

σίας τοῦ Γενικοῦ Ἐπιτελείου Ναυτικοῦ, τὸν καθηγητὴν τοῦ Πανεπιστημίου κ. Παν. Κρητικὸν διὰ τὴν πλουσίαν συλλογὴν τῶν φωτογραφιῶν τοῦ θαλασσοῦ σεισμικοῦ κύματος, καὶ τέλος τοὺς Λιμενάρχας καὶ τοὺς κατὰ τόπους παρατηρητὰς τοῦ Ἀστεροσκοπείου Ἀθηνῶν, οἱ ὅποιοι διὰ τῶν ἐπισταμένων ἐκθέσεων των κατέστησαν δυνατὴν τὴν σύνταξιν τῆς παρούσης μελέτης.

## SUMMARY

The tsunamis are not an infrequent phenomenon in the seismic history of Greece. People, however, are unaware of their destructive efficiency. Due to this event a lot of good observations were made and a series of pictures were taken of the seismic sea wave, which followed the great shock of July 9, 1956 ( $36^{\circ}.9$  N,  $26^{\circ}.0$  E,  $H = 03:11:38$ ,  $M = 7\frac{1}{2}$ ).

They have observed three large waves at intervals of from 5 to 15 minutes. From the course of the wave, the time of arrival and the height it had attained on the Islands Amorgos and Astypalaea, it appears that the wave had started  $36^{\circ}.8$  N,  $26^{\circ}.2$  E from the steep slopes of the submarine trench, which is near the southeast coast of Amorgos Island. From the rather reliable times of the first fall of the water, which were reported from Kalymnos ( $03:25$ ), Patmos ( $03:30$ ) and Grete (Palaeokastron,  $03:46$ ), the average wave speed was estimated at 87 m/sec., 62 m/sec. and 86 m/sec., respectively. The estimated speed is in accordance with the depths encountered in the traveling of the wave. From the tide gauge observations on Leros and the long duration of the tsunami at near-by coasts (24 hours on Astypalaea and Leros), it was concluded that after the principal shock, probably four other submarine landslides were set off by the long sequence of the aftershocks; as the amplitude of the wave on Pholegandros points out, this occurred at least in the case of the second major shock ( $36.8^{\circ}$  N,  $25^{\circ}.2$  E,  $H = 03:24:05$ ,  $M = 7$ ).

**ΘΕΩΡΗΤΙΚΗ ΦΥΣΙΚΗ.—Subharmonics of order one third in the case of cubic restoring force.** Part II, *by Dem. G. Magiros\**. Ἀνεκοινώθη ὑπὸ τοῦ κ. Βασιλ. Αἰγινήτου.

*Introduction*

In this paper we discuss briefly the subharmonics of order one third in the case of a cubic restoring force.

The properly transformed equations, that give the components of the amplitude of the subharmonics, contain, if the amplitude of the external

\* ΔΗΜ. ΜΑΓΕΙΡΟΥ, Περὶ τῶν ὑποαρμονικῶν ταλαντώσεων τάξεως ἑνὸς πρὸς τρία.

force takes the values of an interval given in the paper, the ratios of the coefficients of the damping and the restoring force, and these ratios, under certain condition, can have any values, the coefficients themselves need not necessarily be small, a case very important in many engineering problems. We solve here the problem of subharmonics, their existence and stability, in case of given coefficients of the damping and the cubic restoring force, and the amplitude of the sinusoidal external force. An example is illustrated and the corresponding sketch of the singularities and the integral curves in the whole plane, which is separated into proper regions, is given.

§ I. *The equations for the components of the amplitude of subharmonics.*

The equation to be solved is:

$$(1) \quad \ddot{Q} + \bar{k}\dot{Q} + \bar{c}_1 Q + \bar{c}_2 Q^2 + \bar{c}_3 Q^3 = B \sin 3\tau.$$

By using:

$$(2) \quad \bar{k} = \varepsilon k, \quad 1 - \bar{c}_1 = \varepsilon c_1, \quad \bar{c}_2 = \varepsilon c_2, \quad \bar{c}_3 = \varepsilon c_3,$$

the equation is transformed into:

$$(3) \quad \ddot{Q} + Q = \varepsilon f(Q, \dot{Q}) + B \sin 3\tau,$$

$$(3\alpha) \quad f(Q, \dot{Q}) = -k\dot{Q} + c_1 Q - c_2 Q^2 - c_3 Q^3.$$

In case  $\varepsilon = 0$ , the solution of (3) is given by:

$$(4) \quad Q = x \sin \tau - y \cos \tau - \frac{B}{8} \sin 3\tau,$$

where  $x$  and  $y$  are constants, known for given initial conditions.

