ΜΑΘΗΜΑΤΙΚΑ.— **Characteristic Properties of Linear and Nonlinear Systems,** *by Demetrios G. Magiros* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰω. Ξανθάκη.

INTRODUCTION

In two previous papers (published in: Practica of Athens Academy, June 1976; and as GE Reports: (a) 76SDRO26, 6/28/76; and b) 76SDRO27, 7/1/76, we examined exact and approximate methods of linearization of nonlinear systems.

In the present paper the main characteristic properties of linear and nonlinear systems will be discussed. Some of these properties characterize only the linear systems, and some others only the nonlinear systems. Some properties of NLS disappear by its reduction to a LS, and, therefore, the nature of the problem associated with the NLS, accompanied by the knowledge of the properties of the systems, decide whether the linearization of the NLS is permitted or not.

Nonlinear systems, that is phenomena whose behavior can be described by models which are nonlinear differential equations (NLDE), become increasingly important in many fields, as in astronomy, space flight, automatic control, biology, economics.

Since by linearization of nonlinear systems important features of the phenomena are neglected, it is necessary to know general features, basic striking properties and characteristic peculiarities of linear and nonlinear systems.

We discuss this subject here, and the discussion is illustrated by simple examples.

I. THE PRINCIPLE OF SUPERPOSITION

This principle, first stated by D. Bernoulli (1775) and used by Fourier (1822) in his theorem, holds in linear systems (LS) and characterizes them, but in general, does not hold in nonlinear systems (NLS).

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This principle consists of two properties:

(a): The sum of any number of linearly independent particular solutions of a DE is also a solution of the DE; and

(b): Any constant multiple of a solution is also a solution.

In homogeneous LS the above principle holds and it characterizes completely these systems. By using this principle, one can obtain the general solution of these systems as a linear combination of some easy to get special solutions, the fundamental set of solutions.

In nonhomogeneous LS:

$$\dot{x}_i = Ax_i + h_i(t) \tag{1}$$

where A is a $(n \times n)$ matrix and $h_i(t)$ a «forcing function» or «input», the above principle means that: if x_1 is a solution of system (1) with input h_1, and x_2 another solution with input h_2, then $c_1 x_1 + c_2 x_2$ is solution of (1) with input: $c_1 h_1 + c_2 h_2$. The c_1 and c_2 are arbitrary constants.

• In NLS the principle of superposition does not hold, and, then, its general solution, if it exists, can not be formulated in the simple way as in L. S.

There are NLDE where only the property (b) of the above holds, but not the property (a). For example, for the equation:

$$y''^2 + yy' + y'^2 = 0 \tag{1.1}$$

ot which the terms have the same degree (two) in y, y', y'' we see that:

(i): If y_1 and y_2 are «linearly independent» solutions, then $y = y_1 + y_2$ is not a solution, but $y = c_1 y_1$ and $y = c_2 y_2$ are solutions;

(ii): If y_1 and y_2 are «Linearly dependent» solutions, that is $y_2 = cy_1$, then this equation has $y = y_1 + y_2$ as a solution.

II. THE GLOBAL PROPERTY

The global (or predictability, or provincial) property characterizes the LS, but not the NLS.

• In LS the local behavior of the solutions implies their global behavior. That is, the global behavior can be predicted from the local behavior, then the LS, which may be defined for all values of time, are by nature «provincial».

• In NLS this property does not hold, the global behavior of NLS can not be implied from their local behavior, that is, the «unpredictability» characterizes the NLS. In NLS it may not be possible to extend the solutions beyond a certain time, or these solutions need not be defined for all values of time.

The linearization of a NLS may help to get local properties of NLS. By the following simple example the above are clarified. [4]

Let us take a LS and a NLS :

$$(a): \ \dot{x} = -x, \quad x(0) = x_0 ; \qquad (b): \ \dot{x} = -x + \varepsilon x^2, \quad x(0) = x_0 \tag{2}$$

where ε is a parameter. Their solutions are, respectively:

$$(a): \ x(t) = x_0 e^{-t} ; \qquad (b): \ x(t) = \frac{x_0}{\varepsilon x_0 - (\varepsilon x_0 - 1)\, e^{t}} \tag{2.1}$$

graphically shown in Figure 1.

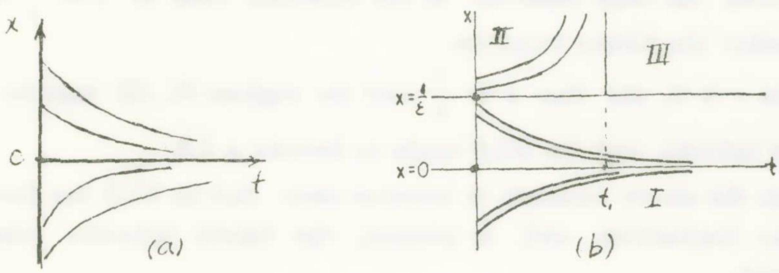

Fig. 1.

