ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **Stability concepts of solutions of differential equations with deviating arguments,** *by Demetrios G. Magiros* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰ. Ξανθάκη.

1. Introduction

The majority of physical and social systems can be expressed by relations between quantities under investigation and their rate of change, that is by differential equations.

The duration of the transmission of the action or signal can not in many cases be neglected, and the processes are governed by «differential equations with deviating arguments», as, e. g., by equations with «retarded arguments», which characterize situations with «delay periods» or «aftereffects». Modern needs of automatic regluation, or probability, of biology, of medicine, and of certain other fields, lead to such equations.

The study of the stability properties of the solutions of these equations is a fundamental problem, and this problem can be treated if, and only if, the «concepts of stability» are clarified and selected appropriately.

In this note we formulate stability concepts of the class of differential equations with deviating arguments, and give somme remarks concerning these stability concepts. The stability concepts depend especially upon the nature of the perturbations, the way the perturbations act on the system, their magnitude, the magnitude of their effect, etc. Definition of stability are given in case of sudden and permanent perturbations, and in case of perturbations of the deviating arguments. By the remarks, we clarify questions on the stability concepts.

2. Definitions

A system of differential equations with retarded arguments, that is variables containing «delay periods», can be given by :

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$$\dot{x}_i(t) = f_i[t, x_j(t - \tau_{jk}(t))] \quad i, j = 1, 2, \ldots, n; \; k = 1, 2, \ldots, m, \; m \leqslant n \tag{1}$$

where the retardations τ_{jk} are positive.

By «solution» of (1) we mean continuous functions $x_i(t)$ which satisfy (1) on $t \geqslant t_0$, and which, on the initial interval: $E_{t_0}: t_0 - \tau_{jk}(t) \leqslant t \leqslant t_0$ become $x_i(t) = \varphi_i(t)$, and $\varphi_i(t)$ are given continuous functions, called «initial functions» of (1).

By «stationary point» of (1) we mean a constant solution x_{i0} of (1) on $t \geqslant t_0$, which is also constant on the initial set E_{t_0}.

The solution of (1) depends on the given arbitrary functions $\varphi_i(t)$. $\varphi_i(t)$ are extensions of the solutions in the interval E_{t_0}, and the solutions are uniquely determined by the initial functions and appropriate properties of f_i.

3. Stability in case of sudden perturbations

«Sudden perturbations» are perturbations «momentarily» applied to the initial functions $\varphi_i(t)$ attached to (1). Let x_φ be the solution of (1) with orbit L corresponding to the initial functions $\varphi_i(t)$ and x_ψ be the solution of (1), after the perturbation, with orbit $\bar{L}$ corresponding to the initial functions $\psi_i(t)$, Fig. (a).

If the points $\bar{P}_{01}$, $\bar{P}_0$, $\bar{P}$ of the perturbed curve $\bar{L}$ correspond to the points P_{01}, P_0, P of the unperturbed curve L, the following distances can be defined, corresponding to time indicated:

$$\begin{aligned} \varrho_{01} &= P_{01}\bar{P}_{01} = |\psi_i(t) - \varphi_i(t) && \text{on} \quad E_{t_0}: t_0 - \tau \leqslant t \leqslant t_0 \\ \varrho_0 &= P_0 \bar{P}_0 = |\psi_i(t_0) - \varphi_i(t_0) && \text{at} \quad t = t_0 \\ \varrho &= P \bar{P} = |x_\psi(t) - x_\varphi(t) && \text{on} \quad t \geqslant t_0 \end{aligned} \tag{2}$$

These distances give the effect of the perturbation at the points P_{01}, P_0, P of L, Fig. (a).

The solution x_φ of (1) is «stable» if, given $\varepsilon > 0$ and $t_0 \geqslant 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that the inequality $\varrho_{01} < \delta$ implies $\varrho < \varepsilon$.

This solutions is «asymptotically stable», if it is stable and, in addition, $\lim_{t \to \infty} \varrho = 0$.

