ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

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ΣΤΑΤΙΣΤΙΚΗ ΦΥΣΙΚΗ.— **Exact Boltzmann distributions for energy dependent anisotropic scattering kernels,** *by C. Syros* *.
Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Περ. Θεοχάρη.

1. INTRODUCTION

The structural method developed previously [1 - 4] for the solution of the linear Boltzmann equation was based on the assumption that the scattering kernel was degenerate. This was necessary because the constant coefficients $\{p_n, q_n\}$ were determined by separating sums having as common factor one part of the factorized scattering kernel. Since the scattering kernels of practical interest consist of several different terms one difficulty arising from this fact was the relatively high order of the appearing matrices. This must be avoided for several practical reasons one of which is pushing further the accuracy of the numerical calculations without going into double precision.

Another more serious reason is the energy dependence as well as the space dependence of the scattering kernel in the case of the dynamics equations of the reactor, a case in which no discussion about degeneracy can be made.

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whilst for $z = 0$ they are equal to

$$S_n(x-\alpha\,;0) = \frac{(x-\alpha)^n}{n!}, \tag{2. 3}$$

where $\alpha = a, b$ are the boundaries of the slab.

From Eq. (2. 1) it is also seen that $\{S_n\}$ are homogeneous of degree n, i.e.,

$$S_n\big(\lambda(x-\alpha), \lambda z\big) = \lambda^n S_n(x-\alpha, z). \tag{2. 4}$$

Another important property of $S_n(x-\alpha, z)$ is expressed by the relation

$$\partial_x S_n(x-\alpha, z) = S_{n-1}(x-\alpha, z). \tag{2. 5}$$

The most important property of these polynomials is that when they are acted on by the operator $z\partial_x + 1$ they are transformed into an z-independent expression, i.e.,

$$(z\partial_x + 1)\, S_n(x-\alpha, z) = \frac{(x-\alpha)^n}{n!}. \tag{2. 6}$$

This is a property valid in spaces of any number of dimensions. The polynomials $S_n(x-\alpha, z)$ are not orthogonal or normalized. In order, however, to develop the method sketched in the introduction we need $\{\sigma_n\}$ to be orthonormal. We use the Schmidt orthogonalization procedure, and we obtain the orthonormal set of polynomials.

Using the definition given in Eq. (2. 1) we have for the n-th polynomial the set of equations

$$\begin{aligned} \sigma_n(x-\alpha), z) &= \sum_{\nu=0}^{n-1} c_{n\nu}\,\sigma_\nu(x-\alpha, z) + c_{nn} S_n(x-\alpha, z), \\ (\sigma_n, \sigma_\nu) &= c_{n\nu} + c_{nn}\big(S_n(x-\alpha, z), \sigma_\nu(x-\alpha, z)\big) \end{aligned} \tag{2. 7}$$

and

$$\begin{aligned} (\sigma_n, \sigma_n) = \sum_{\nu=0}^{n-1} c_{\nu n}^2 &+ 2\sum_{\nu=0}^{n-1} c_{n\nu} c_{nn}\big(S_n(x-\alpha, z), \sigma_\nu(x-\alpha, z)\big) + \\ &+ c_{nn}^2\big(S_n(x-\alpha, z), S_n(x-\alpha, z)\big) = 1, \end{aligned} \tag{2. 8}$$

where

$$(\sigma_n, \sigma_\nu) = \int_a^b dx \int_{-1}^{1} dz\; \sigma_n(x-\alpha, z)\,\sigma_\nu(x-\alpha, z). \tag{2. 9}$$

From Eqs. (2.7) and (2.8) one finds immediately

$$c_{n\nu} = -c_{nn}\left(S_n(x-\alpha, z), \sigma_\nu(x-\alpha, z)\right) \tag{2.10}$$

and

$$c_{nn} = \pm\left[(S_n, S_n) - \sum_{\nu=0}^{n-1}(S_n, \sigma_\nu)^2\right]^{-\frac{1}{2}} \tag{2.11}$$

and consequently

$$c_{n\nu} = (S_n, \sigma_\nu) / c_{nn}. \tag{2.12}$$

Explicitly, we have the expressions

$$(S_n, S_m) = \sum_{\nu=0}^{n}\sum_{\mu=0}^{m}(-)^{\mu+\nu}\frac{[\varepsilon(b-a)]^{m+n-\mu-\nu+1}}{(m+\nu-\mu-\nu+1)(m-\mu)!(n-\nu)!(\mu+\nu+1)}, \tag{2.13}$$

where $\varepsilon = 1$ or -1 corresponding to $\alpha = a$ or b.

