ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **Remarks on stability concepts of solutions of dynamical systems,** *by Demetrios G. Magiros**.

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INTRODUCTION

The study of the stability situation of physical and social phenomena, which are modeled as dynamical systems, is based on a variety of stability concepts, and this variety makes the study complicated and the stability results questionable in many cases.

In this paper, we will give a set of remarks on stability concepts, which may permit a better understanding of the difficulties of current stability problems.

The stability concepts may come from sources of different nature.

Examining the stability of a motion in its orbit and of the orbit of a motion, one can distinguish two basic stability concepts, which contain many other specialized concepts as special cases.

The manner in which a state of a system approaches another state, or deviates from it, the way the perturbations act on a system, or the way one measures their norm and their effect on the system, the type of the mathematical model of the system, etc., are sources for stability concepts of different nature.

All these different stability concepts can be «unified» into the same «stability relationships», and this «unification» of the stability concepts brings a natural simplicity in the understanding of subjects concerning stability, and gives rise to new results.

1. REMARKS ON PERTURBATIONS

It is necessary to make remarks concerning the purturbations of the systems and their solutions. A variety of stability concepts comes from the manner the purturbations act on the system, the way the norm

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of perturbations is taken, and the way their effect on the system is measured.

The perturbations, which can be considered as minor disturbing forces, may act on a system either «momentarily», when only the initial conditions are perturbed, or «permanently», when the system itself is disturbed and the perturbations, during their action, must enter the equations of the system explicitly. The «sudden» and «persistent» perturbations characterize two different classes of stability concepts.

The «norm» of the perturbations can be taken in different ways, and each of them characterizes a special stability concept under persistent perturbations. The perturbations may depend on deviating arguments, when the stability concepts will be related to the deviations of the arguments.

The «effect» of the perturbations is a change of certain quantities pertaining to the original motion and/or to its orbit, and this effect can be visualized by the change of the orbit S of the unperturbed motion $x_i(t)$ into the orbit $\bar{S}$ of the perturbed motion $\bar{x}_i(t)$.

Given a state of a dynamical system, that is a point P on S, if, as a result of the perturbations, $\bar{S}$ is the new orbit, and the point $\bar{P}$ on $\bar{S}$ corresponds to the point P of S, the distance $\varrho = P\bar{P}$ can be taken as the magnitude of the effect of the perturbations at P, Fig. 1.

To a given point P of S, one may make to correspond different points $\bar{P}$ on $\bar{S}$, and each correspondence characterizes a specific stability concept of the motion. Any such correspondence presuposes an assumption, and each assumption comes from a physical reason.

One can distinguish two, the most physical, correspondences between P and $\bar{P}$, when two important stability concepts result, namely, the stability concept in the sence of Liapunov and that in the sense of Poincaré, by using the stability distances $\varrho = P\bar{P}$, called Liapunov and Poincaré distances. In Fig. 2 «Liapunov distances» are shown P and $\bar{P}$ correspond to each other at the «same time». In Fig. 3, P and $\bar{P}$ correspond in such a way that :

$$\varrho = \varrho(P, \bar{S}) = P\bar{P} = \min \left\{ \sum_{i=1}^{n} |\bar{x}_i - x_i|^2 \right\}^{1/2}$$

and $\varrho = P\bar{P}$ is «Poincaré distance».

If the model of the system is expressed by differential equations with deviating arguments, e.g. by differential equations with retarded arguments of retardation $\tau_i(t)$, the «initial functions» $\varphi_i(t)$ of S and $\bar{\varphi}_i(t)$ of $\bar{S}$ define the «initial distance» $\varrho_{01} = P_{01}\bar{P}_{01} = |\bar{\varphi}_i(t) - \varphi_i(t)|$ taken over the interval $t_0 - \tau_i \leqslant t \leqslant t_0$, Fig. 2.

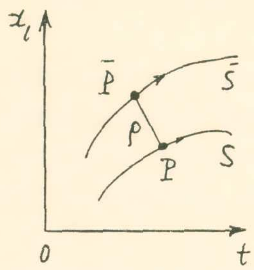

Fig. 1.

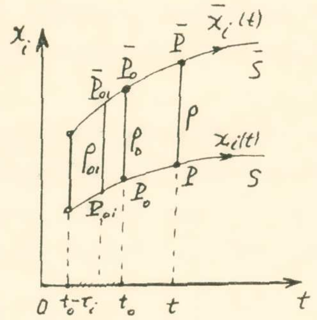

Fig. 2.

