ΜΗΧΑΝΙΚΗ.— **Chaoticlike and other phenomena in the nonlinear dynamic analysis of structures: Quantitative - qualitative criteria,** by *A. N. Kounadis**, διὰ τοῦ Ἀκαδημαϊκοῦ κ. Περικλέους Θεοχάρη.

ABSTRACT

Discrete, dissipative/nondissipative, non-gradient structural systems (due to follower loading) described by nonlinear autonomous ordinary differential equations, are critically discussed. Through a Taylor series of these equations, in their most general form, it is found that the first variation coinciding with the Jacobian matrix is a non-symmetric block-matrix consisting of four submatrices properly identified. Attention is focused on perfect linear elastic or nonlinear elastic structural systems associated with trivial pre-critical equilibrium paths. Using a general approach with qualitative and quantitative criteria the stability of the precritical, critical and postcritical states is thoroughly examined. This analysis is applied to Ziegler's model for which a lot of numerical results are available. New dynamic bifurcations (independent of the structure of the Jacobian matrix) as well as local bifurcations associated with one zero eigenvalue, a double zero eigenvalue and imaginary conjugate eigenvalues (Hopf bifurcation) in a certain region of adjacent equilibria in the neighborhood of a double branching point, are revealed. In this region it is also found that dynamic bifurcations (associated with limit cycles) may occur prior to static bifurcation. New findings for the stability of critical states based on the present nonlinear dynamic analysis contradict previous widely accepted results of the classical (linear) analysis. Local or global bifurcations for both non-gradient and gradient systems even when there is no damping may be associated with chaoslike phenomena. Such phenomena sometimes are quite persistent, a fact which does not allow easy prediction of the long-term response of the system.

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1. Introduction

The nonlinear analysis of engineering structures was long ago recognized as being of paramount importance for establishing accurately and safely their load-carrying capacity. Many instability phenomena of vital concern were brought into light thanks to the postbuckling analysis founded by the Dutch researcher W.T. Koiter fortyfive years ago. This analysis was applied to conservative and later on to non-gradient (nonconservative due to follower loads) structures. However, the postbuckling analysis regarding the latter systems failed in certain cases to predict the exact divergence (buckling) critical load which can be established only by using a nonlinear dynamic analysis [Kounadis $(1989)_4$, $(1990)_2$, $(1991)_2$, Sotiropoulos and Kounadis (1990)].

Dynamic instabilities belong to a much broader class of real-world problems which, being in general nonlinear, fall outside the domain of the classical (linear) dynamic analysis and hence must be tackled in the first instance on a computer. Today we are indeed witnessing a spectacular blossoming of nonlinear dynamic, made possible on the one hand by the wide availability of powerful computers and on the other hand by the qualitative topological approach based on the theoretical Poincaré's strides. These ideas are revolutionizing the theory of dynamical systems in many branches of applied sciences and engineering (e.x. in Physics, Meteorology, Astronomy, Chemistry, Electromagnetics, Biology, Ecology, Economy, etc), while they are having a lesser impact on Mechanics. The study of the interaction between *geometrical nonlinearities* and *damping* revealed new phenomena such as: phenomena due to strange attractors or to simple or multiple attractors, chaoslike and metastability phenomena, phenomena due to sensitivity to initial conditions or to damping. At this point a reasonable question which arises is whether or not such chaoticlike or phenomena of similar nature appear also when a nonlinear dynamic analysis is employed.

The answer to this question was given as a byproduct of a systematic and intensive four year research by the author and his associates who established efficient techniques and qualitative-quantitative criteria for the nonlinear dynamic analysis of structures. In a series of 32 publications (see refs 1–32) among which there are 22 papers in international journals and 10 presentations in international symposia (as invited key-lectures) brought into light for the first time impressive chaoticlike or other interesting phenomena associated with the nonlinear dynamic analysis of structures. New findings contradict widely accepted early results of eminent researchers (such as G. Herrmann, Nemat-Nasser, H. Leipholz, R. Plaut et al) which were derived on the basis of classical (linear) dynamic analysis. Today we are able to explain and analyse qualitatively and quantitatively various phenomena of the dynamic response of structures [Thompson and Stewart (1986), Jackson (1989)] such as the collapse due to flutter (unstable limit cycles) of the Tacoma 854 m span suspension bridge in 1940.

Despite the availability of high speed computer and efficient computational schemes, quite often a large time solution of nonlinear equations of motion may be unreliable due to the accumulation of error. Thus, one has to resort to efficient approximate analytic techniques [Kounadis $(1989)_2$, Kounadis and Mallis (1988), Kounadis $(1992)_1$] or to establish qualitative and quantitative criteria [Kounadis $(1988)_{1,2}$, Raftoyannis and Kounadis (1988), Kounadis, Raftoyiannis and Mallis $(1989)_{1,2}$, Kounadis, Mahrenholtz and Bogacz (1990), Kounadis and Raftoyiannis (1990), Kounadis $(1991)_1$, Kounadis, Mallis and Raftoyiannis (1991), Kounadis $(1992)_3$, Mathrenholtz and Kounadis (1993)].

In addition to the aforementioned qualitative topological approach for reducing the dimension and nonlinearities of dynamical systems, one should mention the old Lyapunov-Schmidt technique and mainly the local techniques of the central manifold [Carr (1981)], of the normal forms [Perko (1991)] and splitting lemma [Gilmore (1981)].

In refs [2, 3, 5, 6, 12, 13, 18, 21, 22, 30, 32] the author and his associates give a qualitative explanation of the mechanism of dynamic buckling for multi-degree-of-freedom systems under conservative loading. These studies extend the well-known research of Budiansky (1967) and Hutchinson dealing with the dynamic buckling of one-degree-of-freedom systems to multi-degree-of-freedom systems. The machanism of dynamic buckling and dynamic instability of nonconservative of divergence and flutter type systems are thoroughly examined in refs [7, 9–11, 14–17, 19, 23–25, 27, 29, 31] in which new phenomena reported for the first time in the literature or contradicting previous ones, are also assessed. Chaoticlike and metastability phenomena, phenomena of sensitivity to damping and to initial conditions appear in both conservative and nonconservative dissipative or nondissipative systems [7, 9, 10, 15–20, 22–25, 27, 29].

The present work, being an extension of previous studies by the author and his associates, deals with the nonlinear dynamic buckling and instability of nonconservative bifurcational nonlinearly elastic dissipative or nondissipative structural systems subjected to follower forces. Using a general theoretical analysis associated with qualitative and quantitative criteria a series of different types of bifurcations is properly identified. Dynamic bifurcations occurring prior to static (divergence) bifurcations, chaoticlike and other phenomena are found in a small region (of abjacent equilibria) in the vicinity of the boundary which separates the region of existence from the region of non-existence of adjacent equilibria. New findings for the stability of critical states in this region as well as in the region of non adjacent equilibria, contradicting previous widely accepted results, are revealed. Local and global bifurcations for both conservative or nonconservative dissipative or non-dissipative systems may be associated with chaoslike phenomena. These phenomena may, quite often, be very persistent, a fact which does not permit to predict easily the long-term response of the system.

2. Mathematical formulation

Consider a general n-degree-of-freedom dissipative discrete system under a partial follower loading λ of constant magnitude associated with the nonconservativeness parameter η. The nonlinear Ordinary Differential Equations (ODE) of Lagrange governing its motion at any time t in terms of generalized displacement q_i and generalized velocities $\dot{q}_i\,(i=1,...,n)$ are given by

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right)-\frac{\partial K}{\partial q_i}+\frac{\partial U}{\partial q_i}+\frac{\partial F}{\partial \dot{q}_i}-Q_i=0 \quad (i=1,...,n) \tag{1}$$

where the dots denote differentiation with respect to time t; $K=(1/2)\alpha_{ij}\dot{q}_i\dot{q}_j$ is the positive definite function of the total kinetic energy of a *natural system* [Meirovitch (1970)] with diagonal elements being functions of masses m_i [i.e. $\alpha_{ii}=\alpha_{ii}(m_i)$, $i=1,...,n$] and non-diagonal elements being functions of both m_i and q_i [i.e. $\alpha_{ij}=\alpha_{ij}(m_i, q_i)$ for $i\neq j$ and $i, j=1,...,n$]; $U=U(q_i)$ is the positive definite function of the strain energy, being a nonlinear analytic function of q_i; $F=(1/2)c_{ij}\dot{q}_i\dot{q}_j$ is the non-negative definite (linear viscous) dissipative function of Rayleigh with coefficients which might be functions of q_i [i.e. $c_{ij}=c_{ij}(q_i)$ with $i, j=1,...,n$]; $Q_i=Q_i(q_i;\eta;\lambda)$ designate generalized, in general, nonpotential forces being nonlinear analytic functions of q_i and η, and linear functions of λ. Clearly, the tensor summation convention of Einstein is employed herein with summation ranging from one to n. Note that for a certain value of the nonconservativeness parameter η the forces Q_i become potential (conservative) forces. In view of these assumptions the system under consideration is classified as a dissipative pseudo-conservative system [Huseyin (1978)].

The loading λ and parameter η are the main *control* parameters for *static* and *dynamic bifurcations* as well as for the *stability of equilibria and limit cycles*. As *dynamic* bifurcation is defined a sudden qualitative change of the system response occurring at a certain value of a smoothly varying parameter. From a point of view of *topology* dynamic bifurcations correspond to those values of a control parameter for which the response of the system becomes *structurally* unstable [Andronov and Pontryagin (1937)]; namely the phase portrait is changed to a topologically nonequivalent portrait by a smooth change of the control parameter. In the following it is assumed that the bifurcation points (either static or dynamic) lie on a trivial fundamental (prebuckling) equilibrium path.

The analysis deals with dissipative systems, since the precise modeling of a real structural system should include damping in addition to geometrical nonlinearities. The presence of internal friction, in the most general sense of the term, has as a consequence the existence of an *attractor*; that is the existence of an asymptotic limit (as $t \to \infty$) of the solutions such that the initial conditions-the point of departure-have no direct influence. In mechanics when friction entails continuous decrease of the energy, the corresponding systems are called for this reason *dissipative*. Many phenomena in nonlinear dynamics are the corollary of the interaction between geometrical nonlinearities and damping.

From the above expression of the total kinetic energy K one can readily obtain

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) = \alpha_{ij}\ddot{q}_j + \dot{\alpha}_{ij}\dot{q}_j \quad \left(\text{with } \dot{\alpha}_{ij} = \frac{\partial \alpha_{ij}}{\partial q_k}\dot{q}_k\right) \tag{2}$$

Writing $\dot{\alpha}_{ij}\dot{q}_j$ for $i=1,\dots,n$ we get a column matrix whose i^{th} element is

$$\dot{\alpha}_{ij}\dot{q}_j = [\dot{q}_1,\dots,\dot{q}_n]\cdot\begin{bmatrix} \frac{\partial \alpha_{i1}}{\partial q_1} & \frac{\partial \alpha_{i2}}{\partial q_1} & \cdots 0 \cdots & \frac{\partial \alpha_{in}}{\partial q_1} \\ \frac{\partial \alpha_{i1}}{\partial q_2} & \frac{\partial \alpha_{i2}}{\partial q_2} & \cdots 0 \cdots & \frac{\partial \alpha_{in}}{\partial q_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \alpha_{i1}}{\partial q_n} & \frac{\partial \alpha_{i2}}{\partial q_n} & \cdots 0 & \frac{\partial \alpha_{in}}{\partial q_n} \end{bmatrix}\cdot\begin{bmatrix}\dot{q}_1\\ \cdot \\ \cdot \\ \cdot \\ \dot{q}_n\end{bmatrix} \tag{3}$$

where, as is known a priori, $\partial\alpha_{ii}/\partial q_k = 0$ for $i,k=1,\dots,n$.

Similarly, it follows that

$$\frac{\partial K}{\partial q_i} = \frac{1}{2}[\dot{q}_1,\dots,\dot{q}_n]\begin{bmatrix} 0 & \frac{\partial \alpha_{12}}{\partial q_i} & \frac{\partial \alpha_{13}}{\partial q_i} & \cdots\cdots & \frac{\partial \alpha_{1n}}{\partial q_i} \\ & 0 & \frac{\partial \alpha_{23}}{\partial q_i} & \frac{\partial \alpha_{24}}{\partial q_i} \cdots & \frac{\partial \alpha_{2n}}{\partial q_i} \\ & & 0 \cdots\cdots & & \vdots \\ & \text{Symmetric} & & \ddots & \frac{\partial \alpha_{n-1,n}}{\partial q_i} \\ & & & & 0 \end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \cdot \\ \cdot \\ \cdot \\ \dot{q}_n\end{bmatrix} \tag{4}$$

By means of relations (3) and (4) one can determine the column with i^{th} element

$$\dot{\alpha}_{ij}\dot{q}_i\frac{\partial K}{\partial q_i}=\dot{\mathbf{q}}^T\mathbf{S}_i\dot{\mathbf{q}} \qquad (5)$$

where $\mathbf{S}_i$ is a square non-symmetric matrix, while $\mathbf{q}^T=(q_1,...,q_n)^T$ is the transpose of the vector $\mathbf{q}$.

Given that $\partial F/\partial\dot{q}_i=c_{ij}\dot{q}_j$ eqs (1) by means of relations (2) through (5) become

$$\alpha_{ij}\ddot{q}_j+\dot{\mathbf{q}}^T\mathbf{S}_i\dot{\mathbf{q}}+c_{ij}\dot{q}_j+\frac{\partial U}{\partial q_i}-Q_i=0 \quad (i=1,...,n) \qquad (6)$$

Since $[\alpha_{ij}]$ is a positive definite matrix, one can always get

$$\begin{bmatrix}\ddot{q}_1\\ \cdot\\ \cdot\\ \cdot\\ \ddot{q}_n\end{bmatrix}=[\beta_{ij}]\begin{bmatrix}\dot{\mathbf{q}}^T\mathbf{S}_1\dot{\mathbf{q}}\\ \cdot\\ \cdot\\ \cdot\\ \dot{\mathbf{q}}^T\mathbf{S}_n\dot{\mathbf{q}}\end{bmatrix}[\tilde{c}_{ij}]\begin{bmatrix}\dot{q}_1\\ \cdot\\ \cdot\\ \cdot\\ \dot{q}_n\end{bmatrix}[\beta_{ij}]\begin{bmatrix}\frac{\partial V}{\partial q_1}\\ \cdot\\ \cdot\\ \cdot\\ \frac{\partial V}{\partial q_n}\end{bmatrix} \qquad (7)$$

where $[\beta_{ij}]=[\alpha_{ij}]^{-1}$, $[\tilde{c}_{ij}]=[\beta_{ij}]\,[c_{ij}]$ and $\partial V/\partial q_i=\partial U/\partial q_i-Q_i$; hence $V=V\,(q_i;\lambda;\eta)$.

The set of n 2^{nd} order O.D.E. of Lagrange [eqs (7)] can be replaced by a set of 2n 1^{st} order equations as follows: Setting

$$y_1=q_1,\ y_2=q_2,...,\ y_n=q_n \text{ and } y_{n+1}=\dot{q}_1,\ y_{n+2}=\dot{q}_2,...,\ y_{2n}=\dot{q}_n \qquad (8)$$

eqs(7) can be written in the form

$$\dot{y}_i=Y_i(y_1,...,y_{2n};\lambda;\eta),\ i=1,...,2n \qquad (9)$$

where

$$Y_1=y_{n+1},\ Y_2=y_{n+2},\ ...,\ Y_n=y_{2n}$$

and

$$\begin{aligned}Y_{n+1}&=-\beta_{1j}\tilde{y}^T S_j\tilde{y}-\tilde{c}_{1j}y_{n+j}-\beta_{1j}\frac{\partial V}{\partial y_j}\\ Y_{n+2}&=-\beta_{2j}\tilde{y}^T S_j\tilde{y}-\tilde{c}_{2j}y_{n+j}-\beta_{2j}\frac{\partial V}{\partial y_j}\\ &\cdots\cdots\cdots\cdots\cdots\\ Y_{2n}&=-\beta_{nj}\tilde{y}^T\mathbf{S}_j\tilde{y}-\tilde{c}_{nj}y_{n+j}-\beta_{nj}\frac{\partial V}{\partial y_j}\end{aligned} \qquad (10)$$

with $\tilde{y}$ designating the vector with components $y_{n+1}, y_{n+2},...,y_{2n}$.

Eqs (9) can be written in matrix-vector form as follows

$$\dot{y}=\mathbf{Y}\,(y;\lambda;\eta) \qquad (11)$$

where $y = (y_1,...,y_{2n})^T$ is the state vector, being a continuous function of t and λ for fixed η; the nonlinear vector-function $\mathbf{Y} = (Y_1,...,Y_{2n})^T$ is assumed to satisfy the Lipschitz conditions, at least in the domain of interest, for all t and λ. This implies that $y_i(t;\lambda)$ belongs to the class of functions C^2 [i.e. $y(t;\lambda) \epsilon C^2$].

For the above bifurcational system eq. (11) is satisfied by the zero solutions, $y=0$, for all values of λ and η, i.e.

$$\mathbf{Y}(0;\lambda;\eta) = 0 \tag{12}$$

For certain ranges of values of the nonconservativeness parameter η the system exhibits adjacent equilibria [determined via eq. (12)]. Outside these regions of divergence *instability* the system displays a *limit cycle* response which cannot be established unless a nonlinear dynamic analysis is employed. In the first case the critical state is associated with a branching point bifurcating into adjacent equilibria (post-buckling equilibrium path), while in the second case the critical state is associated with a branching point bifurcating into limit cycles. In both cases the system displays in addition to the zero solution another (different from zero) solution.

3. Local analysis

The response of the structural system, in several cases, can be established by using a local (linear) dynamic analysis (local bifurcations), while in other cases it cannot be explored unless a global (nonlinear) dynamic analysis is applied [global bifurcations, Kounadis $(1992)_3$]. For instance, dynamic bifurcations with trajectories passing through *saddle* points or cases where closed orbits become *nonhyperbolic* (at least one characteristic multiplier has unit modulus) can be detected only by using a *global* dynamic analysis [Peixoto (1959)]. The last analysis is the only safe way for exploring chaotic or chaoslike phenomena occurring at large time. While chaos phenomena have been observed in non-gradient dissipative systems due to strange attractors, chaoslike phenomena may also occur in dissipative or non-dissipative systems under potential (conservative) forces due to sensitivity to initial conditions [Kounadis $(1991)_3$, Kalathas and Kounadis (1991)]. The application of the global analysis is also justified from the fact that the state of a system may be stable (unstable) on the basis of a linear (local) dynamic analysis, while unstable (stable) using a nonlinear (global) analysis. Examples of contradictions between local and global analyses appear for instance in cases of dynamic buckling of geometrically imperfect systems under step loading of infinite duration [Kounadis $(1991)_1$].