The orthonormal set of polynomials $\{\sigma_n\}$ is therefore given by the expression

$$\left.\begin{aligned}\sigma_n(x-\alpha, z) &= \sum_{\nu=0}^{n-1}\sum_{\mu=0}^{\nu-1}\ldots\sum_{\varrho=0}^{1} c_{n\nu}c_{\nu\mu}\ldots c_{\lambda\varrho}c_{\varrho_0}S_0(x-\alpha, z) + \\ &+\sum_{\nu=0}^{n-1}\sum_{\mu=0}^{\nu-1}\ldots\sum_{\varrho=0}^{2} c_{n\nu}c_{\nu\mu}\ldots c_{\lambda\varrho}c_{\varrho_1}S_1(x-\alpha, z) + \\ &+\ldots+ \\ &+\sum_{\nu=0}^{n-1} c_{n\nu}c_{\nu\nu}S_\nu(x-\alpha, z) + c_{nn}S_n(x-\alpha, z) \equiv \\ &\equiv \sum_{\nu=0}^{n} u_{n\nu}S_\nu(x-\alpha, z).\end{aligned}\right\} \tag{2.14}$$

The first few polynomials $\{\sigma_n\}$ expressed entirely in terms of the polynomials $\{S_n\}$ have the form

$$\begin{aligned}\sigma_0 &= c_{00}S_0, \\ \sigma_1 &= c_{10}S_0 + c_{11}S_1, \\ \sigma_2 &= (c_{20} + c_{21}c_{10})S_0 + c_{21}c_{11}S_1 + c_{22}S_2, \\ \sigma_3 &= (c_{30} + c_{31}c_{10} + c_{32}c_{20} + c_{32}c_{21}c_{10})S_1 + (c_{31}c_{11} + c_{32}c_{21}c_{11})S_1 + \\ &+ c_{32}c_{22}S_2 + c_{33}S_3, \\ \sigma_4 &= (c_{40} + c_{41}c_{10}) + c_{42}(c_{20} + c_{21}c_{10}) + c_{43}(c_{30} + c_{31}e_{10} + c_{32}c_{20} + \\ &+ c_{32}c_{21}c_{10})S_0 + (c_{41}c_{11} + c_{42}c_{31}c_{11} + c_{43}c_{31}c_{11} + \\ &+ c_{43}c_{32}c_{21}c_{11})S_1 + (c_{42}c_{22} + c_{43}c_{32}c_{22})S_2 + \\ &+ c_{43}c_{33}S_3 + c_{44}S_4 \\ &\equiv \sum_{\nu=0}^{4} u_{4\nu}S_\nu, \text{ etc.}\end{aligned} \tag{2.15}$$

In the above equations S_0 is meant normalized, i.e., $S_0 = [2(b-a)]^{-\frac{1}{2}}$, while all other polynomials are given unnomalized.

It follows from Eq. (2.14) that the new polynomials $\{\sigma_n\}$ satisfy, of course, also the relations

$$(z\partial_x + 1)\,\sigma_u(x-\alpha, z) = g(x-\alpha) \equiv \text{function of } x-\alpha \text{ only} \quad (2\ 16)$$

and consequently they conserve this important property of the polynomials $S_n(x-\alpha, z)$.

