ΘΕΩΡΗΤΙΚΗ ΦΥΣΙΚΗ.— Remarks on a problem of subharmonics*, *by*
Dem G. Magiros. Ἀνεκοινώθη ὑπὸ τοῦ κ. Βασιλ. Αἰγινήτου.

Introduction.

This paper is a supplement of the author's previous paper under the title: «*Subharmonics of order one third in the case of cubic restoring force*», contained in this volume as the author's second work on subharmonics and called in the following «paper B», and its previous «paper A». In the first chapter we discuss the conditions under which the basic equation (1) of the «paper B» accepts the harmonic solution $\left(-\frac{B}{8}\sin 3\tau\right)$ as a stable one, the subharmonic term of the solution being zero. In the second chapter the «inverse problem» of that of the «paper B» is discussed. This «inverse problem», so simple form a mathematical point of view, according to the equations found, seems to be of importance from an engineering point of view.

I. *The singularity of the origin in the general case.*

The basic epuation is:

$$\ddot{Q} + \bar{k}\dot{Q} + \bar{c}_1 Q + \bar{c}_2 Q^2 + \bar{c}_3 Q^3 = B\sin 3\tau, \tag{1}$$

and it can be written in the form:

$$\ddot{Q} + Q = \varepsilon f(Q, \dot{Q}) + B\sin 3\tau, \tag{2}$$

with:

$$f(Q, \dot{Q}) = -k\dot{Q} + c_1 Q - c_2 Q^2 - c_3 Q^3, \tag{2α}$$

if:

$$\bar{k} = \varepsilon k, \quad 1 - \bar{c}_1 = \varepsilon c_1, \quad \bar{c}_2 = \varepsilon c_2, \quad \bar{c}_3 = \varepsilon c_3. \tag{3}$$

The steady state solution is:

$$Q = x\sin\tau - y\cos\tau + \frac{B}{8}\sin 3\tau, \tag{4}$$

and the components x and y of the amplitude r of the subharmonic of order one third are given, according to «paper B», by the two equations:

$$\left\{\begin{aligned} &\mu x + \lambda y - y\left(x^2 + y^2 + \frac{B^2}{32}\right) + \frac{1}{4}Bxy = 0, \\ &\lambda x - \mu y + x\left(x^2 + y^2 + \frac{B^2}{32}\right) + \frac{1}{8}B(-x^2 + y^2) = 0, \end{aligned}\right. \tag{5}$$

* Παρατηρήσεις ἐπὶ προβλήματος τῶν ὑποαρμονικῶν.

where:

$$\lambda = \frac{4}{3}\frac{c_1}{c_3} = \frac{4}{3}\frac{1-\bar{c}_1}{\bar{c}_3}, \qquad \mu = \frac{4}{3}\frac{k}{c_3} = \frac{4}{3}\frac{\bar{k}}{\bar{c}_3}. \tag{5α}$$

From (5) we see that the origin is a singularity of our equation; in other wordes, the function: $Q = \frac{B}{8}\sin 3\tau$ is always a solution of the equation, the harmonic solution.

For the stability of this harmonic solution we take the derivatives given by (24) in the «paper B», established under the conditions:

$$\varepsilon = 1, \qquad T = \max\tau < \frac{r}{4M}. \tag{6}$$

From (24) of the «paper B», in the case $x=0$, $y=0$, we get:

$$\begin{aligned} a_1 &= \frac{1}{2}k, & a_2 &= -\frac{1}{2}c_1 + 3c_3\left(\frac{B}{16}\right)^2, \\ b_1 &= \frac{1}{2}c_1 - 3c_3\left(\frac{B}{16}\right)^2, & b_2 &= -\frac{1}{2}k, \end{aligned} \tag{7}$$

when, from (26) of the «paper A», the characteristic roots are given by:

$$p_{1,2} = -\frac{1}{2}k \pm i\left\{\frac{1}{2}c_1 - 3c_3\left(\frac{B}{16}\right)^2\right\}. \tag{8}$$

We, therefore, have, according to the definitions on the singularities of the «paper A» the following:

A. If the imaginary part of the characteristic roots is not zero, that is if:

$$\frac{c_1}{c_3} \neq b\left(\frac{B}{16}\right)^2, \tag{9}$$

then:

α) for $k>0$, the origin is a «stable spiral point»;

β) for $k<0$, the origin is an «unstable spiral point»;

γ) for $k=0$, the origin is either a «center» or a «spiral point».

