ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **On the linearization of nonlinear models of the phenomena. First part: Linearization by exact methods,** *by Demetrios G. Magiros* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰω. Ξανθάκη.

ABSTRACT

This paper deals with remarks on the linearization methods of nonlinear mathematical models of the phenomena.

We can classify the linear methods into two distinct types, the «exact methods» and the «approximate methods».

The advantages and disadvantages of these methods are clarified by appropriately selected examples.

1. INTRODUCTION

The phenomena, either physical or natural or social, are interrelated variations of certain variable quantities and their rate of change, and their nature is usually discussed by mathematical models, where dominant concepts are the linearities and nonlinearities of the models, which usually are differential equations nonlinear in their variables (NLDE).

By «linearization» of a NLDE we mean a reduction to a linear differential equation (LDE), which is either «equivalent» or «almost equivalent» to the NLDE, that is the solution of the LDE may give the solution of the NLDE either «exactly» or «approximately» by an error of small order.

The linearization is a tool for a simplified and easy discussion of the nonlinear phenomena. This tool is helpful in a few cases, but in many cases, especially of practical importance, the linearized models leave out essential features of the NLDE, or they contain properties which are not properties of the NLDE, when the linearization is not acceptable for an adequate description of the real phenomena.

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The aim of this paper is to provide a sketch of ideas and techniques associated with the notion of linearization of NLDE and to exhibit advandages and disadvantages of the linearization methods by using examples of practical significance.

We distinguish two classes of linearization methods : the «exact methods», and the «approximate methods».

The exact methods give exact solutions, essentially general solutions in closed form, and the approximate methods give approximate solutions within an accepted error.

In this part of the paper we deal with «exact methods».

2. LINEARIZATION BY EXACT METHODS

NLDE may be reduced to LDE by «exact transformations of the variables». By this linearization we may obtain general solutions of the NLDE in a «closed form», by using the solutions of the LDE and the transformation formulae of the variables.

Also by exact linearizations, corresponding to some restrictions of the variables or of the parameters, it is possible to find general solutions of special cases of the original NLDE.

The following examples may be sufficient to show the essence and the applicability of the exact linearization methods and their important results.

In some of the examples use is made of appropriate transformations of the variables, and in some other examples appropriate restrictions of the variables or of the parameters.

Example 1. The Bernoulli equation :

$$y' + ay = by^n \tag{1}$$

where a and b are functions of x, and $b \neq 0$, $n \neq 0$, $\neq 1$, is a monumental example of exact linearization.

We change the variable y into a new variable z, keeping the same independent variable x, by the transformation formula :

$$z = y^{1-n} \tag{1.1}$$

when the linear equation in z results :

$$z' + (1-n)\alpha z = (1-n)b \tag{1.2}$$

By solving (1.2), then using (1.1), we can get the solution of (1) in an exact and closed form.

In the specific example :

$$y' - \frac{2}{x}y = 4x^2y^{1/2}, \quad x \neq 0$$

which is of Bernoulli type, the transformation $z = y^{1/2}$ leads to the LDE :

$$z' - \frac{1}{x}z = zx^2$$

of which, by using $\mu = \frac{1}{x}$ as an integrating factor, we find the general solution : $z = (cx + x^3)$, when the general solution of the original NL equation is : $y = (cx + x^3)^2$.

As we see, the single constant of integration enters the general solution of (1) in a nonlinear way.

Example 2. The Ricatti equation :

$$y' = \alpha y^2 + by + c \tag{2}$$

where α, b, c are functions of x, and $\alpha \neq 0$.

Liouville proved (1841) that this simple NLDE in its general form can not be solved by elementary exact methods.

(a): This equation can be linearized to a LDE of second order.

Transforming (2) by

$$y(x) = \frac{v(x)}{\alpha(x)} \tag{2.1}$$

we get :

$$v' = v^2 + \left(\frac{\alpha'}{\alpha} + b\right)v + \alpha c \tag{2.2}$$

which is NL in the new vrriable v. Now, transforming (2.2) by :

$$v(x) = -\frac{u'(x)}{u(x)} \tag{2.3}$$

we get the LDE of second order in u(x) with variable coefficients :

$$u'' - \left(\frac{\alpha'}{\alpha} + b\right)u' + \alpha cu = 0 \tag{2.4}$$

of which the solution can not, in general, be expressed in exact form in term of a finite number of elementary functions.