The equilibrium point of (2. a) is $x = 0$, and of (2. b) $x = 0$, $x = \frac{1}{\varepsilon}$. For both systems (2), $x = 0$ is «asymptotically stable», but $x = \frac{1}{\varepsilon}$ is «unstable».

The solution (2. 1b) for $x > \frac{1}{\varepsilon}$ becomes infinite for the finite time:

$$t_1 = \log \frac{\varepsilon x_0}{\varepsilon x_0 - 1} . \tag{2.2}$$

In Figure (1. b) we distinguish three regions: I, II, III.

In region I $\left(0 \leqslant t,\ x < \frac{1}{\varepsilon}\right)$ all solutions have the t-axis as an asymptote;

In region II $\left(0 \leqslant t \leqslant t_1,\ x > \frac{1}{\varepsilon}\right)$ the solutions become unbounded for the finite time t_1; and

In region III $\left(t_1 \leqslant t,\ x > \frac{1}{\varepsilon}\right)$ there are no solutions.

We see that:

(i): In region I and close to origin, the solutions (2.1.b) have the same topological behavior as in the linear case (2.1.a), which means that the linearization of (2.b) in the region I gives useful information, and the local behavior of the solutions of both system (2) in I implies their global behavior.

(ii): In region II, the nonlinear term εx^2 causes the existence of a new phenomenon, and the linearization makes this phenomenon disappear.

From the local behavior of the solutions close to $x = \frac{1}{\varepsilon}$ one can not predict the future behavior.

As $\varepsilon \to 0$, the line $x = \frac{1}{\varepsilon}$ and the regions II, III tend to disappear at infinity, and the NLS tends to become a LS.

By the above example it becomes clear that in NLS the linearization has limitations, and, in general, the future behavior cannot be predicted.

III. LIMIT CYCLES

Limit cycles may be a phenomenon of NLS, but never a phenomenon of LS. Periodic phenomena of LS or NLS correspond to closed trajectories, called «cycles» with a period or frequency a finite number.

The cycles may constitute a «continuum spectrum», but may be isolated cycles, called «limit cycles», when in a neighborhood of them no other cycles exist.

The limit cycles must not be confused with some nonperiodic closed trajectories, which are members of a special class of solution-curves of systems, the «separatrices».

If, in a LS, a periodic solution y exists, then, due to the principle of superposition, which holds in LS, cy must be also a periodic solution, and since c is an arbitrary constant, no limit cycles exist in LS.

In some NLS, due to the special nature of their nonlinearities, limit cycles may exist, and for each NLDE the problems of existence of one or several limit cycles, their uniqueness and stability, as well as the construction of their boundaries and the calculation of their periods are important and, in general, difficult problems.

Stable limit cycles correspond to important physical phenomena, as, for example, to self-excited oscillations.

E x a m p l e 1. The system: [8]

$$\left.\begin{aligned} \dot{x} &= y + \frac{x}{\sqrt{x^2+y^2}}\left\{1-(x^2+y^2)\right\} \\ \dot{y} &= -x + \frac{y}{\sqrt{x^2+y^2}}\left\{1-(x^2+y^2)\right\} \end{aligned}\right\} \tag{3}$$

has the unit circumference $x^2+y^2=1$ as a «stable limit cycle». Indeed transforming (3) into polar coordinates and using: $x\dot{x}+y\dot{y}=r\dot{r}$, we have:

$$\dot{r} = 1-r^2, \quad \dot{\theta} = 1 \tag{3.1}$$

of which the solution is:

$$r = \frac{ce^{2t}-1}{ce^{2t}+1}, \quad \theta = t + c_1 \tag{3.2}$$

where $c = \dfrac{1+r_0}{1-r_0}$ and c_1 are the integration constants, r_0 the initial condition. If $r_0 < 1$, then, as $t \to \infty$, $r \to 1$ from inside. If $r_0 > 1$, then, as $t \to \infty$, also $r \to 1$ but from outside, when $x^2+y^2=1$ is a «stable limit cycle» of (3).

E x a m p l e 2. The system:

$$\left.\begin{aligned} \dot{x} &= -y + x(x^2+y^2-1) \\ \dot{y} &= x + y(x^2+y^2-1) \end{aligned}\right\} \tag{3.3}$$

has the unit circumference $x^2+y^2=1$ as an «unstable limit cycle».

This system, in polar coordinates, is written in the form:

$$\dot{r} = r(r^2-1), \quad \dot{\theta} = -1 \tag{3.4}$$

with a solution :

$$r = \frac{1}{\sqrt{1 - ce^{2t}}}, \quad \theta = t + c_1 \tag{3.5}$$

where $c = \dfrac{t_0^2 - 1}{r_0^2}$ and c_1 are the integration constants, and t_0, r_0 the initial conditions. We can check that $x^2 + y^2 = 1$ is an «unstable limit cycle» of (3.3).

