in place of L_a etc.). $K = R^3 - \{0\}$ (R denoting, of course, the real line), each point $P \in K$ carrying a unit mass. $M = R$. For each $P \in K$, $g(0, P)$ = the potential of the field at P, i.e. $-c/(0P)$ where c is a positive constant. $L_0 = R_+ - \{0\}$. For each $l_0 \in L_0$, X_{l_0} = the half line $]\leftarrow, -l_0]$. Then $W_{l_0}(0)$ = the sphere with center 0 and radius c/l_0, deprived of its center.

II. Recall of the definition of a sef.

(See N. S. 1 ; we use its terminology and notations).

Note. It is useful, sometimes, to consider a set A_ϑ in place of the family $(a_{\vartheta_i})_{i \in I}$ considered in N. S. 1 ; we represent, in this case, A_ϑ by the identical mapping of A_ϑ onto itself, but, for reasons of simplicity, continue to write A_ϑ.

III. Definition of a U-sef.

We define, below, a special kind of sef, which will be called U-sef and will be denoted by U, U′ etc.

Definition 1. $U = (A, \Theta, B, s)$, where :

$A = E$,

$T = \{a \mid a \in E\}$,

$H = \{W_{l_a}(a) \mid a \in E \text{ and } l_a \in L_a\}$, where $W_{l_a}(a)$ is as in Ia, with $K = E$.

$$\Theta = \{\{a\} \times W_{l_a}(a) \mid a \in E\},$$

$$B = E \times \{X_{l_a} \mid a \in E \text{ and } l_a \in L_a\} \text{ and}$$

$$s : \Theta \to B \text{ is defined by } s(a, W_{l_a}(a)) = (a, X_{l_a}).$$

E will be called the carrier of U.

IV. Recall of the definition of a mapping between sefs.

(See N. S. 1).

Note. Mappings and homomorphisms between sefs and sofs have been studied in N. S. 1 - N. S. 5. In order to be useful in our present study, they need, however, generalization in one direction and specialization on the other. These we now present.

V. Homomorphisms between U-sefs.

Definition 2. Let us consider two sefs Γ and Γ', with $I \subseteq I'$ and non-empty sets T and T′, not necessarily satisfying the relation $T \subseteq T'$, and a triple of functions $f^{(1)} : A \to A'$, $f^{(2)} : B \to B'$, and $f^{(3)} : T \to T'$. For any $\vartheta = (t, (a_i)_{i \in I_\vartheta}) \in \Theta$, set $f^{(1)}_*(\vartheta) = (f^{(3)}(t), (f^{(1)}(a_i))_{i \in I_\vartheta})$.

The corresponding definition in N. S. 1 amounts to the special case obtained by taking $T \subseteq T'$ and $f^{(3)}$ as the identical mapping (insertion) of T into T′. We then call $(f^{(1)}, f^{(2)}, f^{(3)})$ a mapping $\Gamma \to \Gamma'$ iff for all $\vartheta \in \Theta$, $f^{(1)}_*(\vartheta) \in \Theta'$.

In the special case where $\vartheta = (t, A_\vartheta)$, where A_ϑ a set, $f^{(1)}_*(\vartheta) = (f^{(3)}(t), f^{(1)}(A_\vartheta))$, where $f^{(1)}(A_\vartheta)$ is the usual set.

Let us now apply definition 2 to the special case of two U-sefs U and U′. Here, $T = E$ and $T' = E'$ (see definition 1). For any $f^{(1)} : E \to E'$, we take (this is the specialization alluded to) $f^{(3)} = f^{(1)}$. Since on the other side, all $\vartheta \in \Theta$ (resp. $\vartheta' \in \Theta'$) are here of the form (t, A_ϑ) (resp. $(t', A'_{\vartheta'})$), with A_ϑ (resp. $A'_{\vartheta'}$) a set, for all $\vartheta = (a, W_{l_a}(a))$,

$$f^{(1)}_*(\vartheta) = (f^{(1)}(a), f^{(1)}(W_{l_a}(a)) \in \Theta'.$$

Hence $(f^{(1)}, f^{(2)}, f^{(3)})$ is a mapping iff $f^{(1)}$ transforms the W-neighborhood $W_{l_a}(a)$ of $a \in E$, to some W-neighborhood $W_{l'_{a'}}(a')$ of $a' = f^{(1)}(a) \in E'$.