.(b): We can linearize the Ricatti equation (2) by specific «restrictions», e.g. when we know a particular solution $y_1(x)$.

By the transformation:

$$y(x) = y_1(x) + z(x) \tag{2.5}$$

the equation (2) leads to the Bernoulli equation in z:

$$z' - (b + 2\alpha y_1) z = \alpha z^2 \tag{2.6}$$

which can be solved by linearization, and from its solution and the transformation (2.5) we can get the general solution of (2).

E x a m p l e 3. The NLDE:

$$y = \frac{X + yZ}{Y + xZ} \tag{3}$$

where X, Y, Z are homogeneous functions of x, y, the X, Y of degree μ of homogeneity, and Z of degree ν, can be solved by an exact linearization. Transforming by:

$$y = xz \tag{3.1}$$

and using the homogeneity property of X, Y, Z, equation (3) becomes:

$$\frac{dx}{dz} + A(z)\, x = B(z)\, x^{\nu-\mu+2} \tag{3.2}$$

which is of Bernoulli type. The functions A(z) and B(z) are known functions, calculated in the process to find (3.2).

E x a m p l e 4. The Lagrange equation:

$$y = x\,\varphi(y') + \psi(y') \tag{4}$$

which is linear in x and y, but nonlinear in y', can be solved by an exact linearization.

Differentiating (4) with respect to x, putting $y' = p$, then considering x as dependent variable and p as independent variable, we get:

$$(\varphi(p) - p)\frac{dx}{dp} + x\,\varphi'(p) + \psi'(p) = 0 \tag{4.1}$$

which is linear in x. If $x = x(p)$ is the solution of (4. I), inserting $x = x(p)$ into (4), we have the function:

$$y = x(p)\,\varphi(p) + \psi(p) \tag{4. 2}$$

which is the general solution of (4) in an exact closed form with p as a parameter.

E x a m p l e s 5. The NLDE: (a): $f(y') = 0$, (b): $f(y'') = 0, \ldots$ can be solved by an exact linearization.

.(a): If in (a) we put $y' = k$, when $y = kx + c$, that is $k = \frac{y-c}{x}$ then the general solution of (a) is: $f\left(\frac{y-c}{x}\right) = 0$, $c =$ parameter. E. g. the equation: $(y')^4 - 1 = 0$ has $\left(\frac{y-c}{x}\right)^4 - 1 = 0$ as general solution. We have:

$$\left(\frac{y-c}{x}\right)^4 - 1 = \left\{\left(\frac{y-c}{x}\right)^2 + 1\right\} \cdot \left(\frac{y-c}{x} - 1\right)\left(\frac{y-c}{x} + 1\right).$$

The first factor is not zero, the other factors may be zero, then the general solution of this specific example is the couple of the two families of lines: $y = x + c$, and $y = -x + c$ that is all lines in the x, y —plane parallel to the first and second bisector.

.(b): If in (b) we put $y'' = k$, when $y = \frac{k}{2}x^2 + c_1 x + c_2$, or $k = \frac{2}{x^2}(y - c_1 x - c_2)$ then the general solution of (b) is:

$$f\left(\frac{2}{x^2}(y - c_1 x - c_2)\right) = 0, \quad c_1 \text{ and } c_2 \text{ parametrs.}$$

E x a m p l e 6. We may have a linearization of a complicated NLDE by factorization, e. g., if the NLDE is factorized as:

$$(y^2 + y'^2 + 1) \cdot y'' \cdot (x^3 y''' + x^2 y' - 2xy' + 2y) = 0 \tag{6}$$

This equation is equivalent to the system:

$$\text{(a)}: \ y'' = 0, \qquad \text{(b)}: \ x^3 y''' + x^2 y'' - 2xy' + 2y = 0 \tag{6. 1}$$

The first factor can not be zero. The second equation of (6.1) is of Euler type. The solutions of (6.1) are:

$$(a): \quad y = c_1 x + c_2, \qquad (b): \quad y = c_3 x + c_4 x^{-1} + c_5 x^2 \tag{6.2}$$

where $c_1, \ldots, c_5$ are parameters. Therefore the general solution of (6) is given by both equations (6.2), and from each point of the x, y -plane two solution curves of (6.2) pass, one from the family (6.2.a) and one from the family (6.2.b).