In case  $\varepsilon \neq 0$  we try to determine the steady state solutions of the equation (3), i.e. the constant limits:  $x(\varepsilon, \tau)$ ,  $y(\varepsilon, \tau)$ , according to the pre-

$\varepsilon \rightarrow 0$        $\varepsilon \rightarrow 0$

vious paper, part I.<sup>1</sup>

For this we have to find the functions  $A_0(x, y)$  and  $C_0(x, y)$ , of the paper [I]. These functions come from the equations (27) and (28) of the paper [I], if we put  $n=3$  and  $x$  and  $y$  instead of  $u_1$  and  $u_2$  respectively, when the result is:

<sup>1</sup> D. G. MAGIROS, Subharmonics of any order in case of nonlinear restoring forces. *Praktika of Athens Academy* 32, 1957, pp. 77.

$$(5) \quad \begin{aligned} A_o(x, y) &= \frac{1}{2} \left\{ -kx - c_1 y + \frac{3}{4} c_s y \left( x^2 + y^2 + \frac{B^2}{32} \right) - \frac{3}{16} c_s B xy \right\}, \\ C_o(x, y) &= \frac{1}{2} \left\{ c_1 x - ky - \frac{3}{4} c_s x \left( x^2 + y^2 + \frac{B^2}{32} \right) + \frac{3}{32} c_s B (-x^2 + y^2) \right\}, \end{aligned}$$

and the equations which give the unknown  $x$  and  $y$  are:

$$(6) \quad \begin{aligned} kx + c_1 y - \frac{3}{4} c_s y \left( x^2 + y^2 + \frac{B^2}{32} \right) + \frac{3}{16} Bxy &= 0, \\ c_1 x - ky - \frac{3}{4} c_s x \left( x^2 + y^2 + \frac{B^2}{32} \right) + \frac{3}{32} c_s B (-x^2 + y^2) &= 0. \end{aligned}$$

In the case  $c_s \neq 0$ , (6) can be written as:

$$(7) \quad \begin{aligned} \mu x + \lambda y - y \left( x^2 + y^2 + \frac{B^2}{32} \right) + \frac{1}{4} Bxy &= 0, \\ \lambda x - \mu y - x \left( x^2 + y^2 + \frac{B^2}{32} \right) + \frac{1}{8} B(-x^2 + y^2) &= 0, \end{aligned}$$

with:

$$(7\alpha) \quad \lambda = \frac{4}{3} \frac{c_1}{c_s}, \quad \mu = \frac{4}{3} \frac{k}{c_s}.$$

We ask for «real solutions»  $(x, y)$  of the system (7).

*Remarks.* From prescribed initial conditions of the equation (3), say  $Q_o$  and  $\dot{Q}_o$  at  $\tau=0$ , we have, according to (4),

$$(8\alpha) \quad Q_o = -y, \quad \dot{Q}_o = x - \frac{3}{8} B,$$

when the given initial conditions correspond to the point:

$$(8\beta) \quad x = \dot{Q}_o + \frac{3}{8} B, \quad y = -Q_o,$$

in the  $x, y$ -plane, which is the «starting point».

Starting from the «starting point» and following the corresponding integral curve with the lapse of the time we can terminate to a «final point», which corresponds to the proper steady solution. The coordinates of the «final point» are solutions of the system (7). Given the initial conditions and the amplitude  $B$  of the external force, the «starting point» in the  $x, y$ -plane is defined; conversely, any point of  $x, y$ -plane can be taken as «starting point» by properly choosing the initial conditions and the amplitude  $B$ .

If the «starting point» is selected in coincidence with a «stable final point», no «transient phenomena» must exist.