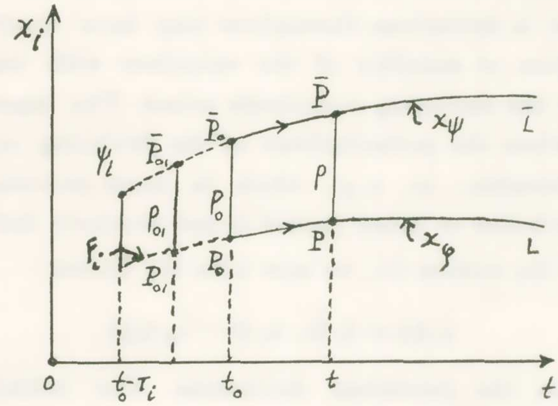

Fig. (a).

These definitions can be accepted for other types of differential equations with deviating arguments, say of neutral types.

4. Stability in case of persistent perturbations

If the system (1) is continuously perturbed by the perturbations R_i, the perturbed system, corresponding to (1), is :

$$\dot{x}_i(t) = f_i\big(t, x_j(t - \tau_{jk}(t))\big) + R_i\big(t, x_j(t - \tau_{jk}(t))\big) \tag{3}$$

R_i are sufficiently small in absolute value. Given $\varepsilon > 0$ and $t_0 \geqslant 0$, if there exist two positive quantities $\delta_1 = \delta_1(t_0, \varepsilon)$, $\delta_2 = \delta_2(t_0, \varepsilon)$ such that: $\varrho_{01} < \delta_1$ valid on E_{t_0}, and $\| R_i \| < \delta_2$ valid on $t \geqslant t_0$, imply $\varrho < \varepsilon$ valid on $t \geqslant t_0$, we say that the solution x_φ of (1) is «stable» with respect to persistent perturbations R_i.

We remark that many basic theorems on stability can be carried out without essential alteration to the case of differential equations with deviating argument, but, up to now, the stability theory for these equations is essential for stationary equations of the first approximation in noncritical cases.

5. Stability in case of perturbations of deviating arguments

In processes with after effects, described by differential equations with deviating arguments, the deviations themselves can not be prescribed exactly, that is deviations themselves may have small disturbances, when the question of stability of the equations with respect to small perturbations of the deviating arguments arises. The important stability problem arises when the perturbations of the deviating arguments have a continuing character, as, e.g., when in some processes with after effects the retardation or delay period is not precisely defined.

Instead of the system (1), we now take the system:

$$\dot{x}_i(t) = f_i\left(t,\ x_j\left(t - \bar{\tau}_{jk}(t)\right)\right) \tag{4}$$

where $\bar{\tau}_{jk}(t)$ are the perturbed deviations. The initial interval is: $E_{\bar{t}_0}$: $\bar{t}_0 - \tau_{jk}(t) \leqslant t \leqslant \bar{t}_0$, $t_0 \leqslant \bar{t}_0$.

Now, the solution of (1), x_φ, defined by $\varphi_i(t)$ on the set E_{t_0}, it said to be «stable with respect to perturbations of deviating arguments», if, for $\varepsilon > 0$, $t_0 \geqslant 0$, we can find $\delta_1 > 0$, $\delta_2 > 0$ such that, if $\varrho_{01} < \delta_1$, for t on E_{t_0}, and if $/\bar{\tau}_{jk}(t) - \tau_{jk}(t)/ < \delta_2$ for t on $t \geqslant t_0$, then $\varrho < \varepsilon$ for t on $t \geqslant t_0$.

In case τ_{jk} and $\bar{\tau}_{jk}$ are constant, we have «stability with respect to a continuously perturbed deviating argument».

In case the difference $(\bar{\tau}_{jk} - \tau_{jk})$ is either positive or negative, we have a «one sided perturbation of a deviating argument» (**1**).

REMARKS (**2**)

The following remarks may give to the reader an opportunity to think more deeply about the difficulties of the subject.

1. The above stability definitions are in the sense of Liapunov or Poincaré, if ϱ_{01}, ϱ_0, ϱ are interpreted as Liapunov or Poincaré distances.

The stability definitions in the sense of Liapunov or Poincaré are equivalent in case of equilibrium solutions, but for any other kind of solutions the stability situation may be different, if we employ different stability concepts, and naturally appears the important subject of the selection of the appropriate stability definition for the stability problem at hand.