For the study of stability of a solution $\mathbf{y}^o$, being associated either with a singular (equilibrium) point $\mathbf{y}^E$ or a limit cycle behavior $\mathbf{y}^R$, we can examine the motion in the vicinity of such a solution by superimposing the disturbance (vector) $\boldsymbol{\xi}$ to $\mathbf{y}^o$. Inserting $\mathbf{y}=\mathbf{y}^o+\boldsymbol{\xi}$ into eq. (11) and expanding $\mathbf{Y}(\mathbf{y}^o+\boldsymbol{\xi};\lambda;\eta)$ into a Taylor series around $\mathbf{y}^o$ we get

$$\dot{\boldsymbol{\xi}}=\delta\mathbf{Y}^o+\frac{1}{2!}\delta^2\mathbf{Y}^o+\frac{1}{3!}\delta^3\mathbf{Y}^o+\ldots \tag{13}$$

where

$$\delta\mathbf{Y}^o=\begin{bmatrix}\delta Y_1^o\\ \cdot\\ \cdot\\ \cdot\\ \delta Y_{2n}^o\end{bmatrix},\ \delta^2\mathbf{Y}^o=\begin{bmatrix}\delta^2 Y_1^o\\ \cdot\\ \cdot\\ \cdot\\ \delta^2 Y_{2n}^o\end{bmatrix},\ \delta^3\mathbf{Y}^o=\begin{bmatrix}\delta^3 Y_1^o\\ \cdot\\ \cdot\\ \cdot\\ \delta^3 Y_{2n}^o\end{bmatrix},\ \text{etc} \tag{14}$$

Note that

$$\delta\mathbf{Y}^o=\begin{bmatrix}(\xi_1\frac{\partial}{\partial y_1}+\ldots+\xi_{2n}\frac{\partial}{\partial y_{2n}})\,Y_1^o\\ \cdot\\ \cdot\\ \cdot\\ (\xi_1\frac{\partial}{\partial y_1}+\ldots+\xi_{2n}\frac{\partial}{\partial y_{2n}})\,Y_{2n}^o\end{bmatrix}=\begin{bmatrix}\frac{\partial Y_1^o}{\partial y_1} & \ldots\ldots & \frac{\partial Y_1^o}{\partial y_{2n}}\\ \cdot & & \cdot\\ \cdot & & \cdot\\ \frac{\partial Y_{2n}^o}{\partial y_1} & \ldots\ldots & \frac{\partial Y_{2m}^o}{\partial y_{2n}}\end{bmatrix}\begin{bmatrix}\xi_1\\ \cdot\\ \cdot\\ \xi_{2n}\end{bmatrix}=\mathbf{Y}_y^o\boldsymbol{\xi}$$

$$\delta^2 Y_i^o=(\xi_1\frac{\partial}{\partial y_1}+\ldots+\xi_{2n}\frac{\partial}{\partial y_{2n}})^2\,Y_i^o,\ \delta^3 Y_i^o=(\xi_1\frac{\partial}{\partial y_1}+\ldots+\xi_{2n}\frac{\partial}{\partial y_{2n}})^3\,Y_i^o,\ \text{etc:} \tag{15}$$

where $\mathbf{Y}_y^o=\mathbf{Y}_y(\mathbf{y}^o;\lambda;\eta)=\partial\mathbf{Y}(\mathbf{y}^o;\lambda;\eta)/\partial\mathbf{y}$ is the Jacobian matrix evaluated at $\mathbf{y}^o$.

It can be readily shown with the aid of relations (9) and (10) that the above Jacobian matrix is a block matrix consisting of four square submatrices of order nxn, that is

$$\mathbf{Y}_y(\mathbf{y}^o;\lambda;\eta)=\begin{bmatrix}0 & \mathbf{I}_n\\ -[\widetilde{V}_{ij}] & -[\tilde{c}_{ij}]\end{bmatrix} \tag{16}$$

where by $\mathbf{0}$ and $\mathbf{I}_n$ we denote the zero and the identity matrix of order nxn, respectively and

$$[\widetilde{V}_{ij}]=[\alpha_{ij}]^{-1}[V_{ij}]=[\beta_{ij}]\,[V_{ij}] \tag{17}$$

From relation (16) it follows that

$$\det\mathbf{Y}_y(\mathbf{y}^o;\lambda;\eta)=\det[\widetilde{V}_{ij}]=\det[\alpha_{ij}]^{-1}\det[V_{ij}] \tag{18}$$

Since $[\alpha_{ij}]$ is always a positive definite matrix the sign of the determinant of the Jacobian depends on the sign of the determinant of the matrix $[V_{ij}]$.

The characteristic equation of the Jacobian matrix (16) is given by

$$|\mathbf{Y}_y(\mathbf{y}^o;\lambda;\eta) - \rho\mathbf{I}_{2n}| = |\rho^2\mathbf{I}_n + \rho[\tilde{c}_{ij}] + [\widetilde{V}_{ij}]| = 0 \tag{19}$$

which is also equivalent to

$$|\rho^2[\alpha_{ij}] + \rho[c_{ij}] + [V_{ij}]| = 0 \tag{20}$$

Eq. (19) upon expansion leads to the following characteristic polynomial

$$\phi(\rho) = \rho^{2n} + a_1\rho^{2n-1} + a_2\rho^{2n-2} + \ldots + a_{2n-1}\rho + a_{2n} = 0 \tag{21}$$

$$\text{where } a_1 = -tr\,\mathbf{Y}_y(\mathbf{y}^o;\lambda;\eta) = \sum_{i=1}^{n}\tilde{c}_{ii},\ a_{2n} = \frac{\det[V_{ij}]}{\det[\alpha_{ij}]} \tag{22}$$

Eq. (19) is equivalent to [Kounadis (1990)$_2$]

$$\phi(\rho) = \prod_{i=1}^{n}(\rho^2 + b_i\rho + c_i) = 0 \tag{23}$$

with the following roots of each factor

$$\left.\begin{array}{l} -0.5b_i \pm \sqrt{(0.5b_i)^2 - c_i} \\ \text{while} \\ \sum_{i=1}^{n} b_i = a_1,\ \prod_{i=1}^{n} c_i = a_{2n}(\lambda;\eta) \end{array}\right\} \tag{24}$$

Clearly, the real parts of the eigenvalues (i.e. $-0.5\,b_i$) depend on the damping coefficients c_{ij} as well as on λ and η, while $a_2,\ldots,a_{2n-1}$ are functions of λ, η and b_i.

Regions of adjacent equilibria

The boundary between the regions of existence and non-existence of adjacent equilibria correspond to a certain value of η, say $\eta = \eta_o$, which is determined as follows:

If the determinant of the Jacobian matrix evaluated at the critical (trivial) state $\mathbf{y} = 0$ [see eq. (12)] is set equal to zero, we obtain the buckling (divergence instability) equation

$$a_{2n} = |\mathbf{Y}_y(0;\lambda_c;\eta)| = 0 \tag{25}$$

which due to relation (22) is equivalent to

$$\det[V_{ij}(0;\lambda_c;\eta)] = 0 \qquad (|\alpha_{ij}| \neq 0) \tag{26}$$

Since the elements V_{ij} are linear functions of the loading λ, eq. (25) or (26) yields an n^{th} degree algebraic polynomial with respect to λ. From these equations one can get, at least implicitly, the relationship

$$\eta = \eta(\lambda_c) \tag{27}$$

Following the procedure outlined by Kounadis (1983) the extreme value $\eta = \eta_o$ which defines the boundary between existence and non-existence of adjacent equilibria is determined by the condition

$$\frac{d\eta}{d\lambda} = \eta'(\lambda_c) = 0$$

or

$$a_{2n\lambda} = |Y_{y\lambda}| = 0 \tag{28}$$

along with eq. (25) or eq. (26). Solving the system of nonlinear eqs (25) and (28) with respect to η and λ_c we choose among all solutions the real solution with the minimum $\lambda_c^o > 0$. Since relation (28) is a necessary condition for extremum in the curve (27), the point (η_o, λ_c^o) is either a maximum or a minimum in this curve. If such a point corresponds to a maximum, adjacent equilibria do not exist for $\eta > \eta_o$, while if it is a minimum, adjacent equilibria exist for $\eta > \eta_o$ (Fig. 1). The critical states outside these regions (of divergence instability) associated with a limit cycle response can be established only by employing a dynamic analysis. As will be shown below the study of the nature of the critical state can be achieved only by using a nonlinear dynamic analysis. Similarly the stability (or instability) of the critical (divergence) states cannot be established unless a nonlinear postbuckling analysis is employed. Note that since $a_{2n}(\lambda_c^o, \eta_o) = a_{2n\lambda}(\lambda_c^o, \eta_o) = 0$ the point (λ_c^o, η_o) is a coincident (double) point in the curve η versus λ_c which is also called compound branching point.

Stability analysis

The stability of the precritical (trivial) states, regardless of whether or not the system is associated with static (divergence) instability or dynamic (limit cycles) instability can be established as follows: Let $Y_y(0;\lambda;\eta)$ be the Jacobian matrix evaluated at $y = 0$ and any real η with corresponding λ less than its critical value. If $\boldsymbol{r}$ is a right eigenvector of this matrix corresponding to the eigenvalue ρ, one can write

$$(\mathbf{Y}_y(0;\lambda;\eta) - \rho \mathbf{I}_{2n})\, \boldsymbol{r} = 0 \tag{29}$$

or due to relation (16) in block-matrix form

$$\begin{bmatrix} -\rho\mathbf{I}_n & \mathbf{I}_n \\ -[\widetilde{V}_{ij}] & -[\tilde{c}_{ij}]-\rho\mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{r}_o \\ \mathbf{r} \end{bmatrix} = 0 \text{ with } \boldsymbol{r}= \begin{bmatrix} \mathbf{r}_o \\ \mathbf{r} \end{bmatrix} \tag{30}$$

From this equation it follows that

$$\left.\begin{aligned} \rho\mathbf{r}_o &= \mathbf{r} \\ [\widetilde{V}_{ij}]\,\mathbf{r}_o + ([\tilde{c}_{ij}] + \rho\mathbf{I}_n)\,\mathbf{r} &= 0 \end{aligned}\right\} \tag{31}$$

Multiplying the second of eqs (31) by ρ and substituting the expression $\rho\mathbf{r}_o$ from the first of these equations we get

$$(\rho^2\mathbf{I}_n + \rho[\tilde{c}_{ij}] + [\widetilde{V}_{ij}])\,\mathbf{r} = 0 \tag{32}$$

which is also equivalent to

$$(\rho^2[\alpha_{ij}] + \rho\,[c_{ij}] + [V_{ij}])\,\mathbf{r} = 0 \tag{33}$$

In case of a conservative loading λ, matrix $[V_{ij}]$ (associated with the second variation of the total potential energy) is positive definite. Then premultiplication of this equation by $\mathbf{r}^T$ yields

$$(\mathbf{r}^T[\alpha_{ij}]\mathbf{r})\rho^2 + (\mathbf{r}^T[c_{ij}]\mathbf{r})\rho + \mathbf{r}^T[V_{ij}]\mathbf{r} = 0 \tag{34}$$

Since all coefficients of this equation are in general positive (scalar), eq. (34) is a 2^{nd} degree algedraic equation with the following roots

$$\rho_{1,2} = \frac{-\mathbf{r}^T[c_{ij}]\,\mathbf{r} \pm [(\mathbf{r}^T[c_{ij}]\mathbf{r})^2 - 4\,(\mathbf{r}^T[\alpha_{ij}]\mathbf{r})\,(\mathbf{r}^T[V_{ij}]\mathbf{r})]^{1/2}}{2\,(\mathbf{r}^T[\alpha_{ij}]\mathbf{r})} \tag{35}$$

If $[c_{ij}]$ is positive definite both roots are either negative or in case of structural (small) damping complex conjugate with negative real parts (i.e. the Jacobian is a stability matrix). Parodi (1952) has shown that if $[\alpha_{ij}]$, $[c_{ij}]$ and $[V_{ij}]$ are positive definite matrices the eigenvalues associated with eq. (33) have negative real parts. Then all prebuckling (trivial) states $\mathbf{y}=0$ are asymptotically stable; namely the motion of the dissipative system converges towards the origin (trivial state) which acts as *point attractor*.

However, the nxn matrix $[V_{ij}]$ is, in general, non-symmetric. In case of divergence instability its determinant (being an n^{th} degree algebraic polynomial in λ) vanishes for n (distinct) buckling loads if η does not coincide with a double point (compound branching). Then for λ less than the smallest buckling load λ_c (prebuckling state) we have the following inequality

$$\det [V_{ij}(0;\lambda;\eta)] > 0 \quad (\lambda < \lambda_c) \tag{36}$$

For a load λ slightly greater than λ_c it follows that

$$\det [V_{ij}(0;\lambda;\eta)] < 0 \quad (\lambda > \lambda_c) \tag{37}$$

provided that $\eta \neq \eta_o$ (i.e. there is no extremum). In view of relation (18), as stated above, the determinant of the Jacobian matrix has the sign of the determinant of matrix $[V_{ij}]$.

Let us first consider the case of a non-dissipative system ($c_{ij}=0$). If matrix $[V_{ij}]$ is strongly asymmetric such that the matrix $[\widetilde{V}_{ij}] = [\beta_{ij}]\,[V_{ij}]$ is not *symmetrizable* the system is called *circulatory*. Nevertheless, the present analysis deals with *pseudo-conservative* systems in which by definition matrix $[\widetilde{V}_{ij}]$ is *symmetrizable*; that is there exists always a positive definite matrix $\mathbf{S}$ such that $[\widetilde{V}_{ij}] = \mathbf{S}\,[\widetilde{V}_{ij}]^T\mathbf{S}^{-1}$. If $\mathbf{r}$ and $\mathbf{q}^T$ are the right and left eigenvectors corresponding to the eigenvalue ρ by virtue of eq. (32) for $[c_{ij}]=0$ one can obtain

$$\left.\begin{aligned} &(\rho^2\mathbf{I}_n + [\widetilde{V}_{ij}])\,\mathbf{r} = 0 \ \text{or} \ (\rho^2\mathbf{S}^{-1} + [\widetilde{V}_{ij}]^T\mathbf{S}^{-1})\,\mathbf{r} = 0 \\ &\mathbf{q}^T(\rho^2\mathbf{I}_n + [\widetilde{V}_{ij}]) = 0 \ \text{or} \ (\rho^2\mathbf{S}^{-1} + \mathbf{S}^{-1}[\widetilde{V}_{ij}])\,\mathbf{S}\mathbf{q} = 0 \end{aligned}\right\} \tag{38}$$

Since $[\widetilde{V}_{ij}]^T\mathbf{S}^{-1} = \mathbf{S}^{-1}[\widetilde{V}_{ij}]$ is a symmetric matrix it follows that $\mathbf{r} = \mathbf{S}\mathbf{q}$ and $\mathbf{q}^T = \mathbf{r}^T\mathbf{S}^{-1}$. Multiplication of the first of eqs (38) by $\mathbf{q}^T = \mathbf{r}^T\mathbf{S}^{-1}$ yields

$$\rho^2 = -\frac{\mathbf{r}^T[\widetilde{V}_{ij}]^T\mathbf{S}^{-1}\mathbf{r}}{\mathbf{r}^T\mathbf{S}^{-1}\mathbf{r}} \tag{39}$$

If the above quadratic forms are associated with positive definite matrices, both eigenvalues are purily imaginary. The origin (trivial state) is a *center*; namely the motion is bounded associated with closed trajectories around the center.

In case that $[c_{ij}] \neq 0$, both matrices $[\tilde{c}_{ij}]$ and $[\widetilde{V}_{ij}]$ are symmetrizable, while the former has positive eigenvalues if $[c_{ij}]$ is positive definite. In view of the fact that $\mathbf{S}$ is not uniquely determined [Gantmacher (1964)] the question which now arises is whether it is possible to find a positive definite transformation matrix $\mathbf{S}$ which renders both the above matrices symmetrizable. Unfortunately this is not always possible; fact which, however, does not imply that there is no asymptotic instability. If such an $\mathbf{S}$ matrix exists then one can write $[\widetilde{V}_{ij}] = \mathbf{S}\,[\widetilde{V}_{ij}]^T\mathbf{S}^{-1}$ and $[\tilde{c}_{ij}] = \mathbf{S}\,[\tilde{c}_{ij}]^T\mathbf{S}^{-1}$. Instead of eqs (38) we have now

$$\left.\begin{aligned}(\rho^2 \mathbf{S}^{-1} + \rho[\tilde{c}_{ij}]^T \mathbf{S}^{-1} + [\widetilde{V}_{ij}]^T \mathbf{S}^{-1})\,\mathbf{r} = 0 \\ (\rho^2 \mathbf{S}^{-1} + \rho \mathbf{S}^{-1}[\tilde{c}_{ij}] + \mathbf{S}^{-1}[\widetilde{V}_{ij}])\,\mathbf{S}\mathbf{q} = 0\end{aligned}\right\} \quad (40)$$

If $[c_{ij}]$ is positive definite matrix the symmetric matrix $[\tilde{c}_{ij}]^T\mathbf{S}^{-1}$ is also positive definite.

Premultiplication of the first of eqs (40) by $\mathbf{r}^T$ leads to an algebraic equation of 2^{nd} degree with both roots having negative real part provided that the symmetric matrix $[\widetilde{V}_{ij}]\mathbf{S}^{-1}$ is positive definite [see also Chetayev (1961)]. Note also that according to KTC (Kelvin-Tait-Chetayev) theorem the above dissipative system is asymptotically stable (unstable) if the corresponding nondissipative system ($[c_{ij}] = 0$) is stable (unstable).

In case that $\mathbf{S}$ is a positive definite matrix which renders symmetrizable only matrix $[\widetilde{V}_{ij}]$, then $[\tilde{c}_{ij}]^T\mathbf{S}^{-1}$ is not a symmetric matrix However, one can follow the above procedure by writing the last matrix as the sum of a symmetric and a skew symmetric matrix. If the symmetric matrix (associated with a positive definite $[c_{ij}]$ matrix) is positive definite the dissipative system is associated with complex conjugate eigenvalues with negative real parts; namely the origin (trivial state) is asymptotically stable. A recent work on symmetrizable nonconservative systems is given by Inman and Olsen (1988).

In the region of non-existence of adjacent equilibria there are no real values of λ for which the determinant of the non-symmetric matrix $[\widetilde{V}_{ij}]$ vanishes. For λ less than the dynamic critical load inequality (36) is valid too. The development outlined above for the case of divergence instability is also valid. Therefore, in both cases (i.e. divergence and dynamic instability) all eigenvalues of the Jacobian matrix have negative real parts (all b_i in eq. (23) are positive); namely the precritical trivial state (origin) is asymptotically stable (associated with a point attractor response).

Recall also the more tedious in use criterion for the asymptotic stability of Routh-Hurwitz which is satisfied if all Routh-Hurwitz determinants Δ_i are positive (i.e. $\Delta_i > 0$). Note that $\Delta_k = a_k \Delta_{k-1}$ [Gantmacher (1964)] and

$$\Delta_{k-1} = (-1)^{\frac{k(k-1)}{2}} \prod_{i<j}^{1,\ldots,k} (\lambda_i + \lambda_j) \qquad (k = 1,\ldots,2n) \quad (41)$$

This is Orlando's formula which is very useful for the subsequent analysis.

Critical states

In case of divergence instability the buckling equation (25) or (26), due to relation (24), yields that one (at least) of c_i's vanishes, say c_j. This implies that one root of eq. (24) is zero and the other is equal to $-b_j$ (<0). Therefore, at $\lambda=\lambda_c$ one (at least) pair of complex conjugate eigenvalues is transformed to a zero eigenvalue and to a negative eigenvalue. For λ slightly greater than λ_c the determinant of $[\widetilde{V}_{ij}]$ (and the corresponding determinant of the Jacobian matrix $[\mathbf{Y}_y]$, being a nonlinear analytic function of λ, changes sign becoming negative [see eq. (37)] provided that the case $\eta\neq\eta_o$ (extremum in the curve η vs λ_c) is excluded. Then, one of c_i's becomes negative which due to eq. (24) implies that the previous zero root becomes now positive. Therefore the trivial state (origin) becomes unstable.