P r o p o s i t i o n 1. Given two U-sefs U and U′, a mapping $f = (f^{(1)}, f^{(2)}, f^{(3)}) : U \to U'$ is a homomorphism iff, for all $a \in E$,

$f^{(2)}(a, X_{1_a}) = (f^{(1)}(a), X_{1'_{f^{(1)}(a)}})$ (δ), where the index $1'_{f^{(1)}(a)}$ is the index of $W_{1'_{f^{(1)}(a)}}(f^{(1)}(a))$.

Proof. a). Set $a' = f^{(1)}(a)$. Suppose f is a homomorphism. Then for all $a \in E$,

(i) $f^{(2)}(s(a, W_{1_a}(a))) = s'(f^{(1)}(a, W_{1_a}(a))$; but

(ii) $s'(f^{(1)}(a, W_{1_a}(a))) = s'(a', W_{1'_{a'}}(a')) = (a', X_{1'_{a'}})$, and

(iii) $f^{(2)}(s(a, W_{1_a}(a))) = f^{(2)}(a, X_{1_a})$.

Equalities (i), (ii), and (iii) imply (δ).

b). Suppose (δ) holds. This and the definition of the mapping $f = (f^{(1)}, f^{(2)}, f^{(3)})$ between U-sefs imply (i). Hence f is a homomorphism. |

Corollary. Let U and U′ be two U-sefs and $f = (f^{(1)}, f^{(2)}, f^{(3)})$ a homomorphism $U \to U'$. Then, given $a' \in E'$ and a W-neighborhood $W_{1'_{a'}}(a')$, for every (a, X_{1_a}) such that $f^{(2)}(a, X_{1_a}) = (a', X_{1'_{a'}})$, $f^{(1)}(W_{1_a}(a)) = W_{1'_{a'}}(a')$.

Proof. Since f is a homomorphism, $a' = f^{(1)}(a)$ and $f^{(1)}(W_{1_a}(a))$ is some W-neighborhood of a′, with the index of which must coincide the index of $X_{1'_{a'}}$, i.e. $1'_{a'}$, which has been given. Hence, this W-neighborhood is $W_{1'_{a'}}(a')$. |

P r o p o s i t i o n 2. Let U and U′ be two U-sefs and f a homomorphism $U \to U'$, with $f^{(2)}$ surjective. Then, for every $a' \in E'$ and for every W-neighborhood $W_{1'_{a'}}(a')$ of a′, there exists at least one couple $(a, W_{1_a}(a))$ such that $a' = f^{(1)}(a)$ and $f^{(1)}(W_{1_a}(a)) = W_{1'_{a'}}(a')$.

Proof. $s'(a', W_{1'_{a'}}(a')) = (a', X_{1'_{a'}})$. Since $f^{(2)}$ is onto, there exists a couple (a, X_{i_a}) such that $f^{(2)}(a, X_{1_a}) = (a', X_{1'_{a'}})$. Then, taking $W_{1_a}(a)$ from $\{a\} \times W_{1_a}(a) = g^{-1}(X_{1_a})$, we have, according to the preceding corollary, $f^{(1)}(W_{1_a}(a)) = W_{1'_{a'}}(a')$.

VI. Notions from generalized topology ; remarks.

We again make the suppositions of Ia, with $K = E$. Ander these circumstances, one can define on E a generalized topology (with the meaning given to this term in Mamuzic's book **(2)**) as follows: In the sequel, A, A_1 and A_2 denote always subsets of E. $A^c = E - A$.

Definition 3. A point a A will be called an interior point of A, iff at least one W-neighborhood of a is contained in A. The set of all the interior points of A will be called the interior of A and will be denoted by i (A).