2. The above classes of stability concepts can give any subclass of stability concepts by appropriate restrictions of the distances, the time, and other quantities. So, e.g., one can speak about «eventually uniform stability», if δ_1 is independent of t_0, that is $\delta_1 = \delta_1(\varepsilon)$, and t_0 has a minimum $\alpha(\varepsilon)$, that is $\alpha(\varepsilon) \leqslant t_0 \leqslant t$. Nonexistence of $\alpha(\varepsilon)$, that is $\alpha(\varepsilon) \equiv 0$, corresponds to «uniform stability».

3. In case of persistent perturbations, the selection of the kind of the norm of the perturbations, specifies the stability concept. So, one may have «total stability», or «integral stability» or «stability in the mean» under suitable norm of the perturbations (3).

4. The stability, as defined above, is a property of the solution different from its boundedness property, although in some cases there may exist regions where these two properties are equivalent and one implies the other.

5. All the above stability concepts are of mathematical type and the results, theorems or criteria, based on them, may not interpret the reality. In order that these results have a practical usefulness, which must be the ultimate purpose of the investigations, the investigation must be accompanied by some additional requirements, as, e.g., to know the region of the practically permitted deviations of the solutions, that is the «ε-region», the corresponding region of the initial conditions, that is the «δ_1-region», and the «δ_2-region» of the norm of the perturbation R_i.

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Ὁ Ἀκαδημαϊκὸς κ. **Ἰω. Ξανθάκης** κατὰ τὴν ἀνακοίνωσιν τῆς ἀνωτέρω ἐργασίας εἶπε τὰ κάτωθι :

«Ἡ ἀνακοίνωσις τοῦ κ. Μαγείρου ἀναφέρεται εἰς τὰς ἐννοίας τῆς εὐσταθείας τῶν λύσεων διαφορικῶν ἐξισώσεων μὲ ἀποκλινούσας μεταβλητάς.

Μία μεγάλη κατηγορία φυσικῶν καὶ κοινωνικῶν φαινομένων ἐκφράζεται μαθηματικῶς διὰ διαφορικῶν ἐξισώσεων μή - γραμμικῶν μὲ ἀποκλινούσας μεταβλητάς. Ἀφ᾽ ἑτέρου, ὡρισμένα προβλήματα αὐτομάτου ἐλέγχου, προβλήματα πιθανοτήτων, ἡ ἀνάπτυξις εἰς τὸν τομέα τῆς βιολογίας καὶ ἰατρικῆς ὁδηγοῦν εἰς ἐξισώσεις τοῦ ἐν λόγῳ τύπου.

Τὸ ζήτημα τῆς εὐσταθείας τῶν λύσεων τῶν ἐξισώσεων αὐτῶν εἶναι θεμελιῶδες. Ἡ σπουδὴ δὲ τῆς εὐσταθείας βασίζεται ἐπὶ διαφόρων ἀντιλήψεων καὶ ὑποθέσεων περὶ εὐσταθείας, αἱ ὁποῖαι ἐξαρτῶνται ἐκ τοῦ τρόπου τῆς δράσεως τῶν «διαταράξεων» καὶ ἐκ τοῦ εἴδους μετρήσεως τοῦ μεγέθους τῶν διαταράξεων.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν ἐκτίθενται αἱ ἔννοιαι εὐσταθείας τῶν λύσεων τῶν ἐν λόγῳ ἐξισώσεων καὶ διατυποῦνται παρατηρήσεις τινὲς ἐπ᾽ αὐτῶν. Διατυποῦνται ἐπίσης οἱ ὁρισμοὶ ὅταν αἱ διαταράξεις εἶναι «αἰφνίδιαι» ἢ «συνεχῶς δρῶσαι», καθὼς καὶ ὅταν, αἱ ἴδιαι αἱ ἀποκλίσεις τῶν μεταβλητῶν ὑπόκεινται εἰς διαταράξεις.

Ἡ εἰσαγωγὴ τῆς ἐννοίας τῆς εὐσταθείας κατὰ Liapounov καὶ Poincaré παρέχει δύο διακεκριμένας γενικὰς κατηγορίας τῶν ἀντιλήψεων εὐσταθείας».