The dissipative system although locally unstable might be globally stable in case of existence either of a *point attractor* (due to a stable post-buckling path) or a *limit cycle attractor*. This can be established only by employing a nonlinear analysis. Clearly, due to the fact that the Jacobian matrix is singular (i.e. $\delta\mathbf{Y}^o=0$) one has to take into account variations of higher order in eq. (13). A very efficient linearized technique for discussing the global stability (see introduction) of the critical (trivial) state which can be used is associated with the *centre manifold theory*.

Note also that a double zero eigenvalue occurs when

$$a_{2n-1}=a_{2n}=0 \tag{42}$$

Since

$$a_{2n-1}=b_1c_2c_3...c_n+c_1b_2c_3c_4...c_n+...+c_1c_2...c_{n-1}b_n \tag{43}$$

condition (42) may occur for suitable values of b_i's (i.e. of the damping coefficients c_{ij}). For instance if $b_1=c_1=0$ then $a_{2n-1}=a_{2n}=0$.

In conclusion the vanishing of one eigenvalue is associated with a *static* bifurcation. A static bifurcation is also a dynamic bifurcation since the latter implies a sudden qualitative change of the system response as the loading control parameter varies smoothly (provided that there is no other dynamic bifurcation prior to the static one).

In case of non-existence of adjacent equilibria the critical state $\lambda=\lambda_{cr}$ occurs when the Routh-Hurwitz determinant Δ_{k-1} in relation (41) vanishes, that is

$$\Delta_{k-1}=0 \qquad (k=2n) \tag{44}$$

which implies that the sum of —at least— two roots of equation $\phi(\rho)=0$ is zero. Since the determinant of the Jacobian —as shown above— is positive (i.e. $a_{2n}>0$) the cases of a double zero root or a pair of opposite roots are excluded. Hence eq. (44) implies that the equation $\phi(\rho)=0$ has one —at least— pair of pure imaginary roots. Namely, the critical behavior is associated with the vanishing of the real part of one —at least— pair of complex eigenvalues. This, due to eq. (24), implies that one of b_i's becomes zero, say $b_j=0$; hence the purily imaginary eigenvalues are $\pm i\sqrt{c_j}$ $(i=\sqrt{-1})$.

Since $-b_i$ $(i=1,...,n)$, being a function of λ, is negative for $\lambda<\lambda_{cr}$ and equal to zero for $\lambda=\lambda_{cr}$, then for $\lambda>\lambda_{cr}$ (if $db_i/d\lambda\neq 0$) it becomes positive. This means instability of the trivial state (origin). However, the dissipative system may be globally stable exhibiting a limit cycle attractor. This is a typical case of a *Hopf* (dynamic) bifurcation [Nemytskii and Stepanov (1960)].

In view of the above development one can observe the following: While the vanishing of the determinant of the Jacobian matrix (one, at least, eigenvalue becomes zero) is associated with a *static* bifurcation (being also a dynamic one), the vanishing of the Routh-Hurwitz determinant Δ_{2n-1} (one —at least— pair of eigenvalues is purily imaginary) is associated with a *dynamic* bifurcation. The stability or instability of such a dynamic critical state cannot be explored unless a nonlinear dynamic analysis is employed. Since the Jacobian determinant is singular $(\delta \mathbf{Y}^{o}=0)$ one has to take into account variations of higher order in eq. (13). Among the aforementioned local (linearization) techniques for establishing the nature of the dynamic critical state (origin) the more efficient, as stated above, is the centre manifold technique. When the compound branching (double) point (λ_c^o, η_o) is associated with a double zero eigenvalue, the divergence critical state coincides with the dynamic critical state

$$\Delta_{2n-1}=0 \tag{45}$$

since $a_{2n}=a_{2n-1}=0$ yields $\Delta_{2n-1}=a_{2n-1}\Delta_{2n-2}=0$. However, in general, $a_{2n-1}\neq 0$ which implies a *discontinuity* between the static (divergence) critical load λ_c and the dynamic (associated with limit cycles) critical load λ_{cr}.

In addition to the study of the nature of the critical state (static or dynamic) it is worth discussing the transition from point attractors (regions of adjacent equilibria) to limit cycle attractors (region of non-existence of adjacent equilibria). For instance, a reasonable question is to investigate whether for $\lambda>\lambda_c$ a limit cycle attractor may occur in some region of adjacent equilibria in the vicinity of the compound branching point (λ_c^o, η_o). Certainly this is impossible to occur in case that all postbuckling modes are independent to each other (Fig. 2a). Clearly, the dissipative system exhibits a point attractor (due to the stable postbuckling path). However, the response may be different for $\lambda > \lambda_c$ when there is only one postbuckling path passing through two consecutive branching points which correspond to the first and second buckling (divergence) loads $\lambda_c^{(1)}$ and $\lambda_c^{(2)}$, respectively (Fig. 2b). It was found numerically, that there is a certain $\lambda^* > \lambda_c^{(1)^c}$ below which the system displays a *point attractor*, while above $\lambda_c^{(1)}$ it experiences a limit cycle attractor [Kounadis $(1990)_2$].

The above phenomenon of existence of one postbuckling equilibrium path associated with the first and second buckling load occurs only in non-gradient systems either discrete [Kounadis $(1990)_2$] or continuous [Kandakis and Kounadis (1992)] for which there exists a coincident critical point (λ_c^o, η_o) determined through the solution of eqs (25) and (28).

The above phenomenon together with the nature of the critical (trivial) state cannot be explored unless a nonlinear (global) dynamic analysis is employed. These questions are discussed below by using the *center manifold technique*. Before closing this section it is worth noticing that the compound branching point (λ_c^o, η_o) although associated with a limit cycle response (while it is an equilibrium point) it is not a Hopf bifurcation (being always related to purely imaginary eigenvalues).

Center manifold technique

According to the center manifold theory eq. (11) must be put in the standard form [Carr (1981), Perko (1991)]

$$\left.\begin{aligned} \dot{x} &= \mathbf{A}\mathbf{x} + \mathbf{f}(x,y,\epsilon) \\ \dot{y} &= \mathbf{B}\mathbf{y} + \mathbf{g}(x,y,\epsilon) \qquad (x,y,\epsilon)\in\mathbb{R}^n \mathrm{x} \mathbb{R}^m \mathrm{x} \mathbb{R}^p \end{aligned}\right\} \tag{46}$$

where

$$\left.\begin{aligned} \mathbf{f}(0,0,0) &= 0, \quad D\mathbf{f}(0,0,0) = 0 \\ \mathbf{g}(0,0,0) &= 0, \quad D\mathbf{g}(0,0,0) = 0 \end{aligned}\right\} \tag{47}$$

A is an nxn matrix having eigenvalues with zero real parts, **B** is an mxm matrix having eigenvalues with negative real parts, while **f** and **g** are C^r functions $(r \geq 2)$ in some neighborhood of $(x,y,\epsilon) = (0,0,0)$ with corresponding Jacobians $D\mathbf{f}(0,0,0)$ and $D\mathbf{g}(0,0,0)$; Finally $\epsilon\in\mathbb{R}^p$ is a vector of p parameters.

An invariant manifold is called *center (local) manifold* for eqs (46) if it can be represented as follows

$$W^c(0) = \{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\epsilon}) \epsilon \mathbb{R}^n x \mathbb{R}^m x \mathbb{R}^p | \boldsymbol{y} = \mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon}), |x| < \delta_1, |x| < \delta_2, \mathbf{h}(0,0) = 0, D\mathbf{h}(0,0) = 0\} \quad (48)$$

for δ_1 and δ_2 sufficiently small.

In parametrized systems we include the parameter ε as a new dependent variable as follows

$$\left.\begin{aligned} &\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\epsilon}) \\ &\dot{\boldsymbol{y}} = \mathbf{B}\mathbf{y} + \mathbf{g}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\epsilon}) \qquad (\boldsymbol{x},\boldsymbol{y},e) \epsilon \mathbb{R}^n x \mathbb{R}^m x \mathbb{R}^p \\ &\dot{\boldsymbol{\epsilon}} = \mathbf{0} \end{aligned}\right\} \quad (49)$$

Under the above conditions there exists a C^r center manifold for eqs (46) $\boldsymbol{y} = \mathbf{h}(x,\varepsilon)$ for $\boldsymbol{x}$ and $\boldsymbol{\epsilon}$ sufficiently small. Then, the dynamics of eqs (46) restricted to the center manifold, is for u sufficiently small, governed by the following n-dimensional vector field

$$\left.\begin{aligned} &\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} + \mathbf{f}(\mathbf{u},\mathbf{h}(\mathbf{u},\boldsymbol{\epsilon}),\boldsymbol{\epsilon}) \\ &\dot{\boldsymbol{\epsilon}} = \mathbf{0} \qquad (\mathbf{u},\boldsymbol{\epsilon}) \epsilon \mathbb{R}^n x \mathbb{R}^p \end{aligned}\right\} \quad (50)$$

According to the 2nd theorem of the center manifold theory if the zero (origin) solution of eq. (50) is stable (asymptotically stable) or unstable then the zero solution of the original system (46) is stable (asymptotically stable) or unstable. Let us suppose that the zero solution is stable. Then if $(\boldsymbol{x}(t), \boldsymbol{y}(t))$ is a solution of eqs (46) for given ε with $(\boldsymbol{x}(0), \boldsymbol{y}(0))$ sufficiently small, there is a solution of eq. (49) such that as $t \to \infty$

$$\left.\begin{aligned} \boldsymbol{x}(t) &= \mathbf{u}(t) + 0(e^{-\gamma t}) \\ \boldsymbol{y}(x) &= \mathbf{u}(t) + 0(e^{-\gamma t}) \end{aligned}\right\} \quad (51)$$

where $\gamma > 0$ is a constant.

Supposing now that all the above assumptions of the center manifold theory are satisfied, the next step is to compute the center manifold.

From the existence theorem we can suppose that there exists a (local) center manifold for eq. (49) given in relation (48).

Substituting $\boldsymbol{y} = \mathbf{h}(x,\varepsilon)$ into the second of eqs (46) and taking into account that $\dot{\boldsymbol{y}} = D\mathbf{h}(x,\varepsilon)\dot{\boldsymbol{x}}$ where $\dot{x}$ is taken from the first of eqs (46), we obtain

$$D\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon})[\mathbf{A}\boldsymbol{x} + \mathbf{f}(\boldsymbol{x},\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon}),\boldsymbol{\epsilon})] = \mathbf{B}\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon}) + \mathbf{g}(\boldsymbol{x},\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon}),\boldsymbol{\epsilon})$$

or $$N(\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon})) \equiv D\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon})[\mathbf{A}\boldsymbol{x} + \mathbf{f}(\boldsymbol{x},\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon}),\boldsymbol{\epsilon})] - \mathbf{B}\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon}) - \mathbf{g}(\boldsymbol{x},\mathbf{h}(\boldsymbol{x},\boldsymbol{\epsilon}),\boldsymbol{\epsilon}] = 0 \quad (52)$$

Unfortunately it is more difficult to solve the last quasilinear partial differential equation than the original one; however, on the basis of the third theorem of the center manifold theory we can approximate the center manifold to any degree of accuracy.

According to this theorem: Let $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a C^1 mapping with $\phi(0) = D\phi(0) = 0$ such that $N(\phi(x)) = 0\ (|x|^q)$ as $x \rightarrow 0$ for some $q > 1$. Then

$$|\mathbf{h}(\boldsymbol{x}) - \phi(\boldsymbol{x})| = 0\ (|\boldsymbol{x}|^q) \text{ as } \boldsymbol{x} \rightarrow 0. \tag{53}$$

Eq. (13), obtained from the original dynamical system (11) through a Taylor's series around y^o, can be written with the aid of relations (12), (14), (15) and (16) as follows

$$\left.\begin{bmatrix} \dot{\boldsymbol{x}} \\ = \\ \dot{\boldsymbol{y}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I}_n \\ -[\widetilde{V}_{ij}] & -[\tilde{c}_{ij}] \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \\ \boldsymbol{y} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{f}}, \\ \\ \tilde{\mathbf{g}}, \end{bmatrix} \begin{matrix} x \in \mathbb{R}^n \\ \\ y \in \mathbb{R}^n \end{matrix} \right\} \tag{54}$$

or
$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbf{y} + \tilde{\mathbf{f}} \\ \dot{\boldsymbol{y}} &= -[\tilde{c}_{ij}]\mathbf{y} - [\widetilde{V}_{ij}]\mathbf{x} + \tilde{\mathbf{g}} \end{aligned} \tag{55}$$

where

$$\mathbf{x} = \begin{bmatrix} \xi_1 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} \xi_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \xi_{2n} \end{bmatrix}, \quad \tilde{\mathbf{f}} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{g}} = \begin{bmatrix} \frac{1}{2}\delta^2 Y_{n+1} + \frac{1}{6}\delta^3 Y_{n+1} \\ \cdot \\ \cdot \\ \frac{1}{2}\delta^2 Y_{2n} + \frac{1}{6}\delta^3 Y_{2n} \end{bmatrix} \tag{56}$$

By virtue of relations (56), eqs (55) can be put into the standard form of eq. (46), that is

$$\left.\begin{aligned} \dot{\boldsymbol{x}} &= 0.\boldsymbol{x} + \mathbf{f}(\boldsymbol{x},\boldsymbol{y},\varepsilon) \\ \dot{\boldsymbol{y}} &= -[\tilde{c}_{ij}]\boldsymbol{y} + \mathbf{g}(\boldsymbol{x},\boldsymbol{y},\varepsilon) \end{aligned}\right\} \tag{57}$$

where $\mathbf{f} = \boldsymbol{y} + \tilde{\mathbf{f}} = \boldsymbol{y}$ and $\mathbf{g} = -[\widetilde{V}_{ij}]\boldsymbol{x} + \tilde{\mathbf{g}}$, while $\boldsymbol{\epsilon} = (\lambda,\eta)^T$. Note also that the matrices **A** and **B** in eq. (46) are equal to 0 and $-[c_{ij}]$, respectively. Given that $[\beta_{ij}]$ is always a positive definite matrix and $[c_{ij}]$ is either a positive or non-negative definite matrix, then their product $[\beta_{ij}]\,[c_{ij}] = [\tilde{c}_{ij}]$ is a matrix with positive or non-negative eigenvalues. Therefore the matrix $-[\tilde{c}_{ij}]$ has either negative or non-positive eigenvalues.

Before employing the center manifold theorem one should transform the (singular) Jacobian matrix $Y_y(0;\lambda;\eta)$ into its canonical form. Since this matrix is associated with n zero eigenvalues and n complex ones the transformation matrix can be put into Jordan canonical form [Gantmacher (1964)]. Thereafter one should check whether the necessary requirements for employing this technique are satisfied.

4. Illustrative example

Consider Ziegler's two-degree-of-freedom nonlinear dissipative system shown in Fig. 3 for which a lot of numerical results are available. The system, being geometrically perfect, is subjected to a partial follower loading λ of constant magnitude acting at an angle $\eta\theta_2$ with respect to the axis of the upper link (autonomous system). Such a loading is tangential for $\eta=0$ and constant directional (conservative) for $\eta=1$; that is for $\eta\neq 1$ it is nonconservative [Plaut (1976)]. Lagrange equations of motion, in dimensionless form, are given by [Kounadis $(1991)_2$]

$$\left.\begin{aligned} &(1+m)\,\ddot\theta_1+\ddot\theta_2\cos(\theta_1-\theta_2)+\dot\theta_2^2\sin(\theta_1-\theta_2)+b_1\dot\theta_1-b_2\dot\theta_2+\frac{\partial V}{\partial\theta_1}=0\\ &\ddot\theta_2+\ddot\theta_1\cos(\theta_1-\theta_2)-\dot\theta_2^2\sin(\theta_1-\theta_2)+b_2\dot\theta_2-b_2\dot\theta_1+\frac{\partial V}{\partial\theta_2}=0 \end{aligned}\right\} \quad (58)$$

where

$$\left.\begin{aligned} &\frac{\partial V}{\partial\theta_1}=2\theta_1-\theta_2+\delta_1\theta_1^2+\gamma_1\theta_1^3-\delta_2(\theta_1-\theta_2)^2+\gamma_2(\theta_1-\theta_2)^3-\lambda\sin[\theta_1+(\eta-1)\,\theta_2]\\ &\frac{\partial V}{\partial\theta_2}=\theta_2-\theta_1+\delta_2(\theta_1-\theta_2)^2-\gamma_2(\theta_1-\theta_2)^3-\lambda\sin\eta\theta_2 \end{aligned}\right\} \quad (59)$$

The matrices $[\alpha_{ij}]$, $[c_{ij}]$, $[V_{ij}]$, $[\beta_{ij}]=[\alpha_{ij}]^{-1}$, $[\tilde c_{ij}]$ and $[\widetilde V_{ij}]$ evaluated at the trivial state (origin) $y^E=0\ (\theta_1=\theta_2=0)$ are given by

$$[\alpha_{ij}]=\begin{bmatrix} m+1 & 1\\ 1 & 1\end{bmatrix},\quad [c_{ij}]=\begin{bmatrix} c_{11} & c_{12}\\ c_{12} & c_{22}\end{bmatrix}=\begin{bmatrix} b_1+b_2 & -b_2\\ -b_2 & -b_2\end{bmatrix}$$

$$[V_{ij}]=\begin{bmatrix} 2-\lambda & -1-\lambda(\eta-1)\\ -1 & 1-\lambda\eta\end{bmatrix},\quad [\beta_{ij}]=\frac{1}{m}\begin{bmatrix} 1 & -1\\ -1 & 1+m\end{bmatrix}$$

$$[\tilde c_{ij}]=\frac{1}{m}\begin{bmatrix} b_1+2b_2 & -2b_2\\ -b_1-(m+2)b_2 & (m+2)b_2\end{bmatrix},\quad [\widetilde V_{ij}]=\frac{1}{m}\begin{bmatrix} 3-\lambda & \lambda-2\\ \lambda-m-3 & m+2-\lambda(1+m\eta)\end{bmatrix} \quad (60)$$

By virtue of these relations the characteristic equation of the Jacobian matrix (21) is the following

$$\rho^4+a_1\rho^3+a_2\rho^2+a_3\rho+a_4=0 \quad (61)$$

$$\left.\begin{array}{l} \text{where}\quad a_1=\frac{1}{m}\left[b_1+(m+4)b_2\right],\; a_2=\frac{1}{m}\left[b_1b_2+m+5-\lambda(2+m\eta)\right] \\ a_3=\frac{1}{m}\left[b_1(1-\lambda\eta)+b_2(1-2\lambda\eta)\right],\; a_3=\frac{1}{m}(1-3\lambda\eta+\eta\lambda^2) \end{array}\right\} \quad (62)$$

Application of relation (24) gives

$$\rho_{1,2}=-\frac{b_1}{2}\pm\left(\frac{b_1^2}{4}-c_1\right)^{1/2},\; \rho_{2,3}=-\frac{b_2}{2}\pm\left(\frac{b_2^2}{4}-c_2\right)^{1/2} \quad (63)$$

where

$$\left.\begin{array}{l} b_1+b_2=a_1 \\ c_1+c_2+b_1b_2=a_2 \\ b_1c_2+b_2c_1=a_3 \\ c_1c_2=a_4 \end{array}\right\} \quad (64)$$

The boundary between the region of existence and nonexistence of adjacent equilibria is determined by solving the system of equations

$$a_4=\frac{da_4}{d\lambda}=0 \quad (65)$$

from which we obtain

$$\eta_o=\frac{4}{9},\; \lambda_c^o=\frac{3}{2}. \quad (66)$$

Obviously for $\eta>4/9$ the dynamical system displays a divergence instability, while for $\eta<4/9$ there are no adjacent equilibria and the system exhibits a limit cycle (stable or unstable) response. The static (divergence) buckling load λ_c is given through equation $a_4=0$ which gives

$$\lambda_c=\frac{1}{2}\left(3\pm\sqrt{9-\frac{4}{\eta}}\right) \quad (67)$$

The dynamic critical load λ_{cr} is obtained through the equation

$$\Delta_3=(a_1a_2-a_3)\,a_3-a_1^2a_4=0 \quad (68)$$

where a_i $(i=1,...,4)$ are given in relation (62). Condition (68) can also be obtained by inserting into eq. (61) $\rho=\pm iv$ (v=real, $i=\sqrt{-1}$) and thereafter eliminating v.

