1b. ἐξωτερικὴ ὄψις ἀριστερᾶς θυρίδος.

Εἰκ. 2. *Chlamys zenonis COWPER - REED*

Εἰκ. 3. *Chlamys scabrella LAMARCK*

Εἰκ. 4. *Xenophora crispa KOENIG*

Εἰκ. 5. *Turritella biplicata BRONNI*

Εἰκ. 6. *Murex brandaris* var. *torularius LAMARCK*

Εἰκ. 7. *Terebra acuminata acuminata BORS.*

ΜΗΧΑΝΙΚΗ. – **On the Convergence of Series Related to Principal Modes of Nonlinear Systems***, *by* ***Demetrios G. Magiros*****. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰωάνν. Ξανθάκη.

I. INTRODUCTION

a. In previous papers [1], where the principal modes of a «dual mode» nonlinear system have been discussed, the solution is found in the form of series. The object of the present announcement is to give a brief discussion of the convergence of these series. Details of the discussion will appear in a forthcoming paper. The convergence is based on the «A b e l ' s t e s t» of convergence.

b. We state for reference that the solution found is the following series:

$$\text{(a).}\quad x(t)=a_0+2a_1\cos\omega t+\frac{2\lambda_1 a_1^3}{k_3}\cos 3\omega t+2\sum_N a_N \cos N\omega t \tag{1}$$

$$\text{(b).}\quad \psi(t)=\frac{\lambda_3 a_0}{\omega_2^2}+\frac{2\lambda_3 a_1}{\omega_2^2-\omega^2}\cos\omega t+\frac{2\lambda_1\lambda_3 a_1^3}{k_3(\omega_2^2-3^2\omega^2)}\cos 3\omega t+2\sum_N \frac{\lambda_3 a_N}{\omega_2^2-N^2\omega^2}\cos N\omega t$$

where:

$$\text{(a).}\quad a_{N=2n}=(-1)^n a_0 p_2 p_4 \ldots p_{2n}\,,\quad n=1,2,3,\ldots$$

$$\text{(b).}\quad a_{N=2n+1}=(-1)^{n+1}\frac{\lambda_1 a_1^3}{k_3}p_5 p_7 \ldots p_{2n+1}\,,\quad n=2,3,4,\ldots$$

$$\text{(c).}\quad p_N=\frac{3\lambda_1 a_1^2}{k_N}\left[1-\frac{6\lambda_1|a_1|^2}{k_N}+\left(\frac{6\lambda_1|a_1|^2}{k_N}\right)^2-\ldots\right] \tag{2}$$

$$\text{(d).}\quad k_N=-N^2\omega^2+\omega_1^2+\frac{\lambda_2\lambda_3}{N^2\omega^2-\omega_2^2}$$

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The values ω which are either submultiples of ω_2 or zeros of the k's are the sigularities of this solution.

2. THE CONVERGENCE OF THE SERIES

The convergence of the series (1) is deduced from that of the series $\sum_N a_N$. The proof of convergence of $\sum_N a_N$ can be done in two steps, namely by using either the first term of the series (2c) (first step), or all its terms (second step).

The formula (2d), for large values of N, shows that $|k_N|$ is of order N^2 and that the sign of k_N is negative. Then a value N_1 of N can be found such that k_N remains negative for any $N > N_1$ and $|k_N|$ increases with N.

First Step

By taking the integers $\bar{N}$, $\bar{n}$, n, σ such that $\bar{N} = 2\bar{n} > N_1$, $n = \bar{n} + \sigma$, where N_1, $\bar{N}$, $\bar{n}$ are fixed, the coefficients a_N can be written as:

$$
\begin{aligned}
a_{N=2n} &= C_{2\bar{n}} \cdot (3\lambda_1 a_1^2)^\sigma \cdot \frac{1}{k_2(\bar{n}+1) \cdots k_2(\bar{n}+\sigma)} \\
a_{N=2n+1} &= C_{2\bar{n}+1} \cdot (3\lambda_1 a_1^2)^\sigma \cdot \frac{1}{k_2(\bar{n}+1)+1 \cdots k_2(\bar{n}+\sigma)+1}
\end{aligned}
\tag{3}
$$