We may have other cases of exact linearization of NLDE [1], [2].

In the following two examples the exact linearization is of different nature:

E x a m p l e 7. The problem of mechanics of the free rotation of a rigid body with any «mass distribution» is governed by the Euler system of NLDE:

$$\left.\begin{array}{l} I_1 \dot{\omega}_1 + (I_3 - I_2)\,\omega_3\,\omega_2 = 0 \\ I_2 \dot{\omega}_2 + (I_1 - I_3)\,\omega_1\,\omega_3 = 0 \\ I_3 \dot{\omega}_3 + (I_2 - I_1)\,\omega_2\,\omega_1 = 0 \end{array}\right\} \tag{7}$$

where $\omega_1, \omega_2, \omega_3$ are the unknown angular velocity components, and the parameters I_1, I_2, I_3 are the moments of inertia of the given rigid body about the body coordinate system with origin the mass center of the body. We can linearize this nonlinear system, if we accept a «special mass distribution», which corresponds to a «special restriction of the parameters» I_1, I_2, I_3. If the mass distribution of the rotating body is such that the body has an axis of symmetry, say the ω_1-axis, then $I_2 = I_3 = I$, and (7) leads to the linear coupled system:

$$\dot{\omega}_2 = c_1 \omega_3, \qquad \dot{\omega}_3 = -c_1 \omega_2 \tag{7.1}$$

where: $$c_1 = \bar{\omega}_1 (I - I_1)/I = \text{constant}.$$

The above linearization, occurring by a restriction of the parameters, corresponds to a physical problem, which is a special case of the initial problem expressed by (7), and the solution of the linearized system (7.1) is a special case of (7).

Example 8. The «pure Keplerian motion» of a body of mass m around a central body of mass M is due to the attractive Newtonian force of the bodies, in the absence of perturbing forces. This motion is governed by the NLDE in vector form:

$$\ddot{x} + \frac{k^2}{r^3} x = 0 \tag{8}$$

where $k = K(M + m) = \text{constant}$, $x = (x_1, x_2, x_3)$, $r^2 = x_1^2 + {}_2^2 + x_3^2$.

In the following we see two types of exact linearizations of (8), one by «regularization», and the other by «restriction of the variables» [(3)].

(a): Regularization of (8). We use an auxiliary equation, and change the independent variable.

The equation (8) is «singular» with the origin as the singularity.

When the motion of m is close to M, it is a «near collision» motion, when large gravitational forces appear and sharp bends of the orbit.

Such a phenomenon occurs when, e.g., an artificial space vehicle is at its start or at its destination.

By appropriate transformation of (8), it is possible to get a regular equation free ot singularities. This is called: «regularization». In the one-dimension case, (8) becomes:

$$\ddot{x} + \frac{k^2}{x^2} = 0 \tag{8.1}$$

and the energy function of the pure Keplerian motion is:

$$h_k = \frac{k}{x} - \frac{1}{2} \dot{x}^2 \tag{8.2}$$

where the energy h_k is a negative constant.

If, in these equations, instead of the «natural time» t, the «artificial time» τ is taken according to:

$$dt = x\, d\tau \tag{8.3}$$

the equations (8.1) and (8.2) become:

$$xx'' - x'^2 + k^2 x = 0 \tag{8.4}$$

$$x'^2 = 2(k^2 x + h_k x^2) \tag{8.5}$$

where τ is the independent variable.

Inserting now (8.5) into (8.4) we have the LDE

$$x'' + 2h_k x = k^2 \tag{8.6}$$

which is the «regularization» of (8).