However, it is easily seen that complicated boundary conditions can be satisfied only, if we associate each polynomial with the corresponding exponential function $(-z)^n \exp\left[-\frac{x-\alpha}{z}\right]$. We define, therefore, our new orthonormalized functions by the new expression

$$\begin{aligned} \bar{\sigma}_n(x-\alpha, z) &= \sum_{\nu=0}^{n-1} \bar{c}_{n\nu}\,\bar{\sigma}_\nu + \bar{c}_{nn}\, f_n^{\pm}(x-\alpha, z)\,, \\ f_n^{\pm}(x-\alpha, z) &\equiv S_n - (-z)^n \exp\left(-\frac{x-\alpha}{z}\right). \end{aligned} \quad (2.17)$$

Clearly, the Eqs. (2.7) - (2.12) and (2.13) - (2.16) remain formally unaltered but the numerical values of the functions differ from those of $\{\sigma_n\}$. In particular the values of the coefficients $\{\bar{c}_{n\nu}\}$ are not the same with $\{c_{n\nu}\}$. The main thing is, of course, that we have the relations

$$(\bar{\sigma}_n\,,\ \bar{\sigma}_\nu) = \delta_{n\nu}\,. \quad (2.18)$$

3. FORMULATION OF THE SOLUTION IN TERMS OF $\bar{\sigma}_n$

The advantage of using the normalized functions $\bar{\sigma}_n$ instead of $\{f_n^{\pm}(x-\alpha, z)\}$ consists in the possibility of developing the scattering kernel in a series in terms of $\{\sigma_n\}$.

In order to find the correct expression of the solution of Eq. (1.4) in terms of the new functions it is useful to observe that the new solution is obtained from the old one by means of a kind of similarity transformation. To see this we write the solution for a critical system in the form

$$\psi_+(x, z) = \sum_{n=0}^{\infty} f_n^+(x-\alpha, z) \cdot q_n\,; \quad z \geqslant 0 \tag{3.1}$$

or in vector form

$$\psi_+ = \mathbf{F}^+ \cdot \mathbf{Q}, \tag{3.2}$$

where

$$\mathbf{F}^+ \equiv \{ f_0^+, \ f_1^+, \ldots f_n^+, \ldots \} \tag{3.3}$$

and

$$\mathbf{Q} \equiv \begin{bmatrix} q_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_n \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{3.4}$$

and with the similar expression for ψ_- valid

$$\psi_- = \mathbf{F}^- \cdot \mathbf{P} \tag{3.1a}$$

Eq. (3.2) can then be transformed by an appropriate matrix, U, in the following manner

$$\psi_+ = \mathbf{F}^+ U\, U^{-1} \mathbf{Q} = \widetilde{\mathbf{F}}^+ \cdot \widetilde{\mathbf{Q}} \tag{3.5}$$

with

$$\widetilde{\mathbf{F}}^+ = \mathbf{F}^+ U, \quad \widetilde{\mathbf{Q}} = U^{-1} \mathbf{Q}. \tag{3.6}$$

We identify now the components of the row vector $\widetilde{\mathbf{F}}^+$ with the ordered functions $\overleftarrow{\sigma}_n$, i.e.,

$$\widetilde{\mathbf{F}}^+ \equiv \{ \overline{\sigma}_0, \ \overline{\sigma}_1, \ \ldots \ \overline{\sigma}_n \ \ldots \} \tag{3.7}$$

and $\widetilde{\mathbf{Q}}$ is a column vector having as components a new set of constant coefficients.

It is easily verified that the matrix U is defined by the relations

$$U_{kl} = \begin{cases} 0\,; & k > l \\ u_{kl}\,; & k \leqslant l\,, \end{cases} \tag{3.8}$$

where $\{u_{kl}\}$ are the coefficients defined in Eq. (2.14).