Obvious (from the definitions) properties of the interiors.

1) $i(A) \subseteq A$; hence, in particular, $i(\varnothing) = \varnothing$.

2) $i(E) = E$.

3) $A_1 \subseteq A_2$ implies $i(A_1) \subseteq i(A_2)$.

4) $i(A_1 \cap A_2) \subseteq i(A_1) \cap i(A_2)$.

5) $i(A_1) \cup i(A_2) \subseteq i(A_1 \cup A_2)$.

6) $i(i(A)) \subseteq i(A)$.

Properties 4) and 5) immediately generalize to an arbitrary family of subsets of E.

If we set $\tau(A) = (i(A^c))^c$, we immediately see that the structure defined on E, as carrier, by definition 3, coincides with the generalized topology-neighborhood spaces defined in p. 14 of Mamuzic's book. Here, the «Kuratowski's closure operator» is characterized by the following theorem (obvious but important because it amounts to the definition of τ), due to A. Appert (a pioneer on the subject): $a \in \tau(A)$ iff for every $l_a \in L_a$ there exists at least one $x \in A$ such that $g(a, x) \in X_{l_a}$.

In this generalized topology, open sets can be defined by

Definition 4. A will be an open subset (or, open set) of E iff $i(A) = A$, i. e. iff all its points are interior. (See Mamuzic's book, p. 19, for an equivalent definition). Immediate consequences of definition 4: 1) $\varnothing$ and E are open sets of E. 2) The union of any family of open sets of E is an open set of E.

In other words, the set of all the open sets of E constitutes a hypotopology on E. (In the terminology of S. P. Zervos (**3**)).

Remark. One must, however, pay attention to the following essential variation in terminology: For each $a \in E$, $(W_{l_a}(a))_{l_a \in L_a}$ is a «neighborhood base» for this generalized topology on E, in Mamuzic's terminology, while S. P. Zervos (**3**), considers the open sets (these are the same in both authors) as the fundamental notion and then defines neighborhoods as in ordinary topology. We avoid here the use of the term «neighborhood», insisting only on the (more important for generalized topology, it seems to us) notions of interior point, open set, closed set and closure; as well as to W-neighborhoods, but regarded only as a means for defining the above notions.

If E and E′ are two generalized topological spaces, a mapping $f : E \to E'$ will be called open iff it transforms open sets to open sets.

Note. It is striking that such a general structure as that defined on M gives, through the «external» method described above, a hypotopology on E.

Proposition 3. Given a set E, all hypotopologies on E can be defined by the external method of U-sefs.

Proof. Let τ be a hypotopology τ on the set E. We shall construct a U-sef having E as carrier and the elements of τ as the open sets of E, defined by means of this U-sef.

Terminology. The elements of τ will be called T-open sets of E. The open sets of E defined by means of the W-neighborhoods will be called U-open sets of E. *Construction of the U-sef. Proof of the proposition.* Take $A = M = E$ and for every $a \in E$, call F_a the set of all the elements of τ (i. e. the T-open sets of E) which contain a, indexed canonically by itself and so forming a family $(X_{1_a})_{1_a \in L_a}$; finally, define $g : E \times E \to M$ by setting $g(a, x) = x$, i. e. by taking pr_2 as g. Then, $g(a, x) \in X_{1_a}$ iff $x \in X_{1_a}$ and, therefore, $W_{1_a}(a) = X_{1_a}$. Hence, all W-neighborhoods of elements of E are T-open.

Let us consider now the hypotopology defined on E by this external method; we call it U-hypotopology. U-open sets are characterized, as we saw above, by the property: A is U-open iff, for every $a \in A$, there exists at least one W_{1_a} (contained) in A. Since, here, all W-neighborhoods are T-open, every U-open set is a union of T-open sets, hence, it is T-open. So τ is finer than the U-hypotopology; and the converse, also, holds, since every T-open set, i. e. every X_{1_a} is, here, a W-neighborhood of each of its points, hence, also, U-open. Hence the two hypotopologies in question coincide. |

VII. Relations between U-sefs and Hypotopology.

Given two U-sefs defined in III, we shall always suppose that its carrier E is supplied with the hypotopology defined on E by the above described external method.