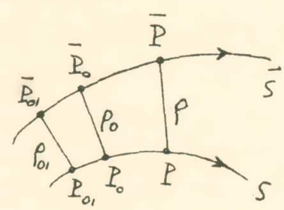

Fig. 3.

The above includes the ordinary differential equations, where $\tau_i = 0$, $\varrho_{01} = \varrho_0 = P_0\bar{P}_0$ at $t = t_0$, Fig. 2.

In case the retardations $\tau_i(t)$ are perturbed, one has a new stability concept characterized by the distance: $\varrho_\tau = |\bar{\tau}_i(t) - \tau_i(t)|$, where $\bar{\tau}_i$ the perturbed regardations (1).

2. REMARKS ON THE RELATIONSHIPS OF THE STABILITY CONCEPTS

Following considerations on «physical stability» and using the precending remarks and notations, any stability concept can be described quantitatively by the «stability conditions»:

$$(a): \varrho_{01} < \delta_1, \quad (b): \varrho < \varepsilon, \quad (c): \lim_{t \to \infty} \varrho = 0, \quad (d): \|p_i\| < \delta_2, \quad (e): \varrho_\tau < \delta_3 \qquad (1)$$

which, then, «unify» all the stability concepts. ε, δ_1, δ_2, δ_3 are positive constants, p_i the perturbations, and, in general, δ_1 and δ_2 depend on t_0 and ε.

By a suitable combination of these relationships, by an appropriate interpretation of the distances involved, and by some restrictions of some quantities of these relationships, one can express any stability concept.

The following remarks may help to clarify the above statements.

The inequality 1(a) is valid for t in the initial inverval $E_{t_0}: t_0 - \tau_i \leqslant t \leqslant t_0$, while the inequalities 1(b), (d), (e) for t in $t \geqslant t_0$, $t_0 \geqslant 0$.

In case of «sudden perturbations», the inequality 1(d) is meaningless, and in this case 1(a), (b) express the «stability», while 1(a), (b), (c) the «asymptotic stability».

In case of «permanent perturbations», when 1(d) is meaningful, the selection of the kind of the norm of perturbations specifies the stability concept, so, one may have «total stability» or «integral stability», or «stability in the mean», under suitable norm of perturbations.

By restricting δ_1 and δ_2 to depend only on ε and not on t_0, we have the «uniform stabilities», as this happens in periodic systems, or in autonomous systems.

Restriction on t_0 to have a minimum, $\min t_0 = \alpha$, implies «eventual stabilities».

If in (1) the distances ϱ are interpreted as Liapunov or Poincaré distances, we have stabilities in the sence of Liapunov or Poincaré.

In case of equilibrium points of a system, when the orbit S shrinks to a point, the distinction between stabilities in the sence of Liapunov and Poincaré is meaningless.

The inequality 1(e) has a meaning in case of perturbed retardations.

In case of periodic motions, when S is a closed curve, the stability concept in Liapunov sense is a narrow concept compared to the stability concept in Poincaré sence. The «isochronism», that is the «constancy of the frequency», characterizes the Liapunov stability, while this notion does not enter in the Poincaré stability. Further we may have that :

. A motion stable in Liapunov sense is also stable in Poincaré sense ;

. A motion unstable in Poincaré sense is also unstable in Liapunov sense ;

. A motion unstable in Liapunov sense may be stable or unstable in Poincaré sense, and

. A motion stable in Poincaré sense may be stable or unstable in Liapunov sense.

The stability is a property of the solution different from its boundedness property, although in some cases there may exist regions where these properties are equivalent and one implies the other. The boundedness of the solution is characterized by the boundedness of $OP = \varrho_1 = |x_i|$, where O the origin of the coordinate system and P point of the orbit; but the stability is characterized by the boundedness of $\varrho = |\bar{x}_i - x_i|$, and it is possible for the orbits x_i and $\bar{x}_i$ to be unbounded as $t \to \infty$, when the stability distance ϱ gets the form $(\infty - \infty)$, when ϱ will be either infinite, or constant, or zero, and the unbounded solution x_i will be either unstable or stable.