Substituting the expressions of a_i $(i=1,...,4)$ into eq. (68) we obtain an algebraic polynomial of 2nd degree with respect to λ_{cr} [Kounadis $(1990)_2$, Kounadis and Avraam (1991)] given below [see eq. (84)]. From this equation it is clear that λ_{cr} depends on both, masses and damping, in contrast with the case of divergence instability where these parameters have no effect on the critical buckling (divergence) load λ_c.

Stability of precritical states

Let us consider the case $\eta=0.5$ (divergence instability) with the following data: $m=2$, $b_1=0.08$, $b_2=0.01$ and $\lambda=0.99<\lambda_c=1$. It is not difficult to show that both matrices $[\widetilde{V}_{ij}]$ and $[\tilde{c}_{ij}]$ are symmetrizable associated with the same positive definite matrix $\mathbf{S}$ which renders $[\widetilde{V}_{ij}]\mathbf{S}$ and $[\tilde{c}_{ij}]\mathbf{S}$ symmetric, that is

$$[\widetilde{V}_{ij}]\mathbf{S}=\frac{1}{2}\begin{bmatrix} 2.01 & -1.01 \\ -4.01 & 2.02 \end{bmatrix}\begin{bmatrix} 0.06 & -0.04 \\ -0.04 & 0.238 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 0.161 & -0.3208 \\ -0.3214 & 0.6412 \end{bmatrix}$$

$$[\tilde{c}_{ij}]\mathbf{S}=\frac{1}{2}\begin{bmatrix} 0.10 & -0.02 \\ -0.12 & 0.04 \end{bmatrix}\begin{bmatrix} 0.06 & -0.04 \\ -0.04 & 0.238 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 0.0068 & -0.0088 \\ -0.0088 & 0.0143 \end{bmatrix} \tag{69}$$

Since both the above symmetric matrices are positive definite the system is associated with two pairs of complex conjugate eigenvalues with negative real parts. Hence the prebuckling (trivial) state is asymptotically stable. The same result could also be obtained by using the Routh-Hurwitz stability criterion. According to this criterion this state is asymptotically stable if all $\Delta_i>0$ or equivalenty if $a_i>0$ (for $i=1,\ldots,4$) and also $\Delta_3>0$. Indeed one can readily find that $a_1=0.07$, $a_2=2.0154$, $a_3=0.02025$, $a_4=0.02525$ and $\Delta_3=2.43\times10^{-3}$.

Let $\mu_i\pm\nu_i\sqrt{-1}$ $(i=1,2)$ be the complex conjugate eigenvalues for $\lambda<\lambda_{cr}$, where $\mu_i=-0.5b_i$ and $\nu_i=[c_i-(0.5b_i)^2]^{1/2}$. Application of Orlando's formula (41) gives

$$\Delta_3=4\mu_1\mu_2[(\mu_1+\mu_2)^2+(\nu_1+\nu_2)^2]\,[(\mu_1+\mu_2)^2+(\nu_1-\nu_2)^2] \tag{70}$$

which for $\Delta_3>0$ yields $\mu_1\mu_2>0$. Since $2(\mu_1+\mu_2)=-b_1-b_2=-a_1<0$ it is clear that both μ_1 and μ_2 are negative and thus all eigenvalues have negative real parts; namely in such a case the asymptotic stability is assured if only $\Delta_3>0$ and $a_1>0$.

Finally, solving the system of eqs (64) for the case $m=2$, $b_1=0.08$, $b_2=0.01$, $\eta=0$ and $\lambda=2<\lambda_{cr}=2.07976$ (implying $\Delta_3=2.5\times10^{-4}>0$, $a_1=0.07>0$) we find due to eqs (63) the following eigenvalues

$$\left.\begin{aligned} \rho_{1,2}&=-\frac{0.05}{2}\pm\left[0.9988-\left(\frac{0.05}{2}\right)^2\right]^{1/2}i=-0.025\pm0.9991i \\ \rho_{2,3}&=-\frac{0.02}{2}\pm\left[0.5006-\left(\frac{0.02}{2}\right)^2\right]^{1/2}i=-0.01\pm0.7075i \end{aligned}\right\}\,,\ i=\sqrt{-1} \tag{71}$$

Critical and postcritical states

For the problem under consideration eqs (56) become

$$x=\begin{bmatrix}\xi_1\\ \\ \xi_2\end{bmatrix},\ y=\begin{bmatrix}\xi_3\\ \\ \xi_4\end{bmatrix},\ \tilde{\mathbf{f}}=\begin{bmatrix}0\\ \\ 0\end{bmatrix},\ \tilde{\mathbf{g}}=\begin{bmatrix}\frac{1}{2}\delta^2 Y_3+\frac{1}{6}\delta^3 Y_3\\ \\ \frac{1}{2}\delta^2 Y_4+\frac{1}{6}\delta^3 Y_4\end{bmatrix} \tag{72}$$

where after a cumbersome manipulation we find

$$\frac{1}{2}\delta^2 Y_3=\frac{1}{m}\left[(2\delta_2-\delta_1)\,\xi_1^2+2\delta_2\xi_2^2-4\delta_2\xi_1\xi_2\right]$$

$$\frac{1}{2}\delta^2 Y_4=\frac{1}{m}\left\{\left[\delta_1-(m+2)\delta_2\right]\xi_1^2-(m+2)\delta_2\xi_2^2+2\,(m+2)\,\delta_2\xi_1\xi_2\right\}$$

$$\begin{aligned}\frac{1}{6}\delta^3 Y_3=&-\frac{b_2\,(m+4)}{2m^2}(\xi_1-\xi_2)^2\xi_4+\frac{\left[2b_1+b_2\,(m+4)\right]}{2m^2}(\xi_1-\xi_2)^2\xi_3-\frac{(\xi_1-\xi_2)}{m}\xi_4^2-\\ &-\frac{(\xi_1-\xi_2)}{m}\xi_3^2-\frac{1}{m}\left[\gamma_1\xi_1^3+2\gamma_2\,(\xi_1-\xi_2)^3\right]+\frac{\lambda}{6m}\{\left[\eta^3-(\eta-1)^3\right]\xi_2^3-\xi_1^3-\\ &-3\,(\eta-1)\,\xi_1^2\xi_2-3\,(\eta-1)^2\xi_2^2\xi_1\}+\frac{(\xi_1-\xi_2)^2}{m^2}\left\{\left(\frac{m+6}{2}-\lambda\right)\xi_1-\left[\frac{m+4}{2}-\lambda\left(1+\frac{m\eta}{2}\right)\right]\xi_2\right\}\end{aligned}$$

$$\begin{aligned}\frac{1}{6}\delta^3 Y_4=&\frac{b_2\,(4+3m)}{2m^2}(\xi_1-\xi_2)^2\xi_4-\frac{(m+2)\,b_1+(4+3m)\,b_2}{2m^2}(\xi_1-\xi_2)^2\xi_3+\frac{(\xi_1-\xi_2)}{m}\xi_4^2\\ &+\frac{(m+1)}{m}(\xi_1-\xi_2)\xi_3^2+\frac{1}{m}\left[\gamma_1\xi_1^3+(m+2)\,\gamma_2\,(\xi_1-\xi_2)^3\right]+\frac{\lambda}{6m}\{\left[\xi_1+(\eta-1)\,\xi_2\right]^3-\\ &-(m+1)\,\eta^3\xi_2^3\}+\frac{(\xi_1-\xi_2)^2}{2m^2}\{\left[(m+2)\,\lambda-4m-6\right]\xi_1-\left[(m\eta+m+2)\,\lambda-3m-4\right]\xi_2\}\end{aligned} \tag{73}$$

Clearly, $\boldsymbol{\epsilon}=(\lambda,\eta,\delta_1,\delta_2,\gamma_1,\gamma_2,b_1,b_2,m)^T$.

Writing eqs (73) for the simplified case with $m=2$, $\delta_1=\delta_2=\gamma_1=\gamma_2=0$ (corresponding to a Hookean material) we find eqs (4a) given by Jin and Matzuzaki (1988). Application of the center manifold technique in the neighborhood of the double critical point ($\eta_o=4/9$, $\lambda_c=3/2$) on the side of divergence instability is comprehensively presented by the aforementioned investigators. However, their analysis refers exclusively to the case of a double zero eigenvalue occurring for $a_3=a_4=0$ [see eqs (42) and (62)]; namely for suitable values of the damping coefficients b_1 and b_2. For this case ($a_3=a_4=0$) they have found several distinct dynamic bifurcations (associated with stable limit cycles) without determining the region of divergence instability where these phenomena may occur. This region can be determined as follows:

Equation $a_3=0$ by virtue of relation (62) yields

$$b=\frac{b_1}{b_2}=\frac{2\lambda_c\eta-1}{1-\lambda_c\eta}=\frac{2(\lambda_c-\frac{1}{2\eta})}{\frac{1}{\eta}-\lambda_c} \tag{74}$$

For $b>0$ the last equation leads to

$$(\lambda_c-\frac{1}{2\eta})\,(\lambda_c-\frac{1}{\eta})<0$$

from which we get

$$\frac{1}{2\eta}<\lambda_c<\frac{1}{\eta} \tag{75}$$

The condition that both roots of equation $a_4=0$ must satisfy inequality (75) yields

$$\eta<\frac{1}{2} \tag{76}$$

Eq. (74) and $a_3=0$ give [Kounadis (1990)$_2$]

$$\lambda_c=\frac{1+2b}{1+b},\quad \eta=\frac{(b+1)^2}{(b+2)\,(2b+1)},\quad (b=\frac{b_1}{b_2}) \tag{77}$$

where λ_c concides with the 1^{st} buckling load of eq. (67) if $b<1$ and with the 2^{nd} buckling load if $b>1$. Hence for the damping ratio given in relation (74) we have critical states (associated with dynamic bifurcations) with a double zero eigenvalue in the small region $4/9<\eta<0.5$ (Fig. 4). It was established by the author [Kounadis (1989)$_4$] that in this region the nonconservative system exhibits one postbuckling equilibrium path passing through the 1^{st} and 2^{nd} branching point (Figs 2b and 5). Note aslo that in case of equal damping coefficients ($b=b_1/b_2=1$) the compound branching point ($\eta_o=4/9$, $\lambda_c=3/2$) is associated with a double zero eigenvalue (because $a_3=a_4=0$). It is worth noticing that although the compound branching point is a *static equilibrium* point it is associated with a *dynamic bifurcation* related to *stable limit cycles* (Fig. 6).

Another important phenomenon occurring in this region ($4/9<\eta<0.5$) is the stability of the postcritical trivial state for a load slightly above the 2^{nd} buckling load. Indeed for $\eta=0.45$, $\lambda=1.7$ ($>\lambda_c=1.666...$), $b_1=0.01$ and $b_2\rightarrow 0$ the trivial state acts as point attractor ($a_1=0.005>0$, $\Delta_3=1.083>0$) after the decay of persistent chaoticlike transients (Fig. 7). Clearly for $\lambda=\lambda_c=1.666...$ and $b=2$ the system exhibits a dynamic bifurcation associated with a double zero eigenvalue. Then for $\lambda>1.666$ we have a stable limit cycle response (Fig. 8).

Another important case is to discuss whether a Hopf bifurcation may occur in the region of existence of adjacent equilibria. Les us consider the case with: $\eta=0.45$, $\lambda=2>\lambda_c^{(2)}=1.6666$, $b_1=0.10$, $b_2\simeq 0$ and $m=2$. Then $\Delta_3=0$ which implies that two eigenvalues are complex conjugate and the other two are purily imaginary. In this case we find: $a_1=0.05$, $a_2=3.5-1.45\lambda$, $a_3=0.05-0.0225\lambda$ and $a_4=0.225\lambda^2-0.675\lambda+0.5$. For $\lambda=1.99$ we get $\Delta_3>0$ with both eigenvalues complex conjugate ($\rho_{1,2}=-0.02424\pm0.72285i$, $\rho_{3,4}=-0.00076\pm0.30220i$), while $\lambda=2.08$ yields $\Delta_3<0$ implying one pair of complex conjugate eigenvalues with negative real part ($\rho_{1,2}=-0.11943\pm0.52320i$) and the other pair with positive real part ($\rho_{3,4}=0.09443\pm0.48187i$). The phase-plane portrait of this dynamic (Hopf) bifurcation in the region of divergence instability is shown in Fig. 9a,b. Note that for $\lambda<2$ the system exhibits a *point attractor*, whereas for $\lambda\geq2$ a *stable limit cycle* response (Fig. 10).

If we choose again $b_1=0.10$ and $b_2\simeq0$ then $\Delta_3=0$ by virtue of eqs (62) gives

$$\Delta_3=\frac{b_1^2}{8}\left[\eta\lambda^2-2(\eta+1)\lambda+4\right]=0 \tag{78}$$

For $\eta=1$ (conservative loading) the last equation yields $\Delta_3>0$ regardless of the value of λ (excluding the case $\lambda=2$ for which $\Delta_3=0$); namely the case of a limit cycle response is impossible. For $\eta\neq1$ eq. (78) vanishes for

$$\lambda_1=2,\ \lambda_2=\frac{2}{\eta} \tag{79}$$

Since at the same time a_4 must be positive, it follows that

$$\frac{b_1}{2}(1-\lambda\eta)>0$$

or

$$\lambda<\frac{1}{\eta} \tag{80}$$

From relation (79) and (80) we obtain again $\eta<1/2$ [see eq. (76)]. Namely, in the small region $4/9<\eta<0.5$ regardless of the value of η the system may exhibit (for certain damping coefficients) a Hopf bifurcation at $\lambda=2$.

There are also other dynamic bifurcations occurring in the same region which, however, cannot be established through the above qualitative analysis. A typical dynamic bifurcation with trajectories passing through the saddle point (trivial state) of Peixoto type is shown in Fig. 11. Note that the associated eigenvalues do not have any characteristic property indicating the limit cycle attractor response. This is a global dynamic bifurcation which cannot be established unless a nonlinear dynamic analysis is employed. An efficient analytic approximate technique for solving eqs (57) and (73) is developed by the author [Kounadis $(1989)_2$, $(1992)_1$].

All the above findings refer to a Hookean material ($\delta_1=\delta_2=\gamma_1=\gamma_2=0$). The author has shown that all critical (divergence) states corresponding to eq. (67) are stable (contrary to the classical analyses) since the following stability condition is always satisfied [Kounadis $(1992)_2$]

$$\lambda_c\left[\left(1+\frac{\eta-1}{1-\eta\lambda_c}\right)^3+\frac{\eta^3(2-\lambda_c)}{(1-\eta\lambda_c)^3}\right]>0 \tag{81}$$

For a nonlinear elastic material with $\delta_1=\delta_2=0$ and $\gamma_1=\gamma_2=\gamma$ the author [Kounadis $(1989)_4$] has derived the following stability condition

$$\gamma f_1+f_2>0 \tag{82}$$

where f_2 is equal to the L.H.S. of inequality (81) and f_1 is given by

$$f_1=6\lambda_c+18(1-\lambda_c)\left[\frac{1}{1-\eta\lambda_c}-\frac{1}{(1-\eta\lambda_c)^2}+\frac{1}{3(1-\eta\lambda_c)^3}\right] \tag{83}$$

Clearly, if inequality (82) is not satisfied (due to γ) the trivial state (origin) is unstable.

From the above development it is clear that a dynamic bifurcation does not imply a static bifurcation. On the contrary the static bifurcation (associated with a zero Jacobian eigenvalue) is also a dynamic bifurcation both occurring at the same loading which is the static buckling (divergence) load. However, such a coincidence exists only in case of bifurcational systems provided that dynamic bifurcation does not occur prior to static one. In case of *limit point* pseudo-conservative systems the author has shown that dynamic buckling (associated with a global dynamic bifurcation) occurs always for a load smaller than the limit point load (being always an upper bound of the dynamic buckling load). Moreover, for non-bifurcational (limit point) undamped systems dynamic buckling occurs via a regular point lying in the vicinity of the unstable postbuckling equilibrium path. Numerical simulation has shown that dynamic buckling occurs when at least one generalized coordinate of the system satisfies the *inflection* point (sufficient) criterion [Kounadis $(1991)_1$]. Therefore, for such systems we have derived the important finding that the nonlinear dynamic analysis yields a critical (buckling) load always less than that derived via a static postbuckling analysis. Thus, the Ziegler's (1968) kinetic criterion for establishing critical loads of nonconservative systems is valid only in case the latter are associated with *trivial fundamental paths* (bifurcational systems) provided that dynamic bifurcation does not exist prior to static one.

The critical condition in the region of non-existence of adjacent equilibria is given by eq. (68) which for $m=2$ yields the following equation for determining the dynamic critical load $\lambda=\lambda_{cr}$

$$A\lambda^2+B\lambda+C=0 \tag{84}$$

where

$$\left.\begin{aligned} A &= \eta b_2^2\,[b^2+4b\,(1+2\eta)+4(4\eta-3)],\ (b=b_1/b_2) \\ B &= b_2^2\,[4\,(5\eta-3)-2b^2\,(1+\eta)-2b\,(7+11\eta)-\eta b b_2^2\,(b^2+8b+12)] \\ C &= b_2^2\,[4+33b+4b^2+bb_2^2\,(b^2+7b+6)] \end{aligned}\right\} \quad (85)$$

Since we are seeking for positive λ_{cr} the discriminant of eq. (84) must be non-negative, that is

$$B^2-4AC\geq 0 \quad (86)$$

which furnishes a relationship between η, b_1 and b_2. A thorough and detailed discussion of eqs (84) and (85) in connection with various physical situations which may occur is given by Kounadis and Avraam (1991). Thus, the subsequent development will be restricted mainly either to new findings or to clarifications of previous ones obtained by the author and his associates.