Proposition 4. Given two U-sefs U and U', all homomorphisms $U \rightarrow U'$ induce a transformation of 1) the interior points of E to interior points of E' and 2) the open sets of E to open sets of E'; i.e. an open mapping $E \rightarrow E'$.

Proof 1) Let a be an interior point of A. Then, there exists at least one $l_a \in L_a$ such that $W_{l_a}(a) \subseteq A$. For every homomorphism $f: U \rightarrow U'$, $f^{(1)}(W_{l_a}(a))$ is a W-neighborhood $W_{l'_{a'}}(a')$ of a', where $a' = f^{(1)}(a)$. Now, $W_{l_a}(a) \subseteq A$ implies that $f^{(1)}(W_{l_a}(a)) \subseteq f^{(1)}(A)$, hence that $W_{l'_{a'}}(a') \subseteq f^{(1)}(A)$. So $f^{(1)}(a)$ is an interior point of $f^{(1)}(A)$.

2) $a' \in f^{(1)}(A)$ implies that for some $a \in A$, $a' = f^{(1)}(a)$. Hence, if A consists exclusively of interior points, then, according to 1), also $f^{(1)}(A)$ consists only of interior points. Hence $f^{(1)}(A)$ is open. So $f^{(1)}$ is an open mapping $E \rightarrow E'$. ❙

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ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν παροῦσαν ἐργασίαν ἐπιτυγχάνεται ὁ μέσῳ τῆς ἐννοίας τῆς sef συνδυασμὸς τῆς γενικευμένης Τοπολογίας πρὸς τὴν Καθολικὴν Ἄλγεβραν, εἰς τὴν περίπτωσιν τῶν γενικωτέρων «τοπολογικῶν» χώρων, οἱ ὁποῖοι εἶναι δυνατὸν νὰ ὁρισθοῦν δι' «ἐξωτερικῶν» μεθόδων. Ἀποδεικνύεται ὅτι οἱ ὁμομορφισμοὶ μεταξὺ τῶν ὁριζομένων sef - χώρων ἐπάγουν ἀνοικτὰς ἀπεικονίσεις μεταξὺ τῶν ἀντιστοίχων ὑποτοπολογικῶν χώρων.

★

Ὁ Ἀκαδημαϊκὸς κ. **Κ. Π. Παπαϊωάννου** ἀνακοινῶν τὴν ὡς ἄνω ἐργασίαν εἶπε τὰ ἑξῆς:

Ὁ πτυχιοῦχος τῶν Μαθηματικῶν τοῦ Πανεπιστημίου Ἀθηνῶν κ. Πέτρος Κρικέλης εἶναι μαθητὴς τοῦ καθηγητοῦ κ. Σ. Π. Ζερβοῦ. Ἡ θεωρία τῶν δομῶν-sef τοῦ κ. Ζερβοῦ, τὴν ὁποίαν παρουσιάσαμεν διὰ πρώτην φορὰν εἰς τὴν Ἀκαδημίαν Ἀθηνῶν κατὰ τὸ ἔτος 1963, παρώρμησε τὸν κ. Κρικέλην εἰς τὴν παροῦσαν ἐργασίαν, ἀποτέλεσμα μακρῶν ἐρευνῶν του. Εἰς τὴν ἐργασίαν αὐτὴν ἐπιτυγχάνεται ὁ σύνδεσμος μεταξὺ τῆς θεωρίας τῶν χώρων περιοχῶν καὶ τῆς θεωρίας τοῦ κ. Ζερβοῦ. Οὕτως, ἐπιτυγχάνει ὁ κ. Κρικέλης τὸν σύνδεσμον τῆς γενικευμένης τοπολογίας καὶ τῆς καθολικῆς ἀλγέβρας, κατὰ τρόπον λίαν ἐνδιαφέροντα.