(b): Restriction of the Variables in (8). By restricting the variables x_1, x_2, x_3 according to:

$$r^2 = x_1^2 + x_2^2 + x_3^2 = \text{constant} \tag{8.7}$$

we linearize (8), when it reduces to the LDE

$$\ddot{x} + vx = 0 \tag{8.8}$$

with $v = kr^{-3/2}$. The solution of (8.8) is:

$$x = a \cos vt + b \sin vt \tag{8.9}$$

with a and b are constant vectors.

The restriction (8.7) specializes the motion of m to be a motion on the surface of the sphere (8.7), and, since the motion of m is only under the influence of a central force, the motion of m is circular on a plane through the origin with period of revolution: $T = \frac{2\pi}{v} = \frac{2\pi}{k} r^{3/2}$ and velocity

$$\dot{x} = v(-c_1 \sin vt + c_2 \cos vt) \tag{8.10}$$

of magnitude:

$$U = vr = \frac{k}{\sqrt{r}} = \text{constant}. \tag{8.11}$$

By considering r of (8.7) as a parameter, the above linearization gives various circular motions of m around M inside a sphere of radius the maximum of the parameter r.

SUMMARY

The mathematical models of the phenomena are usually NLDE, which is very difficult to be solved. Some classes of these equations can be treated by special methods, among which are the linearization methods, either exact or approximate.

In the exact linearization methods one may use appropriate transformation of the variables, when one may have general solutions of the NLDE in closed form, which is an ideal case.

Also, one may use restrictions of the variables or of the parameters of the NLDE, when one may have some special subclasses of the general solutions of the NLDE.

For a single NLDE one may have more than one formula for its general solution, and a NLDE may have, in addition, singular solutions.

ΠΕΡΙΛΗΨΙΣ

Τὰ μαθηματικὰ μοντέλα τῶν διαφόρων φαινομένων εἶναι, συνήθως, μὴ γραμμικαὶ διαφορικαὶ ἐξισώσεις, τῶν ὁποίων ἡ λύσις εἶναι, ἐν γένει, εἴτε ἀδύνατον εἴτε πολὺ δύσκολον νὰ εὑρεθῇ. Διὰ μερικὰς κατηγορίας τῶν μὴ γραμμικῶν ἐξισώσεων δύνανται νὰ ἐφαρμοσθοῦν εἰδικαὶ μέθοδοι, μεταξὺ τῶν ὁποίων εἶναι καὶ ἡ «γραμμικοποίησις», εἴτε ἀκριβὴς (exact) εἴτε κατὰ προσέγγισιν (approximate). Παλαιότερα, ἡ σπουδὴ τῶν μὴ γραμμικῶν ἐξισώσεων διὰ γραμμικοποιήσεως (διαγραφῆς τῶν μὴ γραμμικῶν ὅρων τῶν ἐξισώσεων) ἦτο ἡ δεσπόζουσα μέθοδος, ἰδίως εἰς προβλήματα ἐφαρμογῶν. Πρὸ μερικῶν δεκαετηρίδων ὅμως ἀπεδείχθη ὅτι ἡ γραμμικοποίησις ἐξαλείφει βασικὰς ἰδιότητας τῶν λύσεων τῶν μὴ γραμμικῶν ἐξισώσεων καὶ ὅτι οἱ μὴ γραμμικοὶ ὅροι τῶν ἐξισώσεων παίζουν δεσπόζοντα ρόλον εἰς τὴν ἔρευναν τῶν φαινομένων.

Διὰ τῆς παρούσης ἐργασίας, ὅπως καὶ διὰ μιᾶς ἄλλης ποὺ θὰ ἐπακολουθήσῃ, δίδονται παρατηρήσεις ἐπὶ τῶν μεθόδων γραμμικοποιήσεως, κυρίως ἐπὶ τῆς καταλληλότητος ἢ μὴ τῆς γραμμικοποιήσεως ὡς μεθόδου ἐρεύνης τῶν φαινομένων. Εἰδικῶς, εἰς τὴν παροῦσαν ἐργασίαν ἐξετάζονται ἀκριβεῖς μέθοδοι γραμμικοποιήσεως. Εἰς αὐτὰς τὰς μεθόδους γίνεται χρῆσις καταλλήλων μετασχηματισμῶν τῶν μεταβλητῶν, ὁπότε εἶναι δυνατὸν νὰ ἐπιτευχθοῦν «γενικαὶ λύσεις» τῶν μὴ γραμμικῶν ἐξισώσεων ὑπὸ «κλειστὴν μορφὴν» (closed form), τὸ ὁποῖον εἶναι ἰδεῶδες ἐπίτευγμα.