Let us consider the case $b_1\to 0$ (i.e. $b\to 0$) then eq. (84) yields

$$\eta\,(4\eta-3)\lambda^2+(5\eta-3)\lambda+1=0 \quad (87)$$

from which we get

$$\left.\begin{aligned} \lambda_1 &= \frac{1}{3-4\eta} && \text{with } \eta<3/4 \\ \lambda_2 &= -\frac{1}{\eta} && \text{with } \eta<0 \end{aligned}\right\} \quad (88)$$

Since a_3 must be positive then

$$1-2\lambda\eta>0 \quad (89)$$

This inequality is consistent with the first of eqs (88) if $\eta<1/2$ (region of divergence instability) and with the second of eqs (88) when $\eta<0$ (region of no adjacent equilibria defined *by* $-0.305\leq\eta<0$). Regarding the first case one can obtain the following important finding: For $\eta=0.48$ we get $\lambda_{cr}=0.925925...<\lambda_c=1.09175$. This yields $a_4>0$, while $\lambda<\lambda_{cr}$ implies $\Delta_3>0$, and $\lambda>\lambda_{cr}$ implies $\Delta_3<0$. Namely, in the small region of divergence instability $(4/9<\eta<0.5)$ a Hopf bifurcation (associated with stable limit cycles) occurs for a load $\lambda_{cr}=0.925925...$ less than the divergence buckling load $\lambda_c=1.09175$ (Fig. 12a,b). Note also that for constant $\eta\epsilon[4/9,0.5]$ there appear different types of bifurcations as the loading λ increases from zero. For instance, at $\eta=0.48$ the system exhibits a point attractor for $\lambda<\lambda_{cr}=0.925925...$, a Hopf bifurcation for $\lambda\geq\lambda_{cr}$, a static bifurcation (one eigenvalue zero) for $\lambda>\lambda_c^{(1)}=1.09175$, a stable dynamic global bifurcation of Peixoto type (without any characteristic property in the eigenvalues) for $\lambda < \lambda_c^{(2)} = 1.90825$, a stable static bifurcation for $\lambda = \lambda_c^{(2)}$, a stable dynamic bifurcation (with a double zero eigenvalue) for $\lambda > \lambda_c^{(2)}$ and a stable Hopf bifurcation for $\lambda = 2$.

For $\eta_o = 4/9$, $b = 1$ and $b_1 = b_2 \rightarrow 0$ eq. (84) gives $\lambda_{cr} = 1.5$; namely there is no *discontinuity* in the critical load at the compound branching point contrary to the classical (linear) analysis. The phase-portrait of this point ($\eta_o = 4/9$, $\lambda_c^o = 1.5$) for $b_1 = 0.01$, $b_2 = 0.05$ is shown in Fig. 13a,b. For the case of tangential load ($\eta = 0$) eq. (84) yields [Herrmann and Bungay (1964)]

$$\lambda_{cr} = \frac{4b^2 + 33b + 4}{2\,(b^2 + 7b + 6)} + \frac{1}{2}\,b_1 b_2 \tag{90}$$

which for $b_1 = b_2 = 0.10$ and $b_1 = b_2 \rightarrow 0$ (vanishing damping) gives $\lambda_{cr} = 1.469286$ and $\lambda_{cr} = 1.464286$, respectively. Note that the linear analysis gives $\lambda_{cr} = 0.5(7 - \sqrt{8}) = 2.085786$, that is much higher load than the "exact" latter one. Another significant discrepancy between the classical (linear) and the present nonlinear dynamic analysis is that the critical trivial state according to the first analysis is unstable (associated with an unbounded motion), while according to the latter global analysis this state is stable (see Fig. 14a,b). Clearly, this is due to the omission of geometric nonlinearities which affect decisively the global (long term) response of the system.

The minimum and maximum dynamic critical loads for the case $\eta = 0$ are 1/3 and 2.08578 respectively, associated both with a stable Hopf bifurcation [Kounadis (1992)$_2$]. While the damping ratio $b = b_1/b_2$ may have a considerable effect on the dynamic critical load, in case of $b = 1$ the effect of damping on this load is very small [Kounadis (1990)$_2$]. Note also that the case of existence of two pairs of purily imaginary eigenvalues is not possible to occur because due to eqs (70), (63) and (68) we have $a_3 = a_4 = 0$ (i.e. a double zero eigenvalue).

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ΠΕΡΙΛΗΨΗ

Φαινόμενα χαοτικῆς καὶ ἄλλης φύσεως κατὰ τὴν μὴ γραμμικὴ ἀνάλυση τῶν κατασκευῶν: Ποσοτικὰ - ποιοτικὰ κριτήρια.

Ὁ διακεκριμένος Ὁλλανδὸς ἐρευνητὴς καὶ Ἀκαδημαϊκὸς Koiter μὲ τὴν δημοσιευθεῖσα τὸ 1945 στὰ Ὁλλανδικὰ διδακτορική του πραγματεία περὶ τῆς *ἀρχικῆς μεταλυγισμικῆς συμπεριφορᾶς* τῶν κατασκευῶν (ποὺ ἔγινε ὅμως γνωστὴ μόλις τὸ 1960, μεταφρασθεῖσα τότε σὲ διάφορες ἄλλες γλῶσσες) ἔφερε στὸ φῶς διάφορα φαινόμενα *ἐλαστικῆς ἀστάθειας* (ἀκαριαῖος λυγισμός, εὐαισθησία σὲ ἀρχικὲς ἀτέλειες, πολλαπλὰ διακλαδικὰ σημεῖα κλπ.), τὰ ὁποῖα μόνο μὲ ἐφαρμογὴ *μὴ γραμμικῆς ἀναλύσεως* μποροῦσαν νὰ διαπιστωθοῦν καὶ μελετηθοῦν. Κατὰ τὴ δεκαετία τοῦ 1960 ἄρχισε νὰ γίνεται κατανοητὸ ὅτι ἡ κλασσικὴ (γραμμικὴ) ἀνάλυση τῶν φορέων δὲν εἶναι μόνο *ἀνεπαρκὴς* γιὰ τὴν ὀρθὴ ἐκτίμηση τῆς *φέρουσας ἱκανότητάς* τους (ὁδηγούσα σὲ σπατάλη ὑλικοῦ) ἀλλὰ *ἐνίοτε καὶ ἐπικίνδυνη*. Τὸ τελευταῖο τοῦτο μπορεῖ νὰ συμβεῖ π.χ. στὴν περίπτωση κατασκευῶν εὐαισθήτων σὲ ἀτέλειες, ὅπου ἡ πραγματικὴ τιμὴ τοῦ *κρισίμου φορτίου λυγισμοῦ* δύναται νὰ εἶναι ἡ μισὴ ἢ καὶ μικρότερη ἀκόμη ἐκείνης ποὺ μᾶς δίδει ἡ *κλασσικὴ (γραμμικὴ) ἀνάλυση*.

Ἡ ὑπάρχουσα ἐκ λόγων οἰκονομίας τάση γιὰ *λεπτότερες καὶ ἐλαφρότερες* κατασκευὲς μὲ τὴν *μεγαλύτερη* δυνατὴ *φέρουσα ἱκανότητα* εἶχε ὡς φυσικὸ ἐπακόλουθο τὴν ἐμφάνιση ἀκόμη πιὸ ἔντονων *φαινομένων ἀστάθειας*. Ὡς ἐκ τούτου ἡ ἐφαρμογὴ *μὴ γραμμικῆς ἀναλύσεως* τῶν κατασκευῶν καθίστατο ἐπιτακτική. Αὐτὴ τὴν πλέον πολύπλοκη ἀνάλυση διευκόλυνε ἡ ἐν τῷ μεταξὺ ἀνάπτυξη καὶ διάδοση τῶν ἠλεκτρονικῶν ὑπολογιστῶν σὲ συνδυασμὸ μὲ τὴν ἐπινόηση νέων ὑπολογιστικῶν τεχνικῶν (ὅπως π.χ. εἶναι οἱ μέθοδοι *πεπερασμένων διαφορῶν, πεπερασμένων καὶ συνοριακῶν στοιχείων, οἱ τεχνικὲς διαταραχῆς, ἡ ἀσυμπτωτικὴ ἀνάλυση κλπ.)*.

Ἀλλὰ ἐὰν ἡ *μὴ γραμμικὴ ἀνάλυση* τῶν κατασκευῶν λόγω *στατικῆς φορτίσεως* ἔφερε σὲ φῶς φαινόμενα ἐλαστικῆς ἀστάθειας, ἡ ἐφαρμογὴ αὐτῆς τῆς ἀναλύσεως γιὰ κατασκευὲς ὑποκείμενες σὲ *δυναμικὴ φόρτιση* ἀπεκάλυψε ἕνα κόσμο νέων φαινομένων ποὺ προκάλεσαν ἀληθινὴ ἐπανάσταση στὴ θεωρία τῶν *δυναμικῶν συστημάτων* καὶ *διαφορικῶν ἐξισώσεων* (ποὺ συνδέονται μὲ προβλήματα *ἀρχικῶν τιμῶν*). Σημαντικὸ ρόλο πρὸς τὴν κατεύθυνση αὐτὴ ἔπαιξε ἡ ἀλληλεπίδραση τῆς *γεωμετρικῆς μὴ γραμμικότητος* καὶ τῆς *ἀποσβέσεως*. Σήμερα ἔχει καταστεῖ σαφὲς ὅτι ἡ ἀκριβὴς προσομοίωση ὁποιασδήποτε κατασκευῆς προϋποθέτει ὅτι καὶ οἱ δύο αὐτὲς παράμετροι πρέπει νὰ ληφθοῦν ὑπ' ὄψη στὴ σχετικὴ ἀνάλυση.

Κατὰ τὴν τελευταία δεκαετία παριστάμεθα μάρτυρες μιᾶς ἐντυπωσιακῆς ἀνθίσεως τῆς *μὴ γραμμικῆς δυναμικῆς*, ἡ ὁποία κατέστη δυνατὴ ἀφ' ἑνὸς μὲν χάρις στὰ μεγάλα θεωρητικὰ βήματα τῆς *ποιοτικῆς τοπολογικῆς μεθόδου* τοῦ Poincaré καὶ ἀφ' ἑτέρου χάρις στὴν σημαντικὴ ἐξέλιξη τῶν *ἀναλογικῶν* καὶ ἀκολούθως *ψηφιακῶν* ὑπολογιστῶν. Μὴ γραμμικὰ φαινόμενα, ὅπως χαοτικὰ φαινόμενα (ὀφειλόμενα σὲ παράξενους ἕλκτες, σὲ ἁπλοὺς ἢ πολλαπλοὺς ἕλκτες, φαινόμενα μετευστάθειας καὶ ἀσυνέχειας κρισίμων φορτίων, φαινόμενα εὐαισθησίας σὲ ἀρχικὲς ἀτέλειες ἢ ἀπόσβεση) διαπιστώνονται σὲ διάφορους κλάδους τῶν ἐφηρμοσμένων ἐπιστημῶν (ὅπως π.χ. στὴ Φυσική, Μετεωρολογία, Ἀστρονομία, Χημεία, Ἠλεκτρομαγνητισμό, Βιολογία, Οἰκολογία, Οἰκονομία κλπ.) καὶ ἀργότερα σὲ πολὺ μικρότερη κλίμακα στὴ Μηχανική. Στὸ σημεῖο αὐτὸ γεννᾶται τὸ εὔλογο ἐρώτημα κατὰ πόσο φαινόμενα χαοτικῆς φύσεως ἢ φύσεως ἀνάλογης μὲ τὰ ἀνωτέρω ἐμφανίζονται στὴν περίπτωση *μὴ γραμμικῆς ἀναλύσεως* τῶν κατασκευῶν λόγω δυναμικῆς φορτίσεως.

Ἡ ἀπάντηση ἐδόθη ἔμμεσα κατὰ τὴν πορεία μιᾶς συστηματικῆς καὶ ἔντονης ἐρευνητικῆς προσπάθειας τεσσάρων ἐτῶν τοῦ συγγραφέως τῆς παρουσιαζόμενης μελέτης μὲ μαθητὲς ἢ συνεργάτες του, ἡ ὁποία εἶχε ὡς στόχο τὴν εὕρεση ἀποτελεσματικῶν μεθόδων καὶ ποιοτικῶν-ποσοτικῶν κριτηρίων γιὰ τὴν μὴ γραμμικὴ δυναμικὴ ἀνάλυση τῶν κατασκευῶν. Ἀπὸ μιὰ σειρὰ τριάντα δύο (32) δημοσιεύσεων (βλ. βιβλ. ἀναφ. 1-32) ἀπὸ τὶς ὁποῖες εἴκοσι δύο (22) σὲ ἔγκριτα περιοδικὰ διεθνοῦς κυκλοφορίας καὶ δέκα (10) ἀνακοινώσεις σὲ διεθνὴ συνέδρια (κατόπιν προσκλήσεως) στὴν Εὐρώπη, Η.Π.Α. καὶ Καναδὰ ἦλθαν στὸ φῶς γιὰ πρώτη φορὰ ἐνδιαφέροντα καὶ ἐντυπωσιακὰ χαοτικῆς μορφῆς καὶ ἄλλα φαινόμενα κατὰ τὴν δυναμικὴ ἀνάλυση ἁπλῶν κατασκευῶν τοῦ Πολιτικοῦ Μηχανικοῦ. Συγχρόνως ἀνετράπησαν εὑρήματα διακεκριμένων ἐρευνητῶν (ὅπως π.χ. G. Herrmann, Nemat-Nasser, H. Leipholz, R. Plaut) ποὺ εἶχαν ἐξαχθεῖ βάσει τῆς κλασσικῆς (γραμμικῆς) δυναμικῆς ἀναλύσεως καὶ τὰ ὁποῖα ἀκόμη καὶ σήμερα ἐμφανίζονται σὲ διεθνὴ συγγράμματα. Ἐπίσης σήμερα κατέστη δυνατὴ ἡ ἑρμηνεία καὶ ἡ ποιοτικὴ καὶ ποσοτικὴ ἀνάλυση διαφόρων φαινομένων δυναμικῆς συμπεριφορᾶς τῶν κατασκευῶν, ὅπως π.χ. τῆς καταρρεύσεως τὸ 1940 τῆς κρεμαστῆς γέφυρας Tacoma στὴ Ν. Ὑόρκη, ἀνοίγματος 854 m, ποὺ προῆλθε ἀπὸ *πτερυγισμό* (ἀσταθεῖς ὁριακοὶ κύκλοι).

Ἕνα σημαντικὸ πρόβλημα τῆς μὴ γραμμικῆς δυναμικῆς ἀναλύσεως τῶν κατασκευῶν εἶναι ἡ δυσχέρεια ἐπιλύσεως τῶν ἐντόνως μὴ γραμμικῶν διαφορικῶν ἐξισώσεων κινήσεως σὲ συνδυασμὸ μὲ τὴν συχνὰ ζητούμενη λύση σὲ μεγάλα χρονικὰ διαστήματα (πράγμα ποὺ ἐπιφέρει συσσώρευση σφάλματος σὲ περίπτωση ἐφαρμογῆς ἀριθμητικῶν μεθόδων ἢ ἐπαναληπτικῶν τεχνικῶν). Παρὰ τὴν ὕπαρξη σήμερα ταχύτατων ὑπολογιστῶν καὶ πολὺ ἀποτελεσματικῶν ἀριθμητικῶν σχημάτων, ἡ χρησιμοποίηση καὶ ἄλλων τεχνικῶν (π.χ.) προσεγγιστικῶν ἀναλυτικῶν μεθόδων ἢ κριτηρίων ποιοτικῆς ἢ ποσοτικῆς φύσεως, κρίνεται ἀπολύτως ἐπιβεβλημένη ἀκόμη καὶ γιὰ μὴ γραμμικὰ δυναμικὰ συστήματα τριῶν ἢ τεσσάρων βαθμῶν ἐλευθερίας κινήσεως.

Σημαντικὰ βήματα πρὸς τὴν κατεύθυνση αὐτὴ ἔχουμε τελευταῖα χάρις στὴν πρόοδο τῆς *ποιοτικῆς τοπολογικῆς προσεγγίσεως* μὲ τὴν ὁποία ἐπιτυγχάνεται ἡ σημαντικὴ *μείωση τῆς διαστάσεως* τῶν δυναμικῶν συστημάτων (δηλαδὴ τοῦ ἀριθμοῦ τῶν διαφορικῶν ἐξισώσεων) ἀλλὰ καὶ ἡ *μείωση τοῦ βαθμοῦ μὴ γραμμικότητος*, χωρὶς νὰ διαφεύγει ἡ ποιοτικὴ συμπεριφορὰ τοῦ δυναμικοῦ συστήματος. Αὐστηρὲς μαθηματικὲς τεχνικὲς ἐπάνω σ' αὐτὴ τὴ θεώρηση εἶναι, ἐκτὸς τῆς παλαιᾶς *τεχνικῆς τῶν Lyapunov - Schmidt*, ἡ *θεωρία τοῦ κεντρικοῦ πολλαπλοῦ* (center manifold theory), ἡ μέθοδος τῶν *κανονικῶν μορφῶν* (normal forms) καὶ τὸ *λῆμμα διαχωρισμοῦ* (splitting lemma) στὴ θεωρία τῶν καταστροφῶν.

Σημαντικὴ συμβολὴ πρὸς τὴν κατεύθυνση αὐτὴ γιὰ δυναμικὰ δομικὰ συστήματα μὲ *περισσότερες τῶν δύο ἐλευθεριῶν κινήσεως* ὑπὸ αἰφνιδίως ἐπιβαλλόμενη φόρτιση, διαπιστώνει κανεὶς στὶς προαναφερθεῖσες ἐρευνητικὲς ἐργασίες τοῦ συγγραφέως. Συγκεκριμένα στὶς ὑπ' ἀριθμ. [1, 8, 26] ἐργασίες ἀναπτύσσεται μιὰ ἀποτελεσματικὴ τεχνικὴ ἐπιλύσεως μὴ γραμμικῶν προβλημάτων συνοριακῶν καὶ ἀρχικῶν τιμῶν. Στὶς ὑπ' ἀριθμ. [2, 3, 4, 5, 6, 12, 13, 18, 21, 22, 30, 32] ἐργασίες, ὅπου δίδεται μὲ ποιοτικὴ ἀνάλυση ἡ ἐξήγηση τοῦ μηχανισμοῦ *δυναμικοῦ λυγισμοῦ* μὲ βάση καὶ τὸ θεώρημα τοῦ κεντρικοῦ πολλαπλοῦ (εὐσταθὲς καὶ ἀσταθὲς πολλαπλὸ σημεῖο σέλας), ἀποδεικνύεται ὅτι κατασκευὲς ποὺ εἶναι *στατικὰ εὐσταθεῖς* μπορεῖ κάτω ἀπὸ ὁρισμένες συνθῆκες νὰ καταστοῦν *δυναμικὰ ἀσταθεῖς* (τόσο στὴν περίπτωση συντηρητικῆς, ὅσο καὶ μὴ συντηρητικῆς φορτίσεως). Συμπερασματικῶς ὁ μὴ γραμμικὸς δυναμικὸς λυγισμὸς φορέων μὲ πολλοὺς βαθμοὺς ἐλευθερίας κινήσεως ὑπὸ αἰφνίδια φόρτιση [διὰ τοῦ ὁποίου ἐπεκτείνεται ἡ γνωστὴ ἐρευνητικὴ ἐργασία γιὰ μονοβάθμια συστήματα τῶν διακεκριμένων ἐρευνητῶν Budiansky and Hutchinson [Budiansky (1967)], ἀποτελεῖ συμβολὴ τοῦ συγγραφέως στὸν διεθνὴ ἐπιστημονικὸ χῶρο. Ἀξίζει ἐπίσης νὰ ἀναφερθεῖ ἡ συμβολὴ στὴν ἐξήγηση τοῦ δυναμικοῦ λυγισμοῦ καὶ τῆς δυναμικῆς ἀπώλειας εὐσταθείας *μὴ συντηρητικῶν* συστημάτων τῶν τύπων «ἀποκλίσεως» καὶ «πτερυγισμοῦ» [7, 9-11, 14-17, 19, 23-25, 29, 31]. Σ' αὐτὲς τὶς ἐργασίες ἐκτίθενται πολλὰ νέα εὑρήματα ὁρισμένα τῶν ὁποίων ἀνατρέπουν προηγούμενα εὐρέως ἀποδεκτὰ ἀποτελέσματα. Χαοτικὰ καὶ μετευστάθειας φαινόμενα, φαινόμενα εὐαισθησίας σὲ ἀπόσβεση ἢ ἀρχικὲς συνθῆκες ποὺ ἐμφανίζονται τόσο σὲ μὴ συντηρητικά, ὅσο καὶ συντηρητικὰ δομικὰ συστήματα ἐκτίθενται στὶς ὑπ' ἀριθμ. [7, 9, 10, 15-20, 22-25, 27-29] ἐργασίες. Γιὰ τὴ μελέτη ὅλων τῶν ἀνωτέρω φαινομένων ὁ συγγραφέας ἐχρησιμοποίησε ὡς προσομοιώματα τὸν πρόβολο καὶ τὴν ἁπλὴ ἀμφιέρειστη δοκό.