Where $C_{2\bar{n}}$ and $C_{2\bar{n}+1}$ are constants, and the k's positive. The series $\sum_N a_N$ can be split as follows:

$$
\sum_N a_N = \sum_{N=0}^{N_1-1} a_N + C_{2\bar{n}} \sum_{\sigma=1}^{\infty} (3\lambda_1 a_1^2)^\sigma \frac{1}{K_2(\bar{n}+1) \cdots K_2(\bar{n}+\sigma)} + C_{2\bar{n}+1} \sum_{\sigma=1}^{\infty} (3\lambda_1 a_1^2)^\sigma \frac{1}{k_2(\bar{n}+1)+1 \cdots K_2(\bar{n}+\sigma)+1}
\tag{4}
$$

By imposing the restriction $\lambda_1 < \frac{1}{3a_1^2}$, the Abel's test for the convergence [2b] of the infinite series of the right-hand member of (4) can be applied, since:

(a) the geometrical series $\sum_{\sigma=1}^{\infty} (3\lambda_1 a_1^2)^\sigma$ is convergent, and

(b) the sequences of the products of the inverse of k's are monotonic decreasing and bounded sequences.

Second Step

The coefficients a_N in this case can be expressed as follows:

$$a_{N=2n} = (-1)^n a_0 \frac{(3\lambda_1|a_1|^2)^n}{k_2 \cdots k_{2n}} \left[1 - \frac{6\lambda_1|a_1|^2}{k_2} + \left(\frac{6\lambda_1|a_1|^2}{k_2}\right)^2 - \ldots\right] \ldots \left[1 - \frac{6\lambda_1|a_1|^2}{k_{2n}} + \left(\frac{6\lambda_1|a_1|^2}{k_{2n}}\right)^2 - \ldots\right] \tag{5}$$

$$a_{N=2n+1} = (-1)^{n+1} \frac{a_1}{3} \frac{(3\lambda_1|a_1|^2)^n}{k_3 \ldots k_{2n+1}} \left[1 - \frac{6\lambda_1|a_1|^2}{k_3} + \left(\frac{6\lambda_1|a_1|^2}{k_3}\right)^2 - \ldots\right] \ldots \left[1 - \frac{6\lambda_1|a_1|^2}{k_{2n+1}} + \left(\frac{6\lambda_1|a_1|}{k_{2n+1}}\right)^2 - \ldots\right]$$

Then the series $\sum\limits_N a_N$ can be written as:

$$\sum_N a_N = \sum_{N=2n} A_{2n} \prod_{2n} + \sum_{N=2n+1} A_{2n+1} \prod_{2n+1} \tag{6}$$

where A's are the factors of the right--hand members of (5) outside the brackets, and Π's the products of the brackets, namely:

$$\prod_r^N \left[1 - \frac{6\lambda_1|a_1|^2}{k_r} + \left(\frac{6\lambda_1|a_1|^2}{k_r}\right)^2 - \ldots\right]$$

which for r either even or odd is either $\prod_{2n}$ or $\prod_{2n+1}$, respectively.

The series $\sum A_{2n}$ and $\sum A_{n+1}$ are convergent, according to the preceding step. In addition the Π's of (6) are sequences with monotone and bounded terms, because $\prod\limits_r^N$ can be wiritten as:

$$\prod_r^N \left[1 - \frac{6\lambda_1|a_1|^2}{k_r} + \left(\frac{6\lambda_1|a_1|^2}{k_r}\right)^2 - \ldots\right] = \prod_r^{N1-1} \left[1 - \frac{6\lambda_1|a_1|^2}{k_r} + \left(\frac{6\lambda_1|a_1|^2}{k_r}\right)^2 - \ldots\right] \prod_{N=N1}^N \left[1 + \frac{6\lambda_1|a_1|^2}{k_r} + \left(\frac{6\lambda_1|a_1|^2}{k_r}\right)^2 + \ldots\right] \tag{7}$$

where $\prod\limits_r^{N1-1}$ is a finite fixed number, and $\prod\limits_{N=N1}^{N}$ is convergent as $N \to \infty$, since the series $\sum\limits_{r=N1}^{\infty} \left(\frac{6\lambda_1|a_1|^2}{k_r}\right)^2$ is convergent. [2α]

The above proves that the series (la) is convergent. For the proof of the convergence of the series (lb), we see that the Abel's test of convergence can be applied to the series.

$$\sum_{N=0}^{\infty} \frac{\lambda_3}{N^2\omega^2-\omega_2^{\,2}}\, a_N \,.$$

The nature of the singularities of the series (1) and some subjects related to these singularities will be discussed in another paper.