## § II. Restrictions to the coefficients of the equation (1).

If:

$$(8) \quad A = r^2 + \frac{B^2}{32}, \quad r^2 = x^2 + y^2,$$

the system (7) is written as:



$$(9) \quad \begin{aligned} \mu x + \lambda y - A y &= -\frac{1}{8} B (2xy) \\ \lambda x - \mu y - A x &= -\frac{1}{8} B (-x^2 + y^2). \end{aligned}$$

Squaring and adding (9) we find:

$$(10) \quad \lambda^2 + \mu^2 + A^2 - 2A\lambda = \frac{B^2}{64} r^2;$$

eliminating A between (10) and (8) we have.

$$(11) \quad r^4 + \left(\frac{3}{64} B^2 - 2\lambda\right) r^2 + \left(\lambda^2 + \mu^2 + \frac{B^4}{32^2} - \lambda \frac{B^2}{16}\right) = 0,$$

the roots of which are:

$$(11\alpha) \quad r^2 = \frac{1}{2} \left\{ 2\lambda - \frac{3}{64} B^2 \pm \sqrt{\left(2\lambda - \frac{3}{64} B^2\right)^2 - 4\left(\lambda^2 + \mu^2 + \frac{B^4}{32^2} - \lambda \frac{B^2}{16}\right)} \right\}.$$

The reality of  $r^2$  requires:

$$(12) \quad I \equiv 7B^4 - 2^8 \lambda B^2 + 2^{14} \mu^2 \leq 0.$$

The roots of:  $I=0$  are:

$$(13) \quad B^2 = \frac{2^7}{7} \left( \lambda \pm \sqrt{\lambda^2 - 7\mu^2} \right),$$

then the condition (12) requires the following condition to be fulfilled:

$$(14) \quad \begin{cases} \alpha) & \lambda^2 - 7\mu^2 > 0, \\ \beta) & \frac{2^7}{7} \left( \lambda - \sqrt{\lambda^2 - 7\mu^2} \right) \leq B^2 \leq \frac{2^7}{7} \left( \lambda + \sqrt{\lambda^2 - 7\mu^2} \right). \end{cases}$$

By using (7a) and (2) we find the following restrictions for  $\bar{k}$ ,  $\bar{c}_1$ ,  $\bar{c}_3$ , B:

$$(15) \quad \begin{cases} \alpha) & \left( \frac{1 - \bar{c}_1}{\bar{k}} \right)^2 > 7, \\ \beta) & \frac{2^9}{3 \cdot 7} \left( \frac{1 - \bar{c}_1}{\bar{k}} - \sqrt{\left( \frac{1 - \bar{c}_1}{\bar{k}} \right)^2 - 7 \left( \frac{\bar{k}}{\bar{c}_3} \right)^2} \right) \leq B^2 \leq \frac{2^9}{3 \cdot 7} \left( \frac{1 - \bar{c}_1}{\bar{k}} + \sqrt{\left( \frac{1 - \bar{c}_1}{\bar{k}} \right)^2 - 7 \left( \frac{\bar{k}}{\bar{c}_3} \right)^2} \right). \end{cases}$$

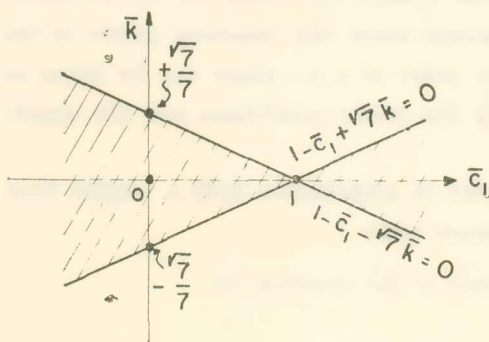


Fig. 1

The inequality (15a) can be written as:

$$(1 - \bar{c}_1 - \sqrt{7} \bar{k}) (1 - \bar{c}_1 + \sqrt{7} \bar{k}) > 0,$$

then only the shaded region in Fig. 1 is valid in the  $\bar{c}_1$ ,  $\bar{k}$ -plane.

From (11) or (11a) we can draw  $r^2$ , versus  $B^2$ , and by using (15b) we can have the arcs of the diagram which are valid in our problem.