Ἡ παροῦσα ἐργασία, ἀποτελοῦσα ἐπέκταση τῶν πάρα πάνω 32 ἐργασιῶν, διαπραγματεύεται τὸν μὴ γραμμικὸ δυναμικὸ λυγισμὸ καὶ τὴ δυναμικὴ ἀστάθεια μὴ συντηρητικῶν διακλαδικῶν μὴ γραμμικῶς ἐλαστικῶν μὲ ἢ χωρὶς ἀπόσβεση δομικῶν συστημάτων, ὑποκειμένων σὲ φόρτιση μεταβαλλόμενης διευθύνσεως. Μὲ ἐφαρμογὴ μιᾶς γενικῆς *θεωρητικῆς ἀναλύσεως* μὲ *ποιοτικὰ* καὶ *ποσοτικὰ κριτήρια* χρησιμοποιώντας ὡς προσομοίωμα ἕνα ἁπλὸ πρόβολο ἀποκαλύπτεται μιὰ ὁλόκληρη σειρὰ *δυναμικῶν διακλαδώσεων*. Γιὰ πρώτη φορὰ εὑρίσκεται ὅτι σὲ περιοχὴ ὑπάρξεως *γειτονικῶν ἰσορροπιῶν* (ὅπου εἶναι ἐφαρμόσιμες οἱ στατικὲς μέθοδοι) εἶναι δυνατὴ ἡ διακλάδωση σὲ *ὁριακοὺς εὐσταθεῖς κύκλους* γιὰ φορτίο μικρότερο τοῦ στατικοῦ λυγισμοῦ. Τὰ παραπάνω φαινόμενα, καθὼς ἐπίσης σημειακοὶ ἢ ὁριακῶν κύκλων ἕλκτες, ποὺ συνδέονται μὲ φαινόμενα ἀκανόνιστης (χαοτικῆς) κινήσεως, ἡ ὁποία διαρκεῖ ἐνίοτε ἐπὶ μακρόν, ἐμφανίζονται σὲ μιὰ μικρὴ περιοχὴ στὴ γειτονιὰ τοῦ συνόρου μεταξὺ στατικῆς καὶ δυναμικῆς ἀστάθειας. Χαρακτηριστικὸ τῆς περιοχῆς αὐτῆς εἶναι ὅτι ὑπάρχει *ἀλληλοεξάρτηση* τῶν μεταλυγισμικῶν δρόμων ἰσορροπίας.

Ὁ Ἀκαδημαϊκὸς κ. **Περικλῆς Θεοχάρης** προσθέτει τὰ ἑξῆς σχετικὰ πρὸς τὴν ἀνωτέρω ἐργασίαν:

Ἀπὸ τὴν ἐπισκόπησιν τῆς διαχρονικῆς ἐξελίξεως τῶν πάσης φύσεως κατασκευῶν τοῦ Πολιτικοῦ Μηχανικοῦ, τοῦ Μηχανολόγου, τοῦ Ἀεροναυπηγοῦ, γίνεται φανερὰ ἡ ἀπὸ μακροῦ χρόνου ὑπάρχουσα ἐκ λόγων οἰκονομίας τάσις πρὸς πραγματοποίησιν *ἐλαφροτέρων* καὶ *λεπτοτέρων* κατασκευῶν ποὺ νὰ δύνανται νὰ φέρουν μὲ ἀσφάλειαν τὸ *μεγαλύτερον* δυνατὸν φορτίον, μὲ ἀποτέλεσμα τὴν ἐμφάνισιν φαινομένων *ἐλαστικῆς ἀσταθείας*, φαινομένων, δηλαδή, ποὺ δύνανται νὰ προκαλέσουν τὴν ἀστοχίαν τῆς κατασκευῆς χωρὶς οἱ τάσεις *νὰ ὑπερβαίνουν τὸ ὅριον ἐλαστικότητος*. Οὐσιῶδες χαρακτηριστικὸν αὐτῶν τῶν φαινομένων εἶναι ὅτι συνδέονται μὲ μὴ γραμμικότητες *γεωμετρικῆς μορφῆς*. Συνεπῶς, εἶναι ὡς ἐκ τῆς φύσεώς τους μὴ γραμμικὰ φαινόμενα, τὰ ὁποῖα δὲν μποροῦν νὰ μελετηθοῦν μὲ τὰς μεθόδους τῆς κλασσικῆς ἀναλύσεως τῶν κατασκευῶν. Ἡ ἐφαρμογὴ τῆς μὴ γραμμικῆς ἀναλύσεως τῶν κατασκευῶν ἐκτὸς τοῦ ὅτι κατέστησε δυνατὴν τὴν ἀκριβῆ ἐκτίμησιν τῆς *φερούσης ἱκανότητος* τῆς κατασκευῆς ἀπεκάλυψε νέον κόσμον φαινομένων, ὅπως π.χ. φαινόμενα σημαντικῆς μειώσεως τῆς φερούσης ἱκανότητος τῶν κατασκευῶν, φαινόμενα μεταλυγισμικῆς ἀντοχῆς, φαινόμενα σταθεροποιήσεως ἢ ἀποσταθεροποιήσεως λόγῳ ἀλληλεπιδράσεως τῶν ἐξωτερικῶν φορτίσεων, φαινόμενα ἁπλῶν ἢ πολλαπλῶν διακλαδώσεων σὲ τετριμμένους ἢ μὴ γραμμικοὺς δρόμους ἰσορροπίας μὲ δευτέραν μεταβολὴν τοῦ συνολικοῦ δυναμικοῦ μεγάλου βαθμοῦ ἀνωμαλίας, φαινόμενα ἀσταθείας ποὺ συνδέονται μὲ διαφόρους τύπους καταστροφῶν. Πρόσφορος κατάταξις τῶν διαφόρων *ἀσταθειῶν* λόγῳ στατικῆς φορτίσεως γίνεται μὲ τὴν βοήθειαν τῆς θεωρίας τῶν καταστροφῶν τῆς εἰσαχθείσης ὑπὸ τοῦ Γάλλου ἐρευνητοῦ René Thom.

Εἰς τὸ σημεῖον αὐτὸ ἀξίζει νὰ γίνῃ ἰδιαιτέρα μνεία διὰ τὸν θεμελιωτὴν τῆς θεωρίας τῆς *ἀρχικῆς μεταλυγισμικῆς συμπεριφορᾶς*, τὸν Ὁλλανδὸν ἐρευνητὴν καὶ Ἀκαδημαϊκὸν W. T. Koiter, ὁ ὁποῖος μὲ τὴν διατριβήν του, τὸ 1945, ἔφερε εἰς φῶς πολλὰ ἀπὸ τὰ ἀνωτέρω φαινόμενα. Κατὰ τὴν δεκαετίαν τοῦ 1960 ἔγινε ἀντιληπτὸν ὅτι ἡ κλασσικὴ (γραμμικὴ) ἀνάλυσις τῶν κατασκευῶν εἶναι ἄλλοτε μὲν *ἀνεπαρκής*, διότι ὁδηγεῖ εἰς σπατάλην ὑλικοῦ, ἄλλοτε δὲ καὶ *ἐπικίνδυνος*, διὰ κατασκευὰς εὐαισθήτους εἰς ἀτελείας. Τοιουτοτρόπως, ἡ ἐφαρμογὴ τῆς μὴ γραμμικῆς ἀναλύσεως καθίστατο ἐπιτακτικὴ διὰ τὴν ὀρθὴν ἐκτίμησιν τῆς φερούσης ἱκανότητος τῶν κατασκευῶν. Ἐδῶ θὰ πρέπει νὰ διευκρινισθῇ ὅτι ἡ μὴ γραμμικὴ ἀνάλυσις διηυκολύνθη ἀφ' ἑνὸς μὲν μὲ τὴν ἐντυπωσιακὴν ἀνάπτυξιν καὶ διάδοσιν τῶν ἠλεκτρονικῶν ὑπολογιστήρων, ἀφ' ἑτέρου δὲ μὲ τὴν ἐπινόησιν νέων ὑπολογιστικῶν τεχνικῶν, ὅπως π.χ. εἶναι αἱ *τεχνικαὶ τῶν διαταραχῶν* καὶ ἡ *ἀσυμπτωτικὴ ἀνάλυσις*.

Ἀλλὰ ἐὰν ἡ μὴ γραμμικὴ ἀνάλυσις τῶν κατασκευῶν λόγῳ *στατικῆς φορτίσεως* ἔφερεν εἰς φῶς φαινόμενα ἐλαστικῆς ἀσταθείας, ἡ ἐφαρμογὴ τῆς ἀναλύσεως αὐτῆς εἰς κατασκευὰς φερούσας *δυναμικὰ φορτία* ἀπεκάλυψε κόσμον ὁλόκληρον *νέων φαινομένων*, ὅπου σημαντικὸν ρόλον παίζει ἡ *ἀλληλεπίδρασις τῆς γεωμετρικῆς μὴ γραμμικότητος* καὶ τῆς *ἀποσβέσεως*.

Χάρις ὅμως εἰς τὴν πρόσφατον πρόοδον τῆς *μὴ Γραμμικῆς Δυναμικῆς*, ἡ ὁποία ἐπετεύχθη μέσῳ τῆς *ποιοτικῆς τοπολογικῆς προσεγγίσεως* τοῦ Poincaré, καὶ μὲ τὴν βοήθειαν τῶν συγχρόνων ἠλεκτρονικῶν ὑπολογιστήρων, ἀνεκαλύφθησαν εἰς διαφόρους κλάδους τῶν ἐφηρμοσμένων ἐπιστημῶν (π.χ. τὴν Φυσικήν, Μετεωρολογίαν, Ἀστρονομίαν, Χημείαν, Ἠλεκτρομαγνητισμόν, Βιολογίαν, Οἰκολογίαν, Οἰκονομίαν κλπ.), νέα δυναμικὰ φαινόμενα, ὅπως π.χ. εἶναι οἱ *ἰδιόμορφοι ἕλκται*, φαινόμενα νέων καθολικῶν διακλαδώσεων, φαινόμενα *μετευσταθείας*, φαινόμενα εὐαισθησίας εἰς ἀπόσβεσιν ἢ ἀρχικὰς συνθήκας κλπ.

Ἀπὸ τὸν τίτλον τῆς παρούσης μελέτης ἀντιλαμβάνεται κανεὶς ὅτι φαινόμενα χαοτικῆς ἢ φύσεως ἀναλόγου μὲ τὰ προαναφερθέντα φαινόμενα, δύνανται νὰ ἐμφανισθοῦν κατὰ τὴν μὴ γραμμικὴν ἀνάλυσιν τῶν κατασκευῶν.

Ἡ παροῦσα ἐργασία ἀποτελοῦσα ἐπέκτασιν προηγουμένων ἐργασιῶν τοῦ συγγραφέως διαπραγματεύεται τὸν *μὴ γραμμικὸν δυναμικὸν λυγισμὸν* ὡς καὶ τὴν *δυναμικὴν ἀστάθειαν μὴ συντηρητικῶν διακλαδικῶν συστημάτων* ἀπὸ μὴ γραμμικῶς ἐλαστικὸν ὑλικόν, μὲ ἢ χωρὶς ἀπόσβεσιν, τὰ ὁποῖα ὑπόκεινται εἰς φόρτισιν *μεταβαλλομένης διευθύνσεως*. Δι' ἐφαρμογῆς τῆς γενικῆς τοπολογικῆς ἀναλύσεως μὲ ποσοτικὰ καὶ ποιοτικὰ κριτήρια ἀνακαλύπτεται ὁλόκληρος σειρὰ *δυναμικῶν τοπικῶν* καὶ *καθολικῶν διακλαδώσεων*. Διὰ πρώτην φορὰν εὑρίσκεται ὅτι διὰ μὴ συντηρητικὰ συστήματα, ἐπιλύσιμα μέσῳ στατικῆς ἀναλύσεως, εἶναι δυνατὸν νὰ ὑπάρξῃ διακλάδωσις σὲ *ὁριακοὺς εὐσταθεῖς κύκλους* διὰ φορτίον *μικρότερον* τοῦ φορτίου τοῦ στατικοῦ λυγισμοῦ. Ἡ διακλάδωσις αὐτὴ εἶναι τύπου Hopf. Σημειοῦται ἐνταῦθα ὅτι λόγῳ ἀσταθῶν ὁριακῶν κύκλων πτερυγισμοῦ τύπου Hopf κατέρρευσε τὸ 1945 ἡ κρεμαστὴ γέφυρα τῆς Tacoma, ἀνοίγματος 854 m τῆς πολιτείας τῆς Ν. Ὑόρκης. Ὁ συγγραφεὺς ἐντοπίζει ἀκόμη μικρὰν μὲ ἐνδιαφέροντα εὑρήματα περιοχὴν *γειτονικῶν ἰσορροπιῶν* εἰς τὴν γειτονίαν *διπλοῦ διακλαδικοῦ σημείου*, ἀποτελοῦντος σύνορον μεταξὺ *στατικῆς* καὶ *δυναμικῆς ἀσταθείας*. Εἰς τὴν μικρὰν αὐτὴν περιοχὴν ἐμφανίζεται ὁλόκληρος σειρὰ νέων διακλαδώσεων, καθὼς αὐξάνει ἡ τιμὴ τῆς ἐξωτερικῆς φορτίσεως. Εἰς τὰ Σχήματα 7 καὶ 10 φαίνεται *στατικὴ* διακλάδωσις συνδεομένη μὲ *σημειακὸν ἕλκτην*, ὅπου μηδενίζεται μία ἰδιοτιμὴ τῆς Ἰακωβιανῆς. Εἰς τὸ Σχῆμα 8 φαίνεται ἄλλος τύπος δυναμικῆς διακλαδώσεως συνδεόμενος μὲ εὐσταθεῖς ὁριακοὺς κύκλους (διπλὴ μηδενικὴ ἰδιοτιμὴ τῆς Ἰακωβιανῆς). Εἰς τὸ Σχῆμα 9 φαίνεται ἄλλος τύπος εὐσταθοῦς δυναμικῆς διακλαδώσεως (τύπου Hopf) τόσον εἰς τὸ ἐπίπεδον φάσεως, ὅσον καὶ εἰς τὸν χῶρον φάσεως. Εἰς τὸ Σχῆμα 11 φαίνεται ἡ καθολικὴ διακλάδωσις συνδεομένη μὲ εὐσταθεῖς ὁριακοὺς κύκλους, ἡ ὁποία προσδιωρίσθη μέσῳ ἀριθμητικῆς ὁλοκληρώσεως τῶν μὴ γραμμικῶν ἐξισώσεων κινήσεως. Εἰς τὸ Σχῆμα 12 βλέπομεν ἄλλον τύπον δυναμικῆς διακλαδώσεως ἡ ὁποία συνδέεται μὲ μεγάλην γωνιακὴν παραμόρφωσιν. Τὸ σημεῖον αὐτὸ εἶναι *διπλοῦν διακλαδικόν*. Εἰς τὸ Σχῆμα 14 παρατηροῦμεν ὅτι ἡ κατακόρυφος θέσις τοῦ προβόλου εἶναι *ἀσταθής*, συμφώνως μὲ τὴν κλασσικὴν ἀνάλυσιν τοῦ Hermann καὶ ἄλλων ἐρευνητῶν, ἐνῶ μὲ τὴν ἐφαρμογὴν τῆς μὴ γραμμικῆς ἀναλύσεως ὁ συγγραφεὺς ἀπέδειξεν ὅτι ἡ θέσις αὐτὴ εἶναι *εὐσταθής*. Τέλος, εἰς τὰ Σχήματα 7 καὶ 10 ποὺ προαναφέραμεν παρατηροῦμεν τὸ χαοτικὸν καθεστὼς προβόλου μὲ ἀπόσβεσιν ὑπὸ μὴ συντηρητικὴν φόρτισιν.

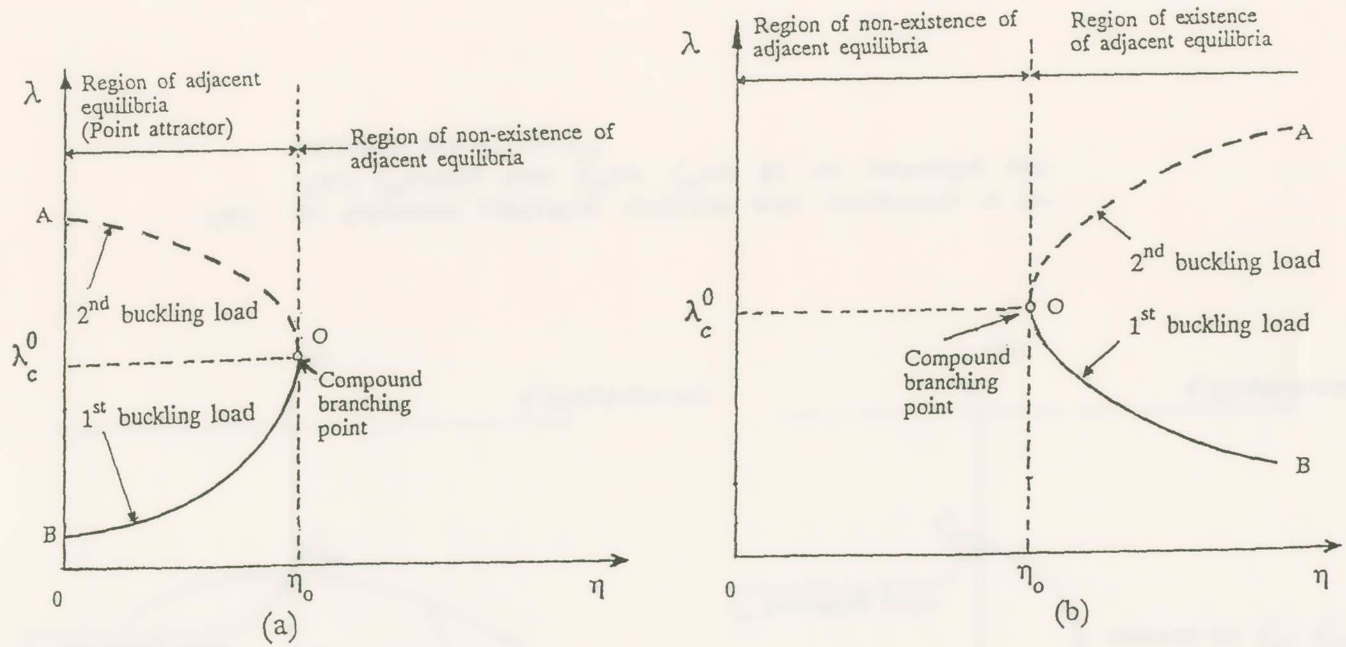

Fig.1. The compound branching point 0 (λ_c^o, η_o), boundary between the regions of existence and non-existence of adjacent equilibria. Point O in the curve AOB may be either a maximum (a) or a minimum (b).