To proceed further it is necessary to make clear that the functions $\{\bar{\sigma}_n(x-a, z)\}$ and $\{\bar{\sigma}_n(x-b, z)\}$ are mutually orthogonal, since

$$\bar{\sigma}_n(x-\alpha, z) = \begin{cases} \left.\begin{array}{l} \sum\limits_{\nu=0}^{n} u_{n\nu} S_n(x-a, z)\,; \quad z \geqslant 0 \\ 0\,; \quad z < 0 \end{array}\right\} \alpha = a \\ \left.\begin{array}{l} 0\,; \quad z > 0 \\ \sum\limits_{\nu=0}^{n} u_{n\nu} S_n(x-b, z)\,; \quad z \leqslant 0 \end{array}\right\} \alpha = b. \end{cases} \tag{3. 9}$$

Consequently, the full range expansion of the scattering kernel takes into account both sets of functions and Eq. (1. 1) becomes

$$K(z, z'; x, x') = \sum_{n=0}^{\infty} k_n \bar{\tau}_n(x, z)\, \bar{\tau}_n(x', z'), \tag{3. 10}$$

where $$\bar{\tau}_n = \frac{1}{\sqrt{2}} [\bar{\sigma}_n(x-\alpha, z) + \bar{\sigma}_n(x-b, z)]$$

and the solution has the form

$$\psi_+(x, z) = \sum_{n=0}^{\infty} \bar{\sigma}_n(x-a, z)\, \bar{q}_n\,; \quad z \geqslant 0$$

and (3. 10a)

$$\psi_-(x, z) = \sum_{n=0}^{\infty} \bar{\sigma}_n(x-b, z)\, \bar{q}_n\,; \quad z \leqslant 0.$$

Accordingly, we write Eq. (1. 3) in the form

$$\begin{aligned} &\sum_{n=0}^{\infty} (z\partial_x + 1)\, \bar{\sigma}_n(x-a, z)\, \bar{q}_n = \\ &= \lambda \sum_{n=0}^{\infty} k_n [\bar{\sigma}_n(x-a, z) + \bar{\sigma}_n(x-b, z)]\, (\bar{q}_n + \bar{p}_n)/2. \end{aligned} \tag{3. 11}$$

By using the properties of the functions $f^{\pm}(x-\alpha, z)$ one easily verifies that

$$(z\partial_x + 1)\, \bar{\sigma}_n = (z\partial_x + 1)\, \sigma_n = \sum_{\nu=0}^{n} u_{n\nu}^{(\alpha)} \frac{(x-\alpha)^\nu}{\nu\,!}\,; \quad (n = 0, 1, \ldots). \tag{3. 12}$$

Writing Eq. (3. 11) both for $\alpha = a$ and $\alpha = b$ and comparing the l. h. sides we deduce the relation

$$\sum_{n=0}^{\infty} \sum_{\nu=0}^{n} u_{n\nu}^{(a)} \frac{(x-a)^\nu}{\nu\,!}\, \bar{q}_n = \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} u_{n\nu}^{(b)} \frac{(x-b)^\nu}{\nu\,!}\, \bar{p}_n\,. \tag{3. 13}$$

From Eq. (3. 13) it follows that

$$\bar{p}_n = \bar{q}_n . \tag{3. 14}$$

To prove this relation we use the homogeneity and the symmetry of the system.

At points symmetric with respect to the middle plane of the slab x and $a + b - x$ the distribution functions for z and for $-z$ must be equal, i. e.,

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{\sigma}_n (x - a, z) \bar{q}_n &= \sum_{n=0}^{\infty} \bar{\sigma}_n (a + b - x - b, -|z|) \bar{p}_n = \\ &= \sum_{n=0}^{\infty} \bar{\sigma}_n (-(x - a), -|z|) \bar{p}_n .\end{aligned} \tag{3. 15}$$

Recalling the definition, Eq. (2. 14), we find that

$$\bar{\sigma}_n (-(x - a), -|z|) = \sum_{\nu=0}^{n} u_{n\nu}^{(b)} f_\nu^- (-(x - a), -|z|) . \tag{3. 16}$$

From Eq. (2. 4) it follows that

$$\bar{\sigma}_n (-(x - a), -|z|) = \sum_{\nu=0}^{n} (-)^\nu u_{n\nu}^{(b)} f_\nu^+ (x - a), |z|) . \tag{3. 17}$$