All the stability concepts included in the relationships (1) are of mathematical type, and the results, theorems or criteria, based on them, may not interpret the reality. Also, one and the same phenomenon may be, mathematically speaking, stable or unstable depending on the stability concept employed in the discussion of the stability of the phenomenon when the selection of the stability concept, appropriate for the phenomenon, arises.

The mathematical stability concepts and the stability criteria based on them, represent a possible functionig of the physical system, and in order all these to have a practical usefulness, and to agree with «practical stability», which must be the ultimate purpuse of the stability investigations, appropriate modifications, changes, supplements of the mathematical stability concepts must accompany the investigations.

The notion of «practical stability» is a subject not yet completely studied, but in many cases is characterized by the knowledge of : (2)

. The size of deviation of the state acceptable for a satisfactory operation of the system ;

. The size of permitted initial conditions that can be controlled ;

. The size of permitted perturbations ;

. The finite time T for the stability investigation.

The nolinearities of the system play a decisive role for practical stability.

3. AN EXAMPLE

We terminate the discussion on stability remarks by using as an example the investigation of the «stability of some precessional phenomena», by which some of the above statements may be clarified.

The rotational motion of a rigid body around its axis of symmetry is governed by the Euler's (ordinary differential) equations, of which the stability of the solutions has been examined under sudden perturbations, then by employing the first three stability relationships (1).

If $\underline{L} = (L_1, L_2, L_3)$ is the external torque vector acting on the body, $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ the angular velocity vector, which characterizes the precession of the body, $\underline{\omega}_0 = (\omega_{01}, \omega_{02}, \omega_{03})$ the initial angular velocity vector, and I_1, $I_2 = I_3 = I$ the moments of inertia, the precessional motion of the body in the following two cases are given by:

$$\left.\begin{array}{l}
\text{(a)}\ \ \omega_1 = \omega_{10} = \text{constant},\ \ \omega_2 = A\cos Q_1,\ \ \omega_3 = A\sin Q_1,\ \ Q_1 = \dfrac{(I_1 - I)\,\omega_{10}}{I}t \\
\qquad \text{(in case,}\ \ L_1 = L_2 = L_3 = 0;\ \ \ I_1,\ I,\ A = \left(\omega_{02}^2 + \omega_{03}^2\right)^{1/2}\ \text{constants)} \\
\text{(b)}\ \ \omega_1 = \dfrac{L_1}{I_1}t,\ \ \omega_2 = A\cos Q_2,\ \ \omega_3 = A\sin Q_2,\ \ Q_2 = Q_1 + \dfrac{(I_1 - I)\,L_1}{2\,I_1 I}t^2 \\
\qquad \text{(in case:}\ \ L_1 = \text{constant},\ \ L_2 = L_3 = 0)
\end{array}\right\} \quad (2)$$

The «regular precession» 2(a) is bounded, while the «helicoid precession» 2(b) is unbounded as $t \to \infty$. The results for their stability situation are the following:

(i) The regular precession 2(a) is «stable» but «not asymptotically stable» in Poincaré sense (orbitally). In Liapunov sense it is «stable» but «not asymptotically stable», if ω_{01} is not affected by the perturbation; and it is «unstable», if ω_{01} is affected by the perturbations.

(ii) The helicoid precession 2(b) is «asymptotically stable» in Poincaré sense, but it is «unstable» in Liapunov sense.

(iii) The stability situation of the above example in Poincaré sense is preferred, because in this case it is proved that the requirements for «practical stability» are satisfied.

We remark that the stability of a system, by using any of the previous concepts, depends, in general, on the selection of the major variables of the system, and on the transformation of the variables. This will be subject of a next paper.

ΠΕΡΙΛΗΨΙΣ

1. Εἰς τὴν παροῦσαν ἐργασίαν δίδονται παρατηρήσεις ἐπὶ τῶν ἀντιλήψεων περὶ εὐσταθείας εἰς φυσικὰ καὶ κοινωνικὰ φαινόμενα, τὰ ὁποῖα μαθηματικοποιοῦνται ὡς δυναμικὰ προβλήματα. Ἡ ποικιλία τῶν ἀντιλήψεων εὐσταθείας κάμνει τὰ προβλήματα εὐσταθείας πολὺ πεπλεγμένα καὶ τ' ἀποτελέσματα τῆς ἐρεύνης βάσει αὐτῶν ὄχι δεκτὰ ἐνίοτε.