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ΠΕΡΙΛΗΨΙΣ

Εἰς προηγουμένας ἐργασίας μας ἔχει εὑρεθῆ ὑπὸ μορφὴν σειρῶν ἡ λύσις προβλήματος τῶν πρωταρχικῶν ταλαντώσεων μὴ γραμμικῶν συστημάτων. Ἐνταῦθα δίδεται σύντομος ἐξέτασις τῆς συγκλίσεως τῶν σειρῶν αὐτῶν. Κατὰ τὴν πορείαν διὰ τὴν ἔρευναν τῆς συγκλίσεως γίνεται χρῆσις τοῦ θεωρήματος τοῦ Abel περὶ συγκλίσεως.

Ἡ πλήρης ἐξέτασις τῆς συγκλίσεως, καθὼς καὶ ἡ φύσις τῶν ἀνωμάλων σημείων τῆς λύσεως θὰ ἐκτεθοῦν εἰς ἐργασίαν, ἡ ὁποία συντόμως δημοσιεύεται ἀλλαχοῦ.

ΑΣΤΡΟΔΥΝΑΜΙΚΗ.– **On the convergence of the solution of a special two-body problem** *, *by* ***Demetrios G. Magiros*** *(**)*. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰωάνν. Ξανθάκη.

1. INTRODUCTION

a. In previous papers [1], where the motion of a projectile a Newtonian center during the action of general thrust vector was investigated, a series solution for this problem was constructed. The purpose of the present note is to give a brief discussion of the determination of the time in-

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terval for which the solution found is valid. Details of the present note and related subjects will appear elsewhere.

b. We state for reference that the differential equation and the initial conditions of the problem in vector form are:

$$\ddot{\underline{r}}(\tau) = -\frac{\mu}{r^3(\tau)}\,\underline{r}(\tau) + \underline{T}(\tau)$$
$$\underline{r}(o) = \underline{r}_0\,,\quad \dot{\underline{r}}(o) = \dot{\underline{r}}_0 + \underline{I}_0 \tag{1}$$

valid in the region D: $|\underline{r}(\tau)| < M_1$, $|\dot{\underline{r}}(\tau)| < M_2$ for any value of time τ in D_1: $o \leq \tau \leq \tau'$. μ is a constant, $\underline{T}$ the thrust, $\underline{I}_0$ the impulse of the thrust for very small time, $\underline{r}$ and $\dot{\underline{r}}$ the displacement and velocity vectors.

If the reference coordinate system is: $(P; \underline{r}_0^*, \underline{s}_0^*, \underline{T}_0^*)$, where P is the position of the projectile when the thrust starts, $\underline{r}_0^*$, $\underline{s}_0^*$, $\underline{T}_0^*$ the unit vectors along $\underline{r}_0$, $\dot{\underline{r}}_0 + \underline{I}_0$, $\underline{I}_0$, respectively, a solution of the form

$$\underline{r}(\tau) = \alpha_1(\tau)\,\underline{r}_0^* + \alpha_2(\tau)\,\underline{s}_0^* + \alpha_3(\tau)\,\underline{T}_0^* \tag{2}$$

can be determined by calculating the scalar functions $\alpha_i(\tau)$, i = 1,2,3, in Mac Laurin's expansions at $\tau = o$.

$$\alpha_i(\tau) = \sum_{n=o}^{\infty} \frac{\alpha_i^{(n)}(o)}{n!}\,\tau^n \;;\; i = 1, 2, 3 \tag{3}$$

The functions $\alpha_i(\tau)$ satisfy the conditions

$$\begin{aligned} \ddot{\alpha}_i + \frac{\mu}{r^3}\,\alpha_i &= T_i \;;\; i = 1, 2, 3 \\ \alpha_1(o) = r_0\,,\ \alpha_2(o) &= \alpha_3(o) = 0 \\ \dot{\alpha}_1(o) = \dot{\alpha}_3(o) &= 0\,,\ \dot{\alpha}_2(o) = s_0 \end{aligned} \tag{4}$$

where $s_0 = |\dot{\underline{r}}_0 + \underline{I}_0|$ and T_1, T_2, T_3, projections of $\underline{T}$ on the $\underline{r}_0^*$, $\underline{s}_0^*$, $\underline{T}_0^*$ – axis. By using (1), (2), (4) we can determine the coefficients of the series (3).