Inserting the above results into Eq. (3. 15) we deduce the relation

$$u_{n\nu}^{(a)} = (-)^\nu u_{n\nu}^{(b)} . \tag{3. 18}$$

Upon taking the k-th order derivative with respect to x of both sides of Eq. (3. 13) and putting $x = a$ and $x = b$ the following equations are obtained:

$$\sum_{n=k}^{\infty} u_{nu}^{(b)} \bar{q}_n = \sum_{n=k}^{\infty} \sum_{\nu=k}^{n} u_{n\nu}^{(a)} \frac{(b - a)^{\nu - u}}{(\nu - u)!} \bar{p}_n \tag{3. 19}$$

and

$$\sum_{n=k}^{\infty} u_{nu}^{(b)} \bar{p}_n = \sum_{n=k}^{\infty} \sum_{\nu=k}^{n} u_{n\nu}^{(a)} \frac{(b - a)^{\nu - u}}{(\nu - u)!} \bar{q}_n . \tag{3 20}$$

These equations are compatible only if Eq. (3. 14) holds true. Hence, Eq. (3. 11) can now be written in the simpler form

$$\sum_{n=0}^{\infty} \sum_{\nu=0}^{n} u_{n\nu} \frac{(x - a)^\nu}{\nu !} \bar{p}_n = \lambda \sum_{n=0}^{\infty} k_n \left[\bar{\sigma}_n (x - a, z) + \bar{\sigma}_n (x - b, z)\right] \bar{p}_n . \tag{3. 21}$$

We next multiply both sides of this equation by $\bar{\sigma}_{n'}(x-a, z)$ or $\bar{\sigma}_{n'}(x-b, z)$, and we integrate over $1 \geqslant z \geqslant 0$ or $0 \geqslant z \geqslant -1$ respectively and also over $a \leqslant x \leqslant b$ and we get the equation

$$\sum_{n=0}^{\infty} (R_{nn'} - \lambda u_n) \bar{p}_n = 0\,; \quad (n' = 0, 1, \ldots), \tag{3.22}$$

where

$$R_{nn'} = \sum_{\nu=0}^{n} \int_a^b \int_0^1 \frac{(x-a)^\nu}{\nu!} u_{n\nu} \bar{\sigma}_{n'}(x-a, z)\, dx\, dz. \tag{3.23}$$

Eq. (3.22) represents an eigenvalue problem from which both the eigenvalues $\{\lambda_i\}$ and the eigenvectors $\{\bar{p}_n^{(i)}\}$ can be determined.

It is easy to see that the integrations in Eq. (3.23) can be carried out analytically, and they lead to the expressions

$$R_{nn'} = \sum_{\nu=0}^{n} \sum_{\varrho=0}^{n'} u_{n\nu}^{(a)} u_{n'\varrho}^{(a)} \left[\sum_{\mu=0}^{\varrho} (-)^\varrho \frac{(b-a)^{\varrho-\mu+\nu}}{(\varrho-\mu)!\,(\varrho-\mu+\nu+1)\,(\mu+1)} - \right.$$
$$\left. - (-)^\nu \sum_{\lambda=0}^{\nu} (b-a)^{\nu-\lambda} E_{\lambda+2}(b-a) \binom{\nu}{\lambda} \lambda! - \nu! \right].$$

4. THE ENERGY - DEPENDENT PROBLEM

We consider the energy dependent Boltzmann equation with anisotropic and energy dependent cross sections $\Sigma_s(E, z)$, $\Sigma_t(E)$ and $\Sigma_t(E)$, where $z = \cos\vartheta$ the scattering cosine. This equation in plane geometry has the form

$$[z\partial_x + \Sigma_t(E)]\, \psi(x, E, z) =$$
$$= \int_a^b dx' \int_{E_1}^{E_2} \int_{-1}^{1} \Sigma_s(E', z')\, K(E', z'; E, z)\, \psi(x', E', z')\, dz'\, dE'. \tag{4.1}$$

The distribution function can be written again in the form of Eq. (3.10a) with the only difference that the argument of the functions $\{\bar{\sigma}_n\}$ will now be space and energy dependent.