Ἐπίσης, μὲ κατάλληλον περιορισμὸν τῶν μεταβλητῶν ἢ τῶν συντελεστῶν τῶν μὴ γραμμικῶν ἐξισώσεων, εἶναι δυνατὸν νὰ ἐπιτευχθοῦν γενικαὶ λύσεις, αἱ ὁποῖαι εἶναι εἰδικαὶ περιπτώσεις τῆς γενικῆς λύσεως τῶν μὴ γραμμικῶν ἐξισώσεων.

Μία μὴ γραμμικὴ ἐξίσωσις ἐνδέχεται νὰ ἔχῃ περισσοτέρας τῆς μιᾶς γενικὰς λύσεις, ὅπως ἐπίσης ἐνδέχεται νὰ ἔχῃ, ἐπὶ πλέον, καὶ ἀνωμάλους (singular) λύσεις.

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Ὁ Ἀκαδημαϊκὸς κ. **Ἰω Ξανθάκης,** παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν, εἶπε τὰ ἑξῆς :

Ἔχω τὴν τιμὴν νὰ ἀνακοινώσω εἰς τὴν Ἀκαδημίαν ἐργασίαν τοῦ κ. Δημητρίου Μαγείρου διὰ τὴν γραμμικοποίησιν μὴ γραμμικῶν Μαθηματικῶν Προτύπων τῶν Φαινομένων.

Ὡς γνωστὸν τὰ μαθηματικὰ πρότυπα (μοντέλα) διαφόρων φυσικῶν φαινομένων μᾶς ὁδηγοῦν συνήθως εἰς μὴ γραμμικὰς διαφορικὰς ἐξισώσεις, τῶν ὁποίων αἱ λύσεις εἶναι, κατὰ γενικὸν κανόνα, εἴτε ἀδύνατοι εἴτε πολὺ δύσκολοι. Λόγῳ τῆς δυσχερείας ταύτης, μέχρι πρό τινων ἐτῶν, διεγράφοντο οἱ μὴ γραμμικοὶ ὅροι τῶν ἀντιστοίχων διαφορικῶν ἐξισώσεων ἰδίως εἰς προβλήματα ποὺ ἀφεώρων ἐφαρμογάς· ἀλλὰ ἡ γραμμικοποίησις τῶν διαφορικῶν ἐξισώσεων διὰ τῆς μεθόδου ταύτης, δηλαδὴ τῆς διαγραφῆς τῶν μὴ γραμμικῶν ὅρων, ἐξαλείφει βασικὰς ἰδιότητας τῶν λύσεων τῶν μὴ γραμμικῶν διαφορικῶν ἐξισώσεων, διότι διαγραφόμενοι μὴ γραμμικοὶ ὅροι διαδραματίζουν ὡς ἐπὶ τὸ πλεῖστον δεσπόζοντα ρόλον εἰς τὴν ἔρευναν τῶν φαινομένων. Εἰς ὡρισμένας κατηγορίας μὴ γραμμικῶν διαφορικῶν ἐξισώσεων δύνανται νὰ ἐφαρμοσθοῦν εἰδικαὶ μέθοδοι, ποὺ στηρίζονται εἰς γραμμικοποποίησιν κατὰ τρόπον ἀκριβῆ ἢ κατὰ προσέγγισιν.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν ὁ κ. Μάγειρος ἀσχολεῖται μὲ ἀκριβεῖς μεθόδους γραμμικοποιήσεως, τὰς ὁποίας ἐφαρμόζει εἰς τὰς ἐξισώσεις Bernouilli, Ricatti, καθὼς καὶ εἰς διαφόρους ἄλλας μορφὰς διαφορικῶν ἐξισώσεων.