Example 3. The NLDE ;

$$\ddot{x} + 3x - 4x^3 + x^5 = 0 \tag{3.6}$$

has infinitely many cycles, and some closed trajectories of special type (separatrices), but no limit cycles. We explain this statement.

The points : $x = 0$, $x = \pm 1$, $x = \pm\sqrt{3}$ on the x-axis are singular points of (3.6); the points : $x = 0$, $x = \pm\sqrt{3}$ are centers, and $x = \pm 1$ saddle points. [7.b]

The phase portrait of (3.6) is shown in Figure 2. There are special solution curves from a saddle point to another one, the «separatrices»

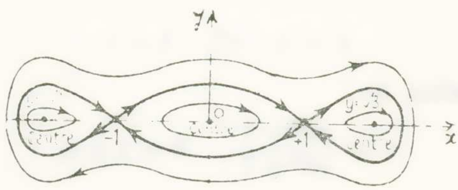

Fig. 2.

of (3.6), which separate the whole x, y-plane into four distinct regions in each of which there is a continuum spectrum of cycles. To go from one saddle point to the other, following a separatrice, theoretically infinite time is needed, and (3.6) has no limit cycles.

IV. SELF-EXCITED OSCILLATIONS

Self-excited (or self-sustained) oscillations are special periodic phenomena corresponding to stable limit cycles. They do not exist in LS, but may exist in NLS. They can be produced in NLS, where the nonlinearities appear in the damping forces, without the influence of external forces, that is in NLS of the form: $\ddot{x} + \varepsilon\varphi(x, \dot{x}) + kx = 0$.

In particular, the form:

$$\ddot{x} + \varepsilon\varphi(\dot{x}) + kx = 0 \tag{4}$$

is very useful. ε and k are constants.

To the nonlinearity of this equation we can give another useful form. We differentiate (4) with respect to the independent variable t then put $\dot{x} = y$, when (4) reduces to:

$$\ddot{y} + \varepsilon\psi(y)\dot{y} + \varkappa y = 0 \tag{4.1}$$

where $\psi(y)$ is the derivative of $\varphi(\dot{x})$ with respect to $\dot{x} = y$.

The above NLE can be transformed into another form where $k = 1$, by changing the independent variable t into a new one τ, according to $\tau = kt$, when, e.g., (4) can be written in the form:

$$x'' + \varepsilon\varphi_1(x') + x = 0 \tag{4.2}$$

where $\varphi_1(x') = \frac{1}{k^2}\varphi(kx')$, and the derivatives are taken with respect to τ.

Electrical systems involving vacuum tubes, mechanical systems of action of solid friction, the Froude's pendulum, and other systems, which can be formulated as special cases of the above NLDE, can execute selfexcited oscillations.

Rayleigh (1883) first studied this kind of oscillations in connection with acoustical phenomena, then Van der Pol (1927) in connection with electrical phenomena.

The Rayleigh equation is:

$$\ddot{x} + (-\alpha + b\dot{x}^2)\dot{x} + kx = 0 \tag{4.3}$$

which is of the form (4). The Van der Pol equation is:

$$\ddot{y} - \varepsilon(1 - y^2)\dot{y} + y = 0 \tag{4.4}$$

wich is of the form (4.1).

The Rayleigh equation (4.3) can be reduced to the Van der Pol equation (4.4), by changing the variables t and x in (4.3) into new variables τ and y, respectively, according to formulae:

$$\tau = \sqrt{k}\,t, \quad y = \sqrt{\frac{3bk}{\alpha}}\,\dot{x} \tag{4.5}$$

and getting $\varepsilon = \dfrac{\alpha}{\sqrt{k}}$.

- We give some special examples.

Example 1. We take a special case of the Rayleigh equation (4.3) with $\alpha = 1$, $b = \frac{1}{3}$, $k = 1$, that is the NLDE:

$$\ddot{x} - \dot{x} + x + \frac{1}{3}\dot{x}^3 = 0. \tag{4.6}$$

The linear part of (4.6):

$$\ddot{x} - \dot{x} + x = 0 \tag{4.7}$$

has $\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ as eigenvalues, then its origin is unstable and the solutions around the origin are spirals that wind away from the origin.

If we add to (4.7) the nonlinearity $\frac{1}{3}\dot{x}^3$, we have (4.6) which in the phase plane can be written in the form:

$$\frac{dy}{dx} = \frac{-x + y - \frac{1}{3}y^3}{y}. \tag{4.8}$$

Applying the Liénard's graphical method, Figure (3a), we find a single stable limit cycle as a closed solution, Figure (3b), corresponding to self-excited oscillations.

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