The form of dependence is chosen such that account is taken both of the homogeneity of the functions $\{f_n^{\pm}(x-\alpha, z)\}$ and of the structure of the l. h. side of the Boltzmann equation, i. e., we have

$$f_n^{\pm}(x-\alpha, z, E) = S_n((x-\alpha)\cdot\Sigma_t(E), z) - (-z)^n e^{-\frac{x-\alpha}{z}\Sigma_t(E)}. \tag{4.2}$$

This definition has the advantage to give a very simple result, when $(z\partial_x + \Sigma_t(E))$ is applied on $f_n^{\pm}(x-\alpha, z, E)$. This is

$$\left(z\partial_x + \Sigma_t(E)\right) f_n^{+} = [\Sigma_t(E)]^n \frac{(x-\alpha)^n}{n!}. \tag{4. 3}$$

To exploit this property we write the function $\sigma_n(x-a, z, E)$ in the form $\sigma_n\left((x-a)\Sigma_t(E), z\right)$ and define

$$f_n^{\pm}(x-a, z, E) = f_n^{\pm}\left((x-a)\Sigma_t(E), z\right). \tag{4. 4}$$

Due to Eq. (4. 4) the orthogonalization of the functions $\{\bar{\sigma}_n\}$ can be done even in the case of the energy dependence in the same way as in Eqs. (2. 17), (2. 18).

We have

$$\begin{aligned} \bar{\sigma}_n\left((x-a)\Sigma_t(E), z\right) = & \sum_{\nu=0}^{n-1} \bar{c}_{n\nu}\,\bar{\sigma}_\nu\left((x-a)\Sigma_t(E), z\right) + \\ & + \bar{c}_{nu}(E)\cdot S_n\left((x-a)\Sigma_t(E), z\right). \end{aligned} \tag{4. 5}$$

It is comfortable to extend this definition now to all coefficients $\bar{c}_{n\nu}$, i.e.,

$$\bar{c}_{n\nu} = \bar{c}_{n\nu}(E). \tag{4. 6}$$

Due to the structure of the energy dependence of the functions it is advisable to define the orthogonalization in the following manner:

$$(\sigma_n, \sigma_{n'}) = \int_{-n}^{1} dz \int_{\bar{a}}^{\bar{b}} \bar{\sigma}_n\left((\bar{x}-\bar{a}), z\right) \sigma_{n'}\left((\bar{x}-\bar{a}), z\right) d\bar{x} = \delta_{nn'} \tag{4. 7}$$

where now $\bar{x} = x\cdot\Sigma_t(E)$ etc.

From the above considerations, Eqs. (4. 6), (4. 7), it follows that the coefficients $\{u_{n\nu}^{(a)}\}$ become now energy dependent and in fact this energy dependence follows from Eqs. (2. 11) and (2. 12) in which, of course, the polynomials $\{S_n\}$ are replaced by the functions $\{f^{\pm}\}$ as defined in Eq. (4. 4). It is useful to observe that the coefficients $\{\bar{c}_{n\nu}(E)\}$ depend on the energy only through $\Sigma_t(E)$. For example

$$\bar{c}_{00}(E) = \left[2(\bar{b}-\bar{a}) + 2\,E_3(\bar{b}-\bar{a}) - \frac{1}{2}E_3\left(2(\bar{b}-\bar{a})\right) - \frac{3}{4}\right]^{-\frac{1}{2}} \tag{4. 8}$$

and

$$\bar{c}_n(E) = \left[(f_1^{+}, f_1^{+})^2 - (f_1^{+}, \bar{\sigma}_0)^2\right]^{-\frac{1}{2}}. \tag{4. 9}$$

In this way the orthogonalization procedure does not involve integrations with respect to the energy, therefore,

$$(f_n^+, f_m^+) = \int_{-1}^{1} dz \int_{\bar{a}}^{\bar{b}} d\bar{x}\, f_n^+ f_m^+ .$$