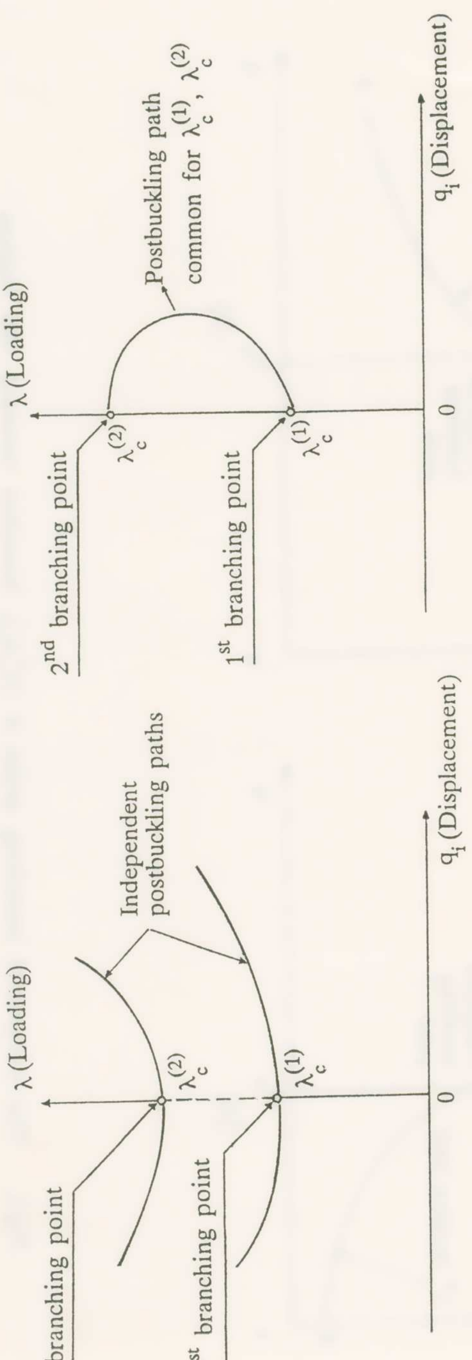

Fig.2. (a) Independent postbuckling equilibrium paths corresponding to the 1st and 2nd buckling loads $\lambda_c^{(1)}$ and $\lambda_c^{(2)}$ and (b) one postbuckling path common for both $\lambda_c^{(1)}$ and $\lambda_c^{(2)}$.

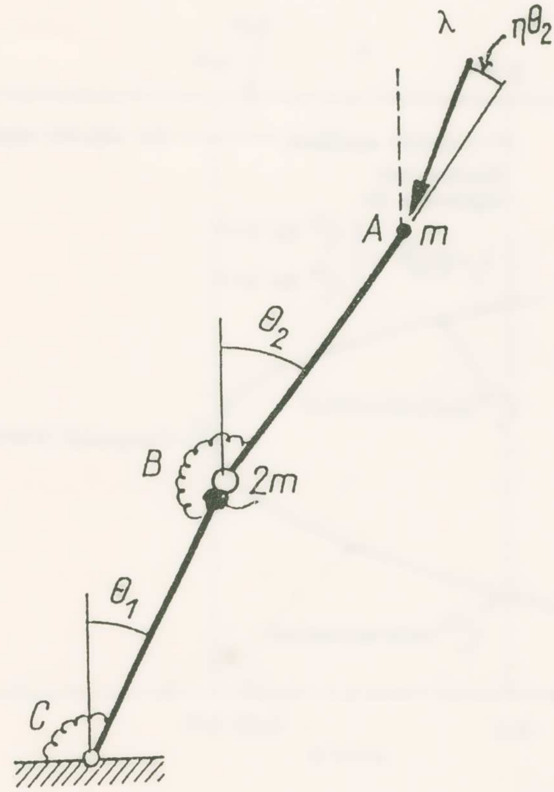

Fig.3. Ziegler's dissipative model under partial follower load.

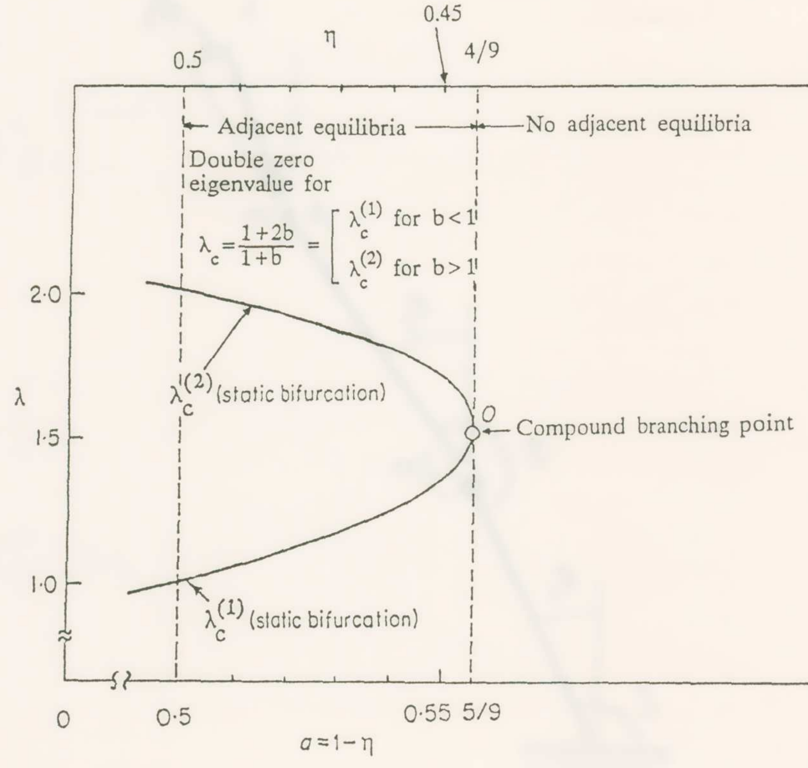

Fig.4. The small region of adjacent equilibria ($4/9<\eta<0.5$) in the neighborhood of the point 0, where a double zero eigenvalue is possible for a suitable damping ratio *b*.

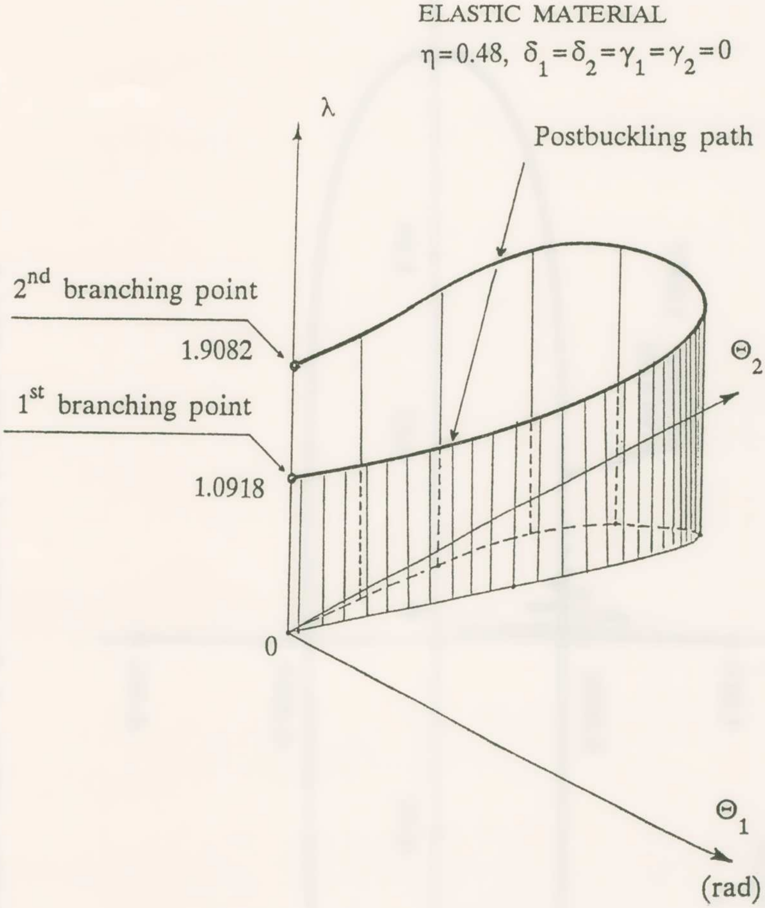

Fig.5. One postbuckling equilibrium path (passing through the 1st and 2nd branching point) for $\eta=0.48\in[4/9,0.5]$.

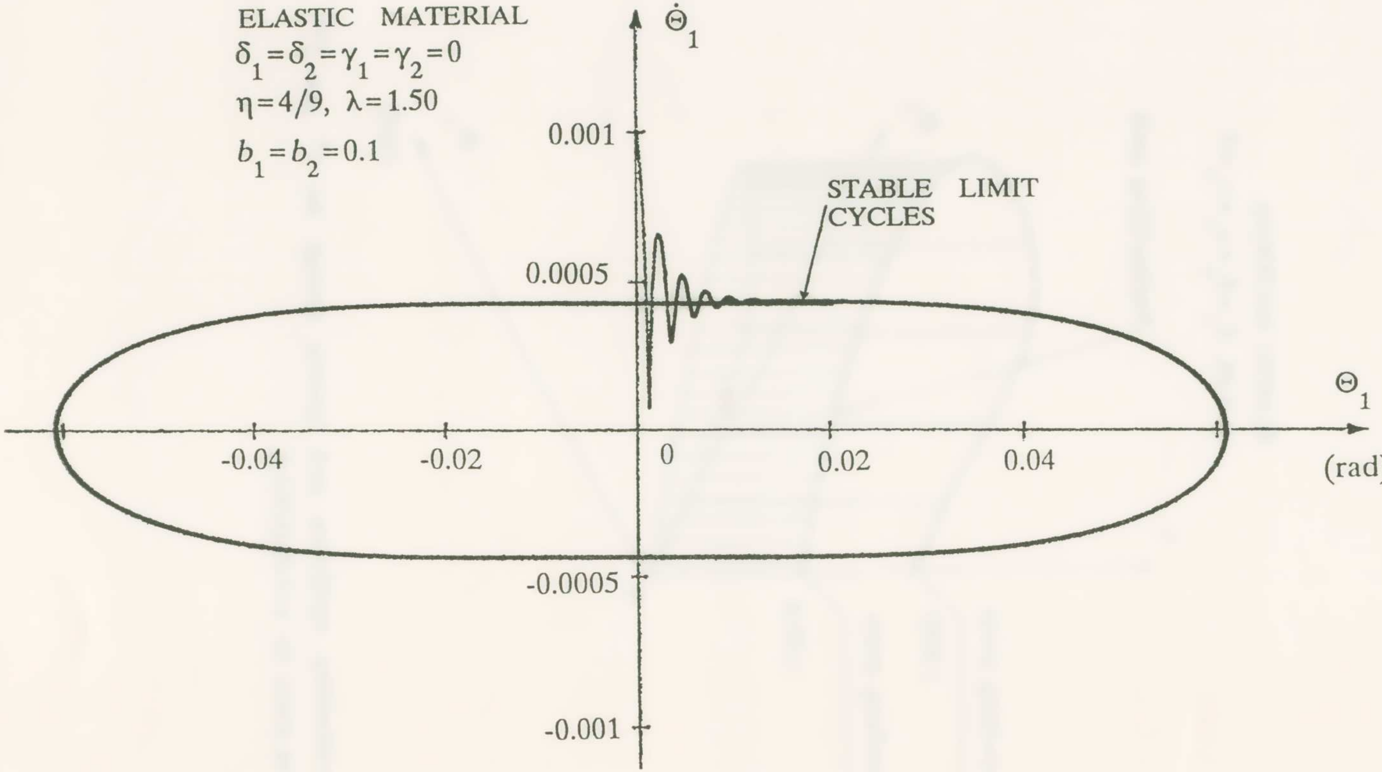

Fig.6. Phase-plane portrait of the compound branching point 0 showing a dynamic bifurcation associated with stable limit cycles.

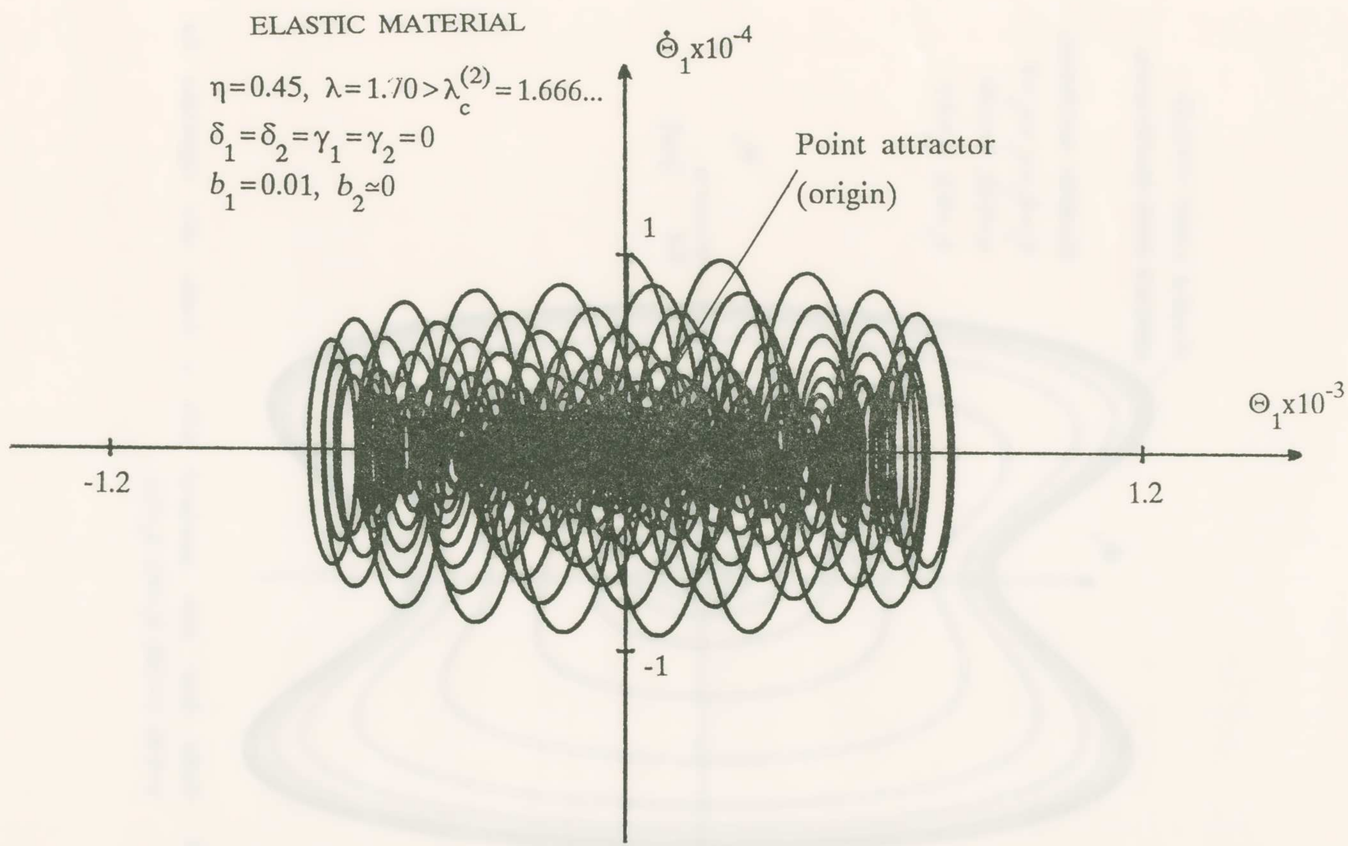

Fig.7. Asymptotically stable origin associated with a persistent chaoslike regime for $\eta=0.45$, $\lambda=1.70>\lambda_c^{(2)}$, $b_1=0.01$ and $b_2\simeq 0$.

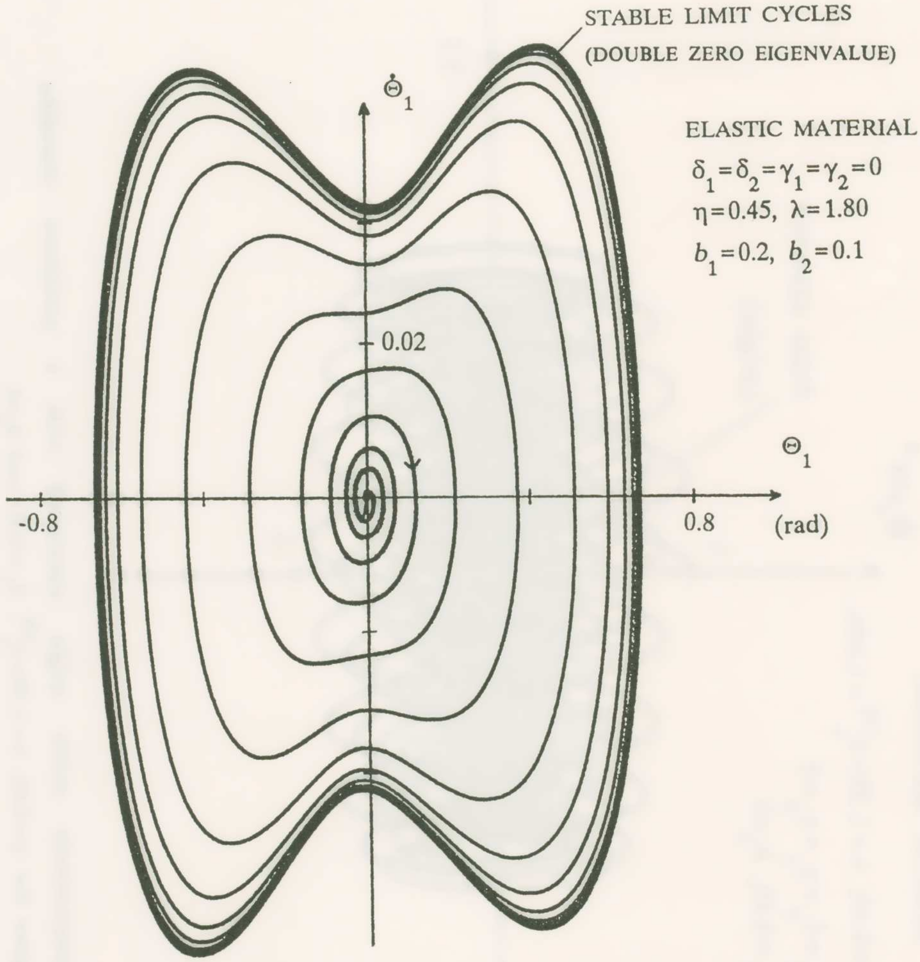

Fig.8. Stable limit cycle associated with a double zero eigenvalue for $\eta=0.45$, $\lambda=1.80$, $b_1=0.2$, $b_2=0.1$.