§ III. *The solutions of the system (9).*

The system (9) can be written as:

$$(16) \quad \begin{aligned} \left(\mu + \frac{1}{4} B y\right) x + (\lambda - A) y &= 0, \\ \frac{1}{8} B x^2 - (\lambda - A) x + \left(\mu - \frac{1}{8} B y\right) y &= 0. \end{aligned}$$

The vanishing of, say, the Sylvester's eliminant, which is the condition for common roots of (16), leads, for non-zero roots, to the cubic:

$$(17) \quad y^3 - 3 \left(\frac{4}{B}\right)^2 \left\{ (\lambda - A)^2 + \mu^2 \right\} y - 2 \left(\frac{4\mu}{B}\right)^3 - 2 \left(\frac{4}{B}\right)^3 \mu (\lambda - A)^2 = 0$$

By knowing the coefficients of (1), we know, from (7α) and (2), λ and μ, then the amplitude r from (11α) in its two values. On the circumference with radius r there are one or three singularities, of which the ordinates y are the real roots of (17), when for their abscissas we apply the Pythagoras theorem. The singularities (x, y) are therefore at most seven, included the origin, which, in every case, is a singularity.

§ IV. *Example:*

$$\text{Given:} \quad \ll B = 4, \quad \bar{k} = \frac{3}{16}, \quad \bar{c}_1 = 1 \frac{3}{4}, \quad \bar{c}_3 = -\frac{1}{2} \gg.$$

We can find is this special case:

$$(18) \quad \left\{ \begin{aligned} \frac{1 - \bar{c}_1}{\bar{c}_3} = \frac{c_1}{c_3} = \frac{3}{2}, \quad \frac{\bar{k}}{\bar{c}_3} = \frac{k}{c_3} = -\frac{3}{8}, \quad \lambda = 2, \quad \mu = -\frac{1}{2}, \\ r_1^2 \approx 2, \quad r_2^2 \approx 1,25, \quad r_1 \approx 1,414, \quad r_2 \approx 1,118, \\ A_1 \approx 2,5, \quad A_1^2 \approx 6,25, \quad A_2 \approx 1,75, \quad A_2^2 \approx 3,06, \end{aligned} \right.$$

then:

$$(19) \quad \begin{aligned} \alpha) \quad y_1^3 - \frac{3}{2} y_1 + \frac{1}{2} &= 0, \\ \beta) \quad y_2^3 - \frac{15}{16} y_2 + \frac{5}{16} &= 0. \end{aligned}$$

Each of these cubic equations has three real unequal roots:

$$(20\alpha) \quad y_{11} \approx 0,9996, \quad y_{12} \approx -1,3645, \quad y_{13} \approx 1,3645,$$

the first, and

$$(21\alpha) \quad y_{21} \approx 0,7032, \quad y_{22} \approx -1,1034, \quad y_{23} \approx 1,0431,$$

the second, when the corresponding abscissas are:

$$(20\beta) \quad x_{11} \approx 0,9996, \quad x_{12} \approx 0,3648, \quad x_{13} \approx -0,3648,$$

$$(21\beta) \quad x_{21} \approx 0,8672, \quad x_{22} \approx 0,1755, \quad x_{23} \approx -0,3991.$$

§ V. *The stability of the solutions.*

For the study of the stability of the solutions the number  $\varepsilon$  enters. The number  $\varepsilon$  must be such that:

$$(22) \quad |\varepsilon| < \frac{r}{4M(\tau - \tau_0)},$$

according to paper [1], § IV, γ.

If in (22) the initial time  $\tau_0 = 0$ , by taking arbitrarily  $\varepsilon = 1$ , this means that the max  $\tau = T$  is taken according to:

$$(23) \quad T < \frac{r}{4M}.$$

Take now the partial derivatives with respect to  $x$  and  $y$  of the functions  $A_0(x, y)$  and  $C_0(x, y)$  given by (5). By establishing the restriction (23), which corresponds to  $\varepsilon = 1$ , these partial derivatives can be written as follows:

$$(24) \quad \begin{aligned} \frac{\partial A_0}{\partial x} &\equiv a_1 = \frac{1}{2} \bar{c}_3 \left\{ -\frac{\bar{k}}{\bar{c}_3} + \frac{3}{2} xy - \frac{3}{16} By \right\}, \\ \frac{\partial A_0}{\partial y} &\equiv a_2 = \frac{1}{2} \bar{c}_3 \left\{ -\frac{1 - \bar{c}_1}{\bar{c}_3} + \frac{3}{4} \left( x^2 + 3y^2 + \frac{B^2}{32} \right) - \frac{3}{16} Bx \right\}, \\ \frac{\partial C_0}{\partial x} &\equiv b_1 = \frac{1}{2} \bar{c}_3 \left\{ \frac{1 - \bar{c}_1}{\bar{c}_3} - \frac{3}{4} \left( 3x^2 + y^2 + \frac{B^2}{32} \right) - \frac{3}{16} Bx \right\}, \\ \frac{\partial C_0}{\partial y} &\equiv b_2 = \frac{1}{2} \bar{c}_3 \left\{ -\frac{\bar{k}}{\bar{c}_3} - \frac{3}{2} xy + \frac{3}{16} By \right\}. \end{aligned}$$

The characteristic roots, which help for finding the type of the singularity, according to § VII of paper [1], is:

$$(25) \quad p_{1,2} = \frac{1}{2} \left\{ a_1 + b_2 \pm \sqrt{(a_1 - b_2)^2 + 4a_2b_1} \right\}.$$

The computation for the singularities of our example, the coordinates of which are given by  $(20\alpha, \beta)$  and  $(21\alpha, \beta)$ , gives:

$$(26) \quad \left\{ \begin{array}{llll} 0: & \text{The origin} & x = 0 & y = 0 & : \text{«stable spiral»} \\ I: & \text{The point} & x_{11} = 0,9996 & y_{11} = 0,9996 & : \text{» »} \\ II: & \text{» »} & x_{12} = 0,3648 & y_{12} = -1,3645 & : \text{» »} \\ III: & \text{» »} & x_{13} = -0,3648 & y_{13} = 1,3645 & : \text{«saddle point»} \\ IV: & \text{» »} & x_{21} = 0,8675 & y_{21} = 0,7032 & : \text{» »} \\ V: & \text{» »} & x_{22} = 0,1755 & y_{22} = -1,1034 & : \text{» »} \\ VI: & \text{» »} & x_{23} = -0,3991 & y_{23} = 1,0431 & : \text{» »} \end{array} \right.$$

The origin corresponds to «harmonic solution», which, as stable, is

acceptable. The points I, II, III are on the circumference with radius  $r_1=1,414$ . The points IV, V, VI, which are on the circumference with radius  $r_2=1,118$ , are «intrinsically unstable».

§ VI. *Non-existence of limiting cycles.*

From (24) we have:

$$(27) \quad \frac{\partial A_0}{\partial x} + \frac{\partial C_0}{\partial y} = -\bar{k}$$

valid in the whole  $x,y$ -plane, and according to the Bendixson's<sup>1</sup> criterion no limit cycles can exist in the whole  $x,y$ -plane.

For  $\bar{k}=0$ , some of the singularities may be «centers», then we may have «closed integral curves».

§ VII. *Sketch corresponding to the above example.*

Applying the «method of isoclines» to the differential equation:

$$(28) \quad \frac{dy}{dx} = \frac{C_0(x,y)}{A_0(x,y)},$$

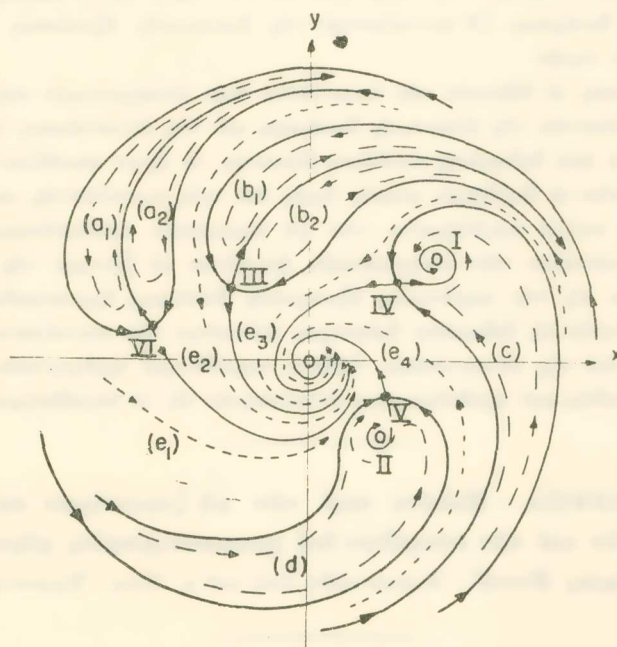


Fig. 2

<sup>1</sup> I. BENDIXSON, *Acta Math.* 24 (1901), 1-88.



where the functions  $A_0(x,y)$  and  $C_0(x,y)$  are given by (5), we can have a figure showing the singularities, the integral curves and the separation into regions corresponding to the above example. In Fig. 2 a sketch of these things is given.