Since,

$$\bar{\sigma}_n\big((x-\alpha)\Sigma_t, z\big) = \sum_{\nu=0}^{n} u_{n\nu}^{(\alpha)}(E)\, f_\nu^{\pm}\big((x-\alpha)\Sigma_t(E), z\big), \tag{4.10}$$

it follows that

$$\left[z\partial_x + \Sigma_t(E)\right]\bar{\sigma}_n = \sum_{\nu=0}^{n} u_{n\nu}^{(\alpha)}(E)\,\frac{(x-\alpha)^\nu}{\nu!}\left[\Sigma_t(E)\right]^\nu. \tag{4.11}$$

Upon writing

$$\psi_+(x, z, E) = \sum_{n=0}^{\infty} \bar{\sigma}_n(\bar{x}-\bar{a}, z)\cdot \bar{q}_n ; \quad (z \geqslant 0). \tag{4.12}$$

and

$$\psi_-(x, z, E) = \sum_{n=0}^{\infty} \bar{\sigma}_n(\bar{x}-\bar{b}, z)\cdot \bar{p}_n ; \quad (z \leqslant 0). \tag{4.13}$$

for the solution $\psi(x, z, E)$, it follows from Eqs. (4.1) and (4.12) - (4.13) that

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{\nu=0}^{n} u_{n\nu}^{(a)}(E)\,\frac{(x-a)^\nu}{\nu!}\left[\Sigma_t(E)\right]^\nu \bar{q}_n = \\ &= \lambda\sum_{n=0}^{\infty} k_n\left[\bar{\sigma}_n\big((x-a)\Sigma_t(E), z\big) + \bar{\sigma}_n\big((x-b)\Sigma_t(E), z\big)\right]\bar{q}_n , \end{aligned} \tag{4.14}$$

where

$$k_n = \int_{E_1}^{E_2} dE \int_{E_1}^{E_2} dE' \int dz \int dz' \int d\tilde{x} \int d\tilde{x}'\, K(\tilde{x}, \tilde{x}'; z, z')\, \bar{\tau}_n(\tilde{x}, z)\, \bar{\tau}_n(\tilde{x}', z'). \tag{4.15}$$

In order to obtain the spectral equation for the determination of the eigenvalues $\{\lambda\}$ and the eigenvectors $\{\bar{q}_n\}$ we multiply both sides of Eq. (4.14) by $\bar{\sigma}_{n'}(\tilde{x}-\tilde{a}, z)$ and we integrate over $[\tilde{a}, \tilde{b}]$ and $[E_1, E_2]$ $[-1,1]$ and obtain the equations

$$\sum_{n=0}^{\infty} (R_{nn'} - \lambda k_n)\,\bar{q}_n = 0 ; \quad n' = 0, 1, 2, \ldots , \tag{4.16}$$

where

$$R_{nn'} = \sum_{\nu=0}^{n} \int_{E_1}^{E_2} dE\, u_{n\nu}^{(a)}(E) \int_{\tilde{a}}^{\tilde{b}} d\tilde{x}\, \frac{(\tilde{x}-\tilde{a})^\nu}{\nu!} \int_{-1}^{1} \bar{\sigma}_{n'}(\tilde{x}-\tilde{a}, z)\, dz . \tag{4.17}$$

It is noticed that the integrations over $-1 \leqslant z \leqslant 1$ and over $\bar{a} \leqslant \bar{x} \leqslant \bar{b}$ can be made analytically and only the subsequent integration over $E_1 \leqslant E \leqslant E_2$ need to be done by computer, this fact makes the present procedure advantageous with respect to other known methods for solving the energy dependent linear Boltzmann equation.

5. THE BOUNDARY CONDITIONS

We wish now to show that the completely determined solutions as given by Eqs. (4. 12) and (4. 13) satisfy indeed the appropriate physically prescribed boundary conditions.