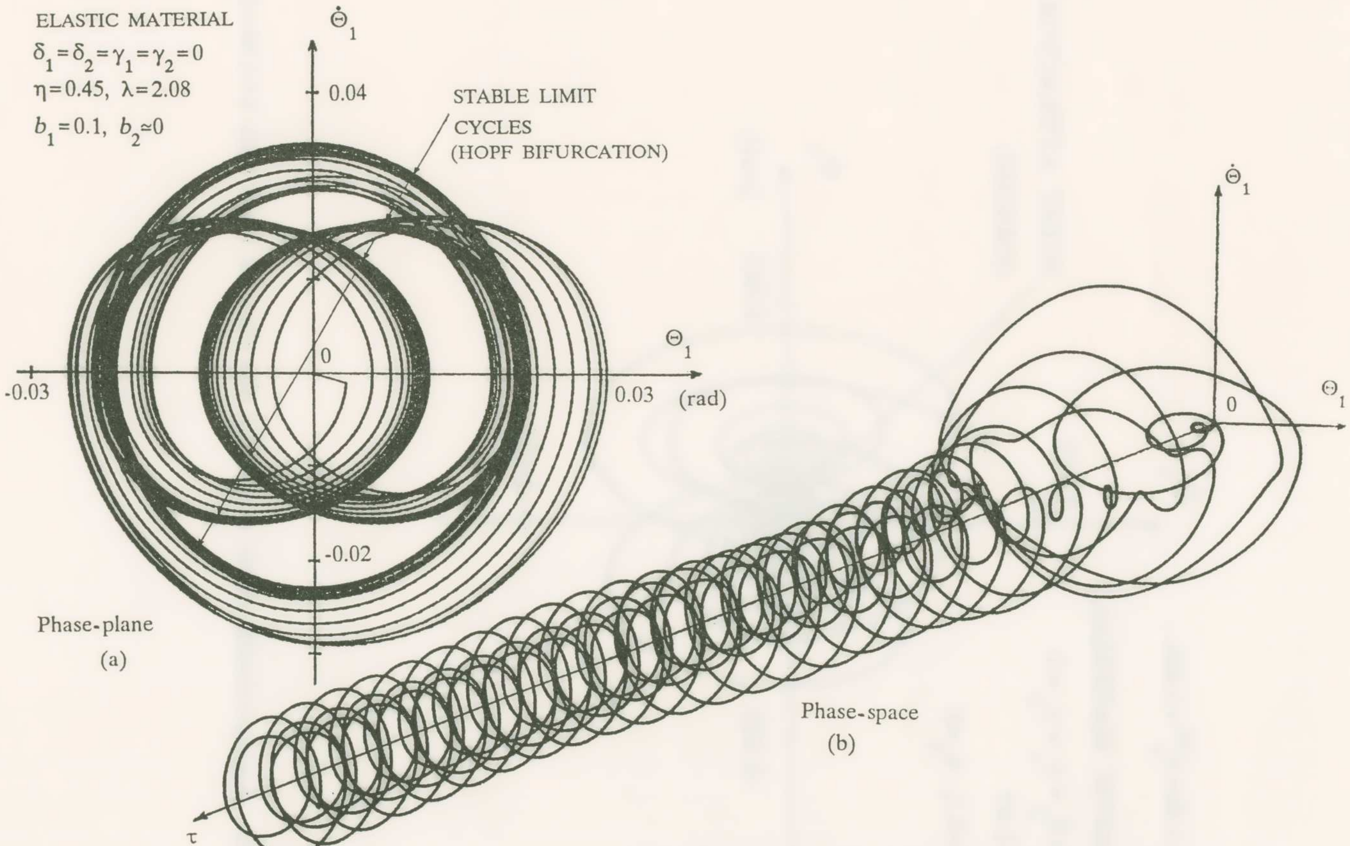

Fig.9 Phase-plane (a) and phase-space (b) portraits of a stable Hopf bifurcation in the region of adjacent equilibria for $\eta = 0.45$, $\lambda = 2.08$, $b_1 = 0.1$, $b_2 \cong 0$

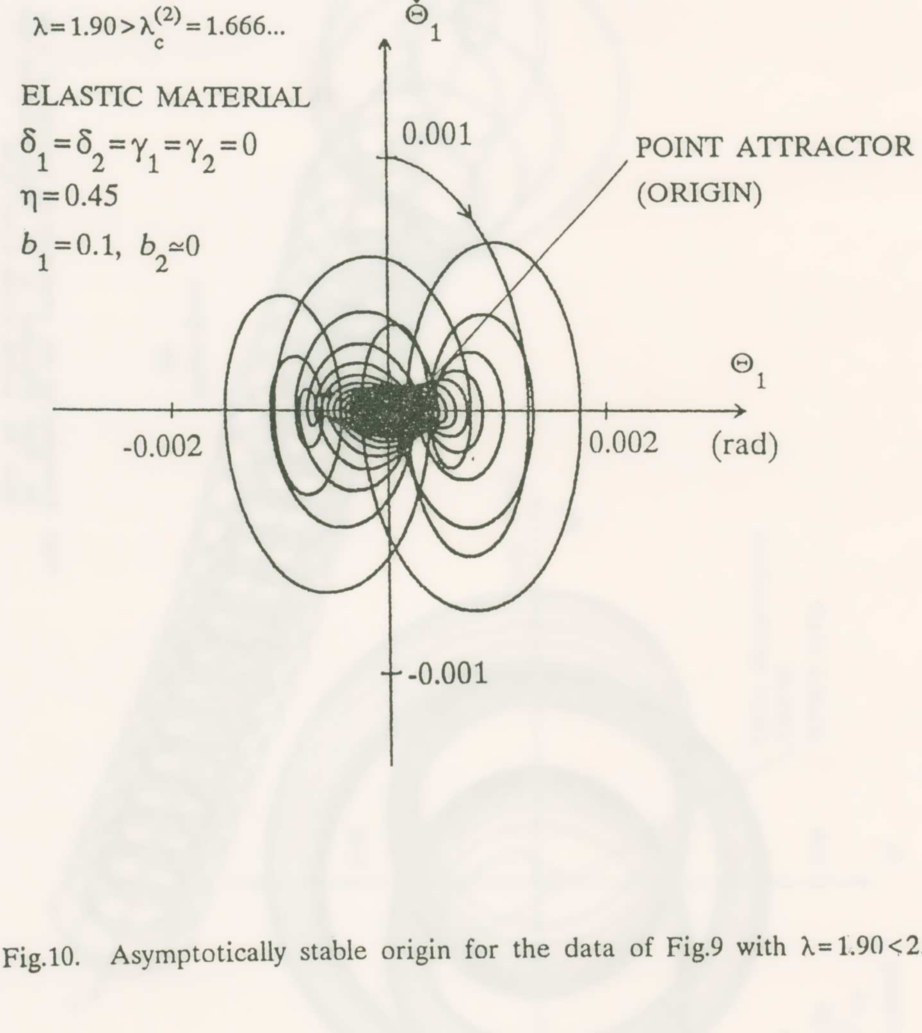

Fig.10. Asymptotically stable origin for the data of Fig.9 with $\lambda = 1.90 < 2$.

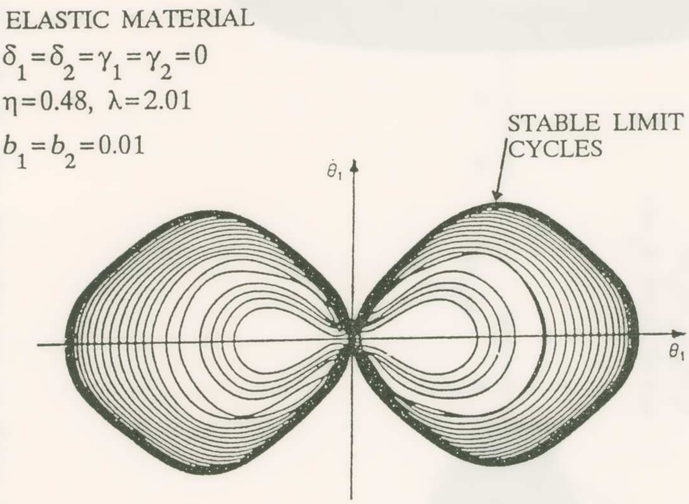

Fig.11. Global stable dynamic bifurcation with trajectories passing through the saddle of the origin for $\eta=0.48$, $\lambda=2.01>\lambda_{cr}=2.007$, $b_1=b_2=0.01$.

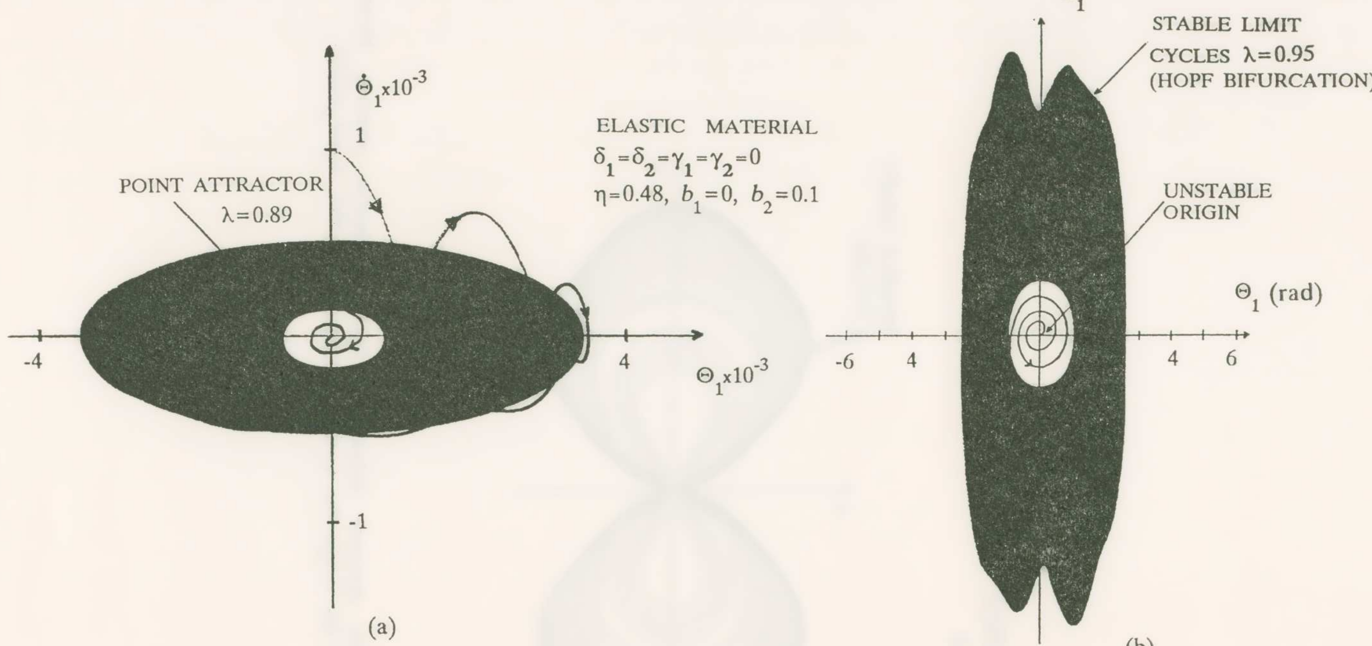

Fig.12. Point attractor for $\lambda=0.89$ (a) and stable limit cycles (b) for $\lambda=0.95$ for $\eta=0.48$, $b_1=0$, $b_2=0.1$.

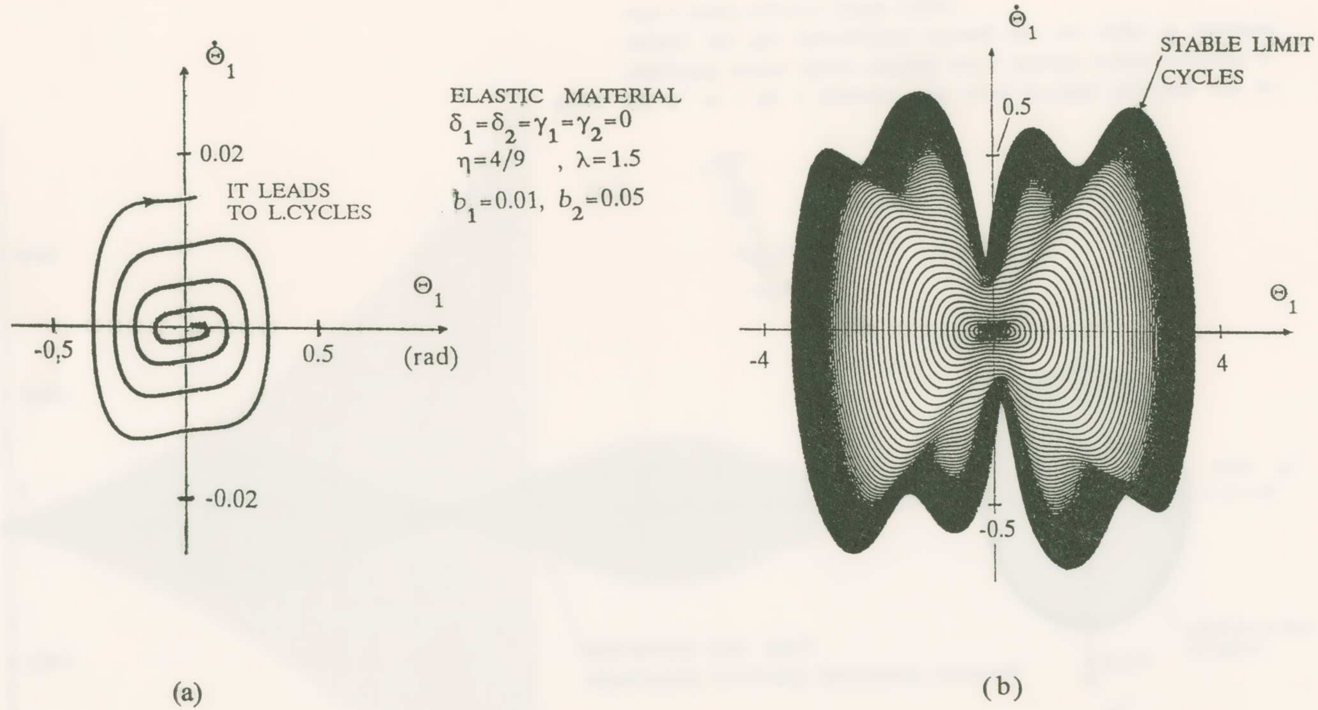

Fig.13. The phase-portrait of the point ($\eta_o=4/9$, $\lambda_c^o=1.5$) associated with an unstable origin (a) and stable limit cycles (b) for $b_1=0.01$, $b_2=0.05$.

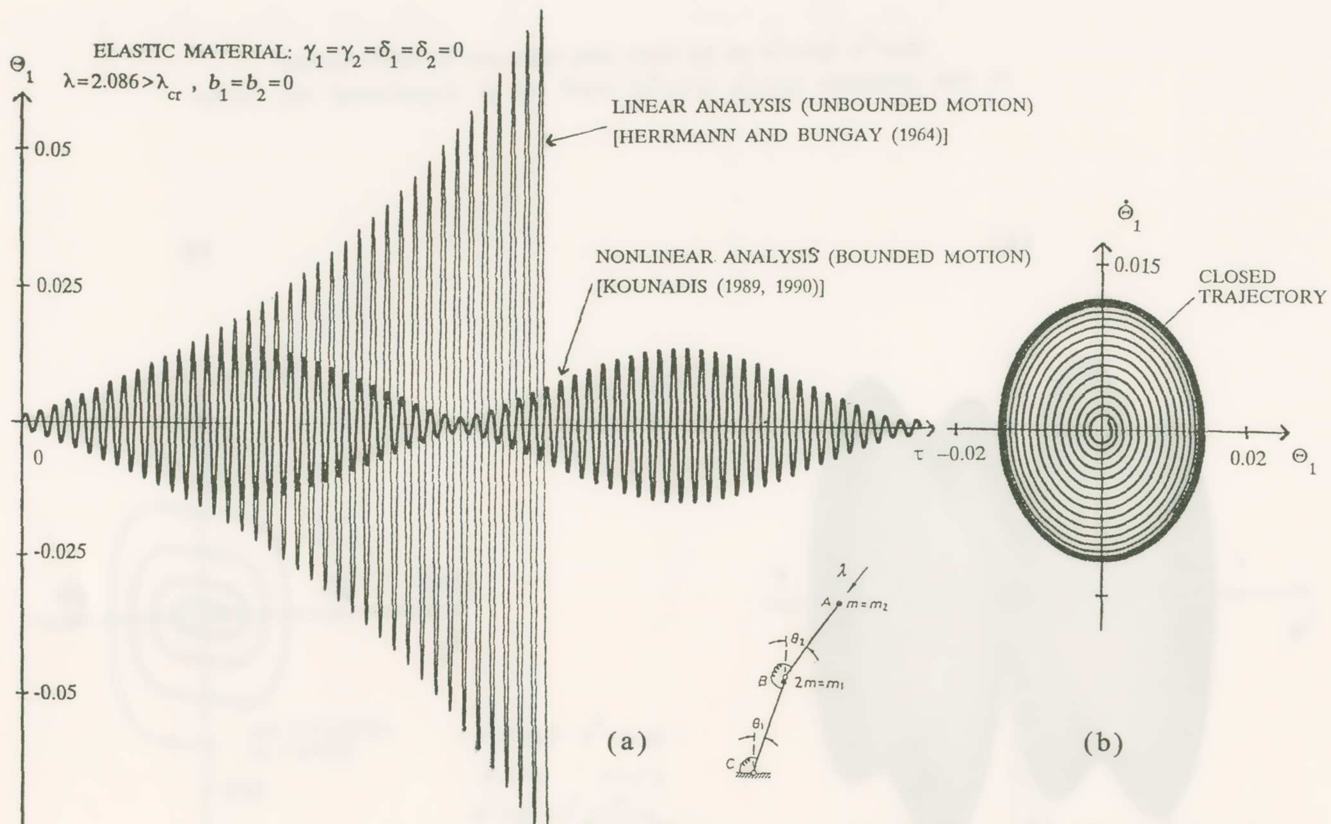

Fig.14. (a) Θ_1 vs τ for a tangential load ($\eta=0$, $\lambda=2.086$) associated with an unbounded motion (linear analysis) and a bounded motion (nonlinear analysis). (b) The phase-portrait showing that the origin is associated with a closed trajectory (stable origin).