B. If the imaginary part is zero, that is if:

$$\frac{c_1}{c_3} = 6\left(\frac{B}{16}\right)^2, \tag{10}$$

then:

α) for $k>0$, the origin is a «nodal stable point»;

β) for $k<0$, the origin is an «unstable nodal point»;

γ) for $k=0$, the origin is not a simple singularity of the kind we know from «paper A» and «paper B», since the characteristic roots are zero.

The result from the above is that: the origin is a stable singularity when $k>0$, of spiral type under the condition (9), and of nodal type under the condition (10). In other words, the function $(Q=-\frac{B}{8}\sin 3\tau)$ is the harmonic stable solution of the equation (1) if $k>0$.

We plot in the: $\frac{1-\bar{c}_1}{\bar{c}_3}$, $\bar{k}$—plane the above results. The left half of this plane corresponds to the instability of the zero-subharmonic, that is to instability to the solution $Q=-\frac{B}{8}\sin 3\tau$; the right half to the stability,

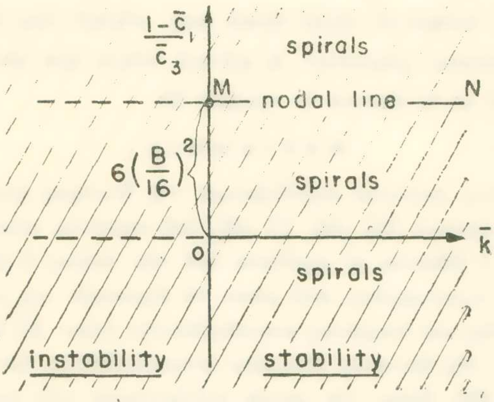

Fig 1

with all «spirals», except the points of the line $\frac{1-\bar{c}_1}{\bar{c}_3}=6\left(\frac{B}{16}\right)^2$ which are «nodals».

The points of the: $\frac{1-\bar{c}_1}{\bar{c}_3}$—axis may be either «centers» or «spirals», except the point M which is not a simple singularity but of advanced order.

The distance of the «nodal line» MN from the $\bar{k}$—axis has a maximum and a minimum, due to the restrictions of B, given in the «paper B», and in the case of free vibrations $(B=0)$ the «nodal line» is the $\bar{k}$—axis itself.

2. *The inverse problem.*

The inverse problem of that of the «paper B» is the following: «*Given the amplitude r of the subharmonic vibrations, find the coefficients of the differential equation, and study the stability of the solutions obtained*».

The solution of this problem correspond to determine the numbers λ and μ in terms of r and B, by using the additional equation:

$$x^2 + y^2 - r^2 = 0. \tag{11}$$

This determination is impossible due to the form of the equations (5).

From (5) and (11) we can have λ and μ in terms of x, y, r, B; then the determination of λ and μ needs to know B and two of x, y, r. If we know B, x, y, then the numbers λ and μ are known by solving the system (5) in λ and μ, (11) being a restriction between x, y, r. By knowing λ and μ, we know the ratios $\frac{1-\bar{c}_1}{\bar{c}_3}$, $\frac{\bar{k}}{\bar{c}_3}$, then any two coefficients from $\bar{c}_1$, $\bar{c}_3$, k can be determined in terms of their third one, which can have arbitrary values, and the «inverse problen» is solved, since the subject of the stability can be treated as is shown in «paper B».

ΠΕΡΙΛΗΨΙΣ

Ἡ ἐργασία αὕτη ἀποτελεῖ συμπλήρωμα τῆς δευτέρας ἐργασίας ἡμῶν ἐπὶ τῶν ὑποαρμονικῶν ταλαντώσεων (βλ. σελ. 77 κἑξ. τοῦ παρόντος τόμου).

Εἰς ταύτην α΄) δίδονται αἱ συνθῆκαι ὑπὸ τὰς ὁποίας ἡ βασικὴ διαφορικὴ ἐξίσωσις δέχεται λύσιν συνισταμένην ἀπὸ μόνον τὸ ἁρμονικόν της μέρος (ἄνευ ὑποαρμονικοῦ) καὶ δὴ εὐσταθὲς καὶ ἑπομένως φυσικῶς δεκτὴν λύσιν· β΄) ἐξετάζεται τὸ «ἀντίστροφον πρόβλημα» τῆς δευτέρας, ἀνωτέρω μνημονευθείσης, ἐργασίας. Τὸ πρόβλημα τοῦτο ἐμφανίζεται ἐδῶ, βάσει τῆς σειρᾶς συλλογισμῶν τῶν προηγουμένων καὶ τῆς παρούσης ἐργασίας, ὡς ἁπλούστατον μαθηματικῶς, ὅμως ἀποτελεῖ πρόβλημα πολλῆς σπουδαιότητος ἀπὸ τεχνικῆς ἀπόψεως.