2. Ἀναφέρομεν μερικὰς πηγάς, ἀπὸ τὰς ὁποίας δυνάμεθα νὰ ἔχωμεν ποικιλίαν διαφόρου φύσεως ἀντιλήψεων περὶ εὐσταθείας:

—Ὁ τρόπος μὲ τὸν ὁποῖον μία κατάστασις ἑνὸς συστήματος πλησιάζει μίαν ἄλλην κατάστασιν ἢ ἀπομακρύνεται ἀπὸ αὐτήν.

—Ὁ τρόπος μὲ τὸν ὁποῖον προκαλοῦνται καὶ δροῦν αἱ διαταραχαὶ ἑνὸς συστήματος, ὁ τρόπος μὲ τὸν ὁποῖον μετροῦμεν τὴν ἔντασιν τῶν διαταραχῶν, καθὼς καὶ τὴν τῶν ἀποτελεσμάτων των ἐπὶ τοῦ συστήματος.

—Ἡ δομὴ τοῦ μαθηματικοῦ μοντέλου τοῦ συστήματος κλπ.

3. Ὅλαι αἱ ἀντιλήψεις εὐσταθείας, ἂν καὶ διαφόρου φύσεως, δύνανται ν' ἀναχθοῦν εἰς τὰς αὐτὰς μαθηματικὰς «σχέσεις εὐσταθείας», αἱ ὁποῖαι δίδουν μίαν «ἑνοποίησιν» τῶν ἀντιλήψεων εὐσταθείας, αὐτὴ δὲ ἡ ἑνοποίησις ὑποβοηθεῖ τὴν κατανόησιν τῶν ἀντιστοίχων προβλημάτων, δύναται δὲ νὰ ὁδηγήσῃ εἰς νέα συμπεράσματα. Μὲ κατάλληλον συνδυασμὸν τῶν «σχέσεων εὐσταθείας», καὶ κατάλληλον ἑρμηνείαν ἢ περιορισμὸν τῶν ποσοτήτων τῶν σχέσεων αὐτῶν, δύναται νὰ προκύψῃ ὁποιαδήποτε ἀντίληψις εὐσταθείας.

4. Αἱ ἀντιλήψεις εὐσταθείας, ποὺ περικλείονται εἰς τὰς «σχέσεις εὐσταθείας» ὑποδεικνύουν ἐνδεχομένην λειτουργίαν τοῦ συστήματος, τὰ δὲ συμπεράσματα, θεωρήματα ἢ κριτήρια, ποὺ βασίζονται ἐπ' αὐτῶν, ἐνδέχεται νὰ μὴ ἑρμηνεύουν τὴν πραγματικότητα κατὰ ἱκανοποιητικὸν τρόπον, ἢ ἐνδέχεται νὰ ἔχωμεν διὰ τὸ αὐτὸ φαινόμενον διαφόρους καταστάσεις εὐσταθείας, ὁπότε γεννᾶται τὸ πρόβλημα τῆς ἐκλογῆς τῆς καταλλήλου καταστάσεως διὰ τὸ φαινόμενον.

5. Αἱ μαθηματικαὶ ἀντιλήψεις περὶ εὐσταθείας, καθὼς καὶ τὰ βάσει αὐτῶν συμπεράσματα, διὰ νὰ ἑρμηνεύουν τὴν πραγματικότητα κατὰ ἱκανοποιητικὸν τρόπον, πρέπει νὰ συμφωνοῦν μὲ τὰ πορίσματα τῆς «πρακτικῆς εὐσταθείας», πρὸς τοῦτο δὲ χρειάζονται κατάλληλον τροποποίησιν καὶ συμπλήρωσιν. Ἡ πρακτικὴ εὐστάθεια δὲν ἔχει πλήρως σπουδασθῆ, ὅμως κύρια χαρακτηριστικά της δύναται νὰ εἶναι ἡ γνῶσις:

— τοῦ μεγέθους τῆς ἀποκλίσεως δεκτῶν καταστάσεων τοῦ συστήματος πρὸς ἱκανοποιητικὴν λειτουργίαν τοῦ συστήματος,
— τοῦ μεγέθους τῶν ἀρχικῶν συνθηκῶν, αἱ ὁποῖαι δύνανται νὰ ἐλεγχθοῦν,
— τοῦ μεγέθους τῶν ἐπιτρεπομένων διαταραχῶν,
— τοῦ χρόνου, ποὺ μελετῶμεν τὴν εὐστάθειαν.
— Αἱ μὴ γραμμικότητες τοῦ συστήματος παίζουν ἀποφασιστικὸν ρόλον διὰ τὴν πρακτικὴν εὐστάθειαν, καὶ δὲν δύνανται νὰ ἀμεληθοῦν.