2. THE RADIUS OF CONVERGENCE OF THE SERIES (3).

The radius of convergence of the series (3) is the reciprocal of the upper limit: [2]

$$\overline{\lim_{n\to\infty}} \left\{ |\alpha_i^{(n)}(o)| \,/\, n! \right\}^{1/n},\; i = 1, 2, 3 \tag{5}$$

The $n^{\underline{th}}$ derivative of $\alpha_i(\tau)$, found from the equation (4), is:

$$\alpha_i^{(n)}(\tau) = -\mu \left[\frac{\alpha_i(\tau)}{r^3(\tau)} \right]^{(n-2)} + T_i^{(n-2)}(\tau) \tag{6}$$

The general term $\left(\alpha_i^{(n)}(\tau) \,/\, n! \right)$, if we rake into account the formula (e) of the Appendix, becomes:

$$\frac{\alpha_i^{(n)}(\tau)}{n!} = -\sum_{m=0}^{n-2} \frac{\alpha_i^{(m)}(\tau)\left[V_1 r^{n-m-3} + V_2 r^{n-m-4} + \cdots + V_{n-m-2} \right]}{(n-1)n\, m!\,(n-m-2)!\, r^{n-m+1}} + \frac{T_i^{(n-2)}(\tau)}{n!} \tag{7}$$

where V's are polynomicals in the derivatives of r up to the order $(n-m-2)$ with coefficients smaller than $(n-m)!$

To find the limit of the n^{th} root of the absolute value of the right-hand member of (7) for $\tau=0$ as $n \to \infty$, we consider each term of it as positive, and then take the $n^{\underline{th}}$ root of each such term. Their sum Sn must satisfy the relation:

$$\left[\left| \alpha_i^{(n)}(0) \right| \,/\, n! \right]^{1|n} \leq Sn \tag{8}$$

Since $r_0 \neq 0$ and all derivatives of r and T_i are bounded at $\tau=0$, the sum $Sn \to 0$ as $n \to \infty$, then the limit of the left-hand member of (8) is zero, and, as a result, the radius of convergence is ∞, and the series (3) converge for any value of τ'.

APPENDIX:

The $n^{\underline{th}}$ derivative of the product of the functions $\sigma(\tau)$, $\varphi(\tau)$ is given by:

$$[\sigma\varphi]^{(n)} = \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}\, \sigma^{(m)}\, \varphi^{(n-m)} \tag{a}$$

and that of:

$$\varphi(\tau) = \frac{1}{r^3(\tau)} \tag{b}$$

by:

$$\varphi^{(n)} = \frac{P(r)}{r^{n+3}} \tag{c}$$

with:

$$P(r) = V_1 r^{n-1} + V_2 r^{n-2} + \cdots + V_n \tag{d}$$

where the V's are polynomials in the derivatives of r up to the order $n^{\underline{th}}$ with coefficients smaller than $(n+2)!$

Combining (a), (b), (c) we can get:

$$[\sigma / r^3]^{(n)} = \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}\, \sigma^{(m)}\, \frac{P(r)}{r^{n-m+3}} \tag{e}$$

where:

$$P(r) = V_1 r^{n-m-1} + V_2 r^{n-m-2} \cdots + V_{n-m} \tag{f}$$

The V's polynomials have coefficients smaller than $(n-m+2)!$

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[2] K. Knopp: «Theory and Applications of Infinite Series», § 18, Section 94, 154-155 Hafner Publishing Co., New York.

ΠΕΡΙΛΗΨΙΣ

Εἰς προηγουμένας ἐργασίας κατεσκευάσθη ἡ λύσις τοῦ προβλήματος τῆς κινήσεως ὀχήματος ὑπὸ τὴν ἐπίδρασιν Νευτωνίου ἑλκτικοῦ κέντρου καὶ μιᾶς γενικῆς ὠστικῆς δυνάμεως. Ἐνταῦθα ἐκτίθεται ἐν συντομίᾳ τὸ ζήτημα τῆς συγκλίσεως τῆς λύσεως αὐτῆς. Λεπτομέρειαι, καθὼς καὶ ἄλλα συναφῆ ζητήματα, θὰ ἐκτεθοῦν εἰς ἐργασίαν, ἡ ὁποία θὰ δημοσιευθῇ ἀλλαχοῦ.

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* Θὰ δημοσιευθῇ κατωτέρω.