The solid lines are the boundaries from the saddle points, the dotted lines are the integral curves. The regions:  $(\alpha_1)$ ,  $(\alpha_2)$ ,  $(\beta_1)$  and  $(\beta_2)$  correspond to no solutions of our example. The regions:  $(e_1)$ ,  $(e_2)$ ,  $(e_3)$  and  $(e_4)$  correspond to «stable harmonic solution»:  $Q = -\frac{1}{2} \sin 3\tau$ . The region (c) correspond to «stable solution»:  $Q = 0,9996 \sin \tau - 0,9996 \cos \tau - \frac{1}{2} \sin 3\tau$ ; and the region (d) correspond to «stable solution»:  $Q = 0,3648 \sin \tau + 1,3645 \cos \tau - \frac{1}{2} \sin 3\tau$ . The amplitude of the subharmonic term in the last two stable solutions is the same:  $r_1 = 1,414$ .

#### ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν ἐργασίαν ταύτην συζητεῖται ἐν συντομίᾳ τὸ ζήτημα τῶν ὑποαρμο-  
νικῶν ταλαντώσεων τάξεως ἐνὸς πρὸς τρία εἰς τὴν περίπτωσιν κυβικῆς συναρτήσεως  
τῆς ἐλαστικῆς δυνάμεως. Οἱ συντελεσταὶ τῆς διαφορικῆς ἐξισώσεως εἶναι ὅχι κατ'  
ἀνάγκην μικρῶν τιμῶν.

Αἱ ἐξισώσεις αἱ δίδουσαι τὰς συνιστώσας τῶν ὑποαρμο-  
νικῶν ταλαντώσεων τῆς ἐλαστικῆς δυνάμεως καὶ τῆς ἀντιστάσεως (damping) καὶ  
τὰ πηλίκα αὐτὰ ὑπὸ δεδομένης συνθήκας δύνανται νὰ ἔχουν οἰασδήποτε τιμὰς, χωρὶς  
νὰ εἶναι ἀναγκαῖον νὰ δεχθῶμεν μικρὰς τιμὰς διὰ τοὺς συντελεστάς, περίπτωσις πολὺ  
σημαντικὴ εἰς πολλὰ προβλήματα τῶν μὴ γραμμικῶν ταλαντώσεων. Εὐρίσκονται  
ἐνταῦθα αἱ συνιστώσαι τῶν ὑποαρμο-  
νικῶν, ἐρευνᾶται τὸ ζήτημα τῆς ὑπάρξεως καὶ  
εὐσταθείας τῶν εἰς τὴν περίπτωσιν ἐξωτερικῆς δυνάμεως ἡμιτονοειδοῦς τύπου ὑπὸ  
πλάτος μεταβλητὸν εἰς δεδομένον διάστημα, διδομένων τῶν συντελεστῶν τῆς ἐλαστι-  
κῆς δυνάμεως καὶ τῆς ἀντιστάσεως. Δίδεται παράδειγμα ἀριθμητικὸν ὡς ἐφαρμογὴ  
τῆς θεωρίας, καθὼς καὶ σχεδιάγραμμα ἀντιστοιχοῦν εἰς τὸ παράδειγμα αὐτό.

ΟΡΓΑΝΙΚΗ ΧΗΜΕΙΑ.—Μελέτη περὶ τῶν μὴ ζυμωσίμων σακχάρων τῶν  
σταφυλῶν καὶ τῶν σταφίδων διὰ χρωματογραφίας χάρτου, ὑπὸ *Λυσ.*  
*Νιννῆ* καὶ *Μαρίας Νιννῆ\**. Ἀνεκοινώθη ὑπὸ τοῦ κ. Ἑμμ. Ἑμμανουήλ.

\* Θὰ δημοσιευθῇ κατωτέρω.