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Ὁ Ἀκαδημαϊκὸς κ. **Ἰωάννης Ξανθάκης**, παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν, εἶπε τὰ ἑξῆς:

Εἰς τὴν ἐργασίαν ταύτην τοῦ κ. Μαγείρου, τὴν ὁποίαν ἔχω τὴν τιμὴν νὰ παρουσιάσω εἰς τὴν Ἀκαδημίαν, ἐκτίθενται ὡρισμέναι ἐνδιαφέρουσαι παρατηρήσεις ἐπὶ τῶν ἀντιλήψεων περὶ εὐσταθείας εἰς φυσικὰ καὶ κοινωνικὰ φαινόμενα, ποὺ ἐκφράζονται μαθηματικῶς ὡς δυναμικὰ προβλήματα.

Αἱ ἀντιλήψεις περὶ εὐσταθείας προέρχονται ἀπὸ πηγὰς διαφόρου φύσεως. Ἡ μελέτη τῆς σταθερότητος μιᾶς κινήσεως ἐπὶ τῆς τροχιᾶς της ἢ ἡ μελέτη τῆς τροχιᾶς μιᾶς κινήσεως μᾶς παρέχει δύο διακεκριμένας βασικὰς ἀντιλήψεις περὶ εὐσταθείας, αἱ ὁποῖαι περιέχουν πλῆθος ἄλλων εἰδικευμένων ἀντιλήψεων ὡς εἰδικὰς περιπτώσεις. Ὁ τρόπος μὲ τὸν ὁποῖον ἡ κατάστασις ἑνὸς συστήματος πλησιάζει πρὸς μίαν ἄλλην, ἢ παρεκκλίνει ἐξ αὐτῆς, ὁ τρόπος μετρήσεως τῆς ἐντάσεως τῶν διαταραχῶν ἑνὸς συστήματος, ἡ δομὴ τοῦ μαθηματικοῦ μοντέλου ἑνὸς συστήματος καὶ ἄλλα εἶναι πηγαὶ ἀντιλήψεων εὐσταθείας διαφόρου φύσεως. Ἡ ποικιλία αὕτη τῶν ἀντιλήψεων εὐσταθείας κάμνει τὰ προβλήματα λίαν πολύπλοκα καὶ τὰ ἀποτελέσματα τῶν ἐρευνῶν βάσει αὐτῶν ἐνίοτε δὲν εἶναι γενικῶς ἀποδεκτά.

Ὅλαι αἱ ἀντιλήψεις περὶ εὐσταθείας, ἂν καὶ διαφόρου φύσεως, δύνανται νὰ ἀναχθοῦν, κατὰ τὸν κ. Μάγειρον, εἰς τὰς αὐτὰς μαθηματικὰς «σχέσεις εὐσταθείας», αἱ ὁποῖαι παρέχουν μίαν ἑνοποίησιν τῶν διαφόρων περιστάσεων. Ἡ ἑνοποίησις αὕτη ὑποβοηθεῖ εἰς τὴν πληρεστέραν κατανόησιν τῶν ἀντιστοίχων προβλημάτων.

Τὰ συμπεράσματα, θεωρήματα ἢ κριτήρια, τὰ στηριζόμενα εἰς τὰς μαθηματικὰς σχέσεις ἀντιλήψεων εὐσταθείας, εἶναι δυνατὸν νὰ μὴ ἑρμηνεύουν κατὰ ἱκανοποιητικὸν τρόπον τὴν πραγματικότητα, ὁπότε παρίσταται ἀνάγκη τροποποιήσεων ἢ συμπληρώσεων τοῦ μαθηματικοῦ προτύπου, οὕτως ὥστε ἡ μαθηματικὴ διατύπωσις τῆς ἀντιλήψεως εὐσταθείας νὰ πλησιάζῃ, ὅσον τὸ δυνατὸν περισσότερον, πρὸς τὴν λεγομένην «Πρακτικὴν Εὐστάθειαν», ἥτις ὅμως δὲν ἔχει ἀκόμη πλήρως μελετηθῆ.