Since the system considered is the critical one (Fig. 1) in the

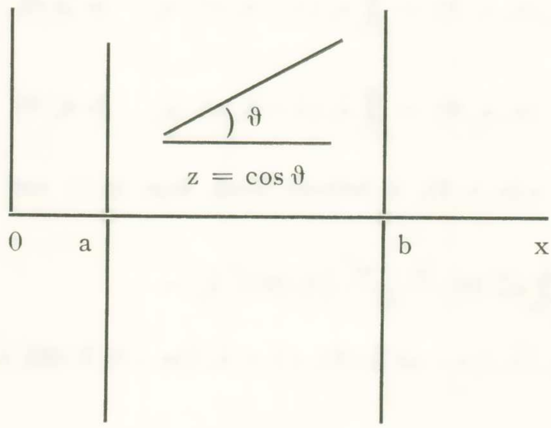

Fig. 1. Plane geometry of a homogeneous system.

present situation the boundary conditions are as follows :

$$\psi_+(a, z, E) = 0 ; \quad \{\forall z \in [0, 1] \wedge \forall E \in [E_1, E_2]\} \tag{5. 1}$$

and

$$\psi_-(b, z, E) = 0 ; \quad \{\forall z \in [-1, 0] \wedge \forall E \in [E_1, E_2]\} \tag{5. 2}$$

First let us show that Eq. (5. 1) is satisfied. This can be written in the form

$$\sum_n \bar{\sigma}_n (0, z)\, q_n = \sum_{n=0}^{\infty} \sum_{\nu=0}^{n} u_{n\nu}^{(a)}(E)\, f_\nu^+ (0, z). \tag{5. 3}$$

From Eq. (4. 4) we see that

$$f_n^+ (0, z, E) = f_n^+ (0, z) \tag{5. 4}$$

or using the definition in Eq. (4. 2) we see that

$$f_n^+(0, z) = \{(-z)^n - (-z)^n = 0. \tag{5. 5}$$

Finally, from the comparison of Eqs. (5. 3) and (5. 5) there follows Eq. (5. 1).

In the same way we see that Eq. (5. 2) is also fulfilled.

Let us next suppose that an inhomogeneous Dirichlet condition is imposed. For example we may assume that

$$\psi_+(a, z, E) = \psi_a(z, E) \tag{5. 6}$$

and

$$\psi_-(a, z, E) = 0. \tag{5. 7}$$

In this case one easily finds that

$$\psi_+(x, z, E) = \psi_a(z, E)\, e^{-\frac{\widetilde{x}-\widetilde{a}}{z}} + \sum_{n=0}^{\infty} \sigma_n(\widetilde{x}-\widetilde{a}, z)\, q_n \tag{5. 8}$$

and

$$\psi_-(x, z, E) = \sum_{n=0}^{\infty} \sigma_n(\widetilde{x}-\widetilde{b}, z) \cdot p_n . \tag{5. 9}$$

It is easily seen that indeed both Eqs. (5. 6) and (5. 7) are satisfied.

ΠΕΡΙΛΗΨΙΣ

Ἡ μέθοδος τῶν πολυωνύμων $\{S_n\}$, τὰ ὁποῖα εἶχον χρησιμοποιηθῆ εἰς προηγουμένας ἐργασίας, ἐφαρμόζονται ἐν προκειμένῳ ὑπὸ τὴν κανονικοποιημένην μορφήν. Ἡ χρησιμότης τῆς μορφῆς αὐτῆς ἐπιδεικνύεται εἰς τὴν περίπτωσιν τοῦ ὑπολογισμοῦ συναρτήσεων κατανομῆς νετρονίων, ὅταν αἱ ἐνεργοὶ διατομαὶ ἐξαρτῶνται ἐκ τῆς ἐνεργείας. Τοῦτο ἐπιτυγχάνεται διὰ τῆς ἀναπτύξεως τῆς συναρτήσεως σκεδάσεως εἰς σειρὰν τῇ βοηθείᾳ τῶν νέων ὀρθοκανονικῶν συναρτήσεων.

REFERENCES

1. C. Syros, Let. Nuov. Cim. **6**, 273, 1973.
2. ———, Acad. Roy. Belg., Bull. Cl. Sc. **6**, 500, 1973.
3. ———, Acad. Roy. Belg., Bull. Cl. Sc. **9**, 926, 1973.
4. ———, Ann. Nucl. Energy **4**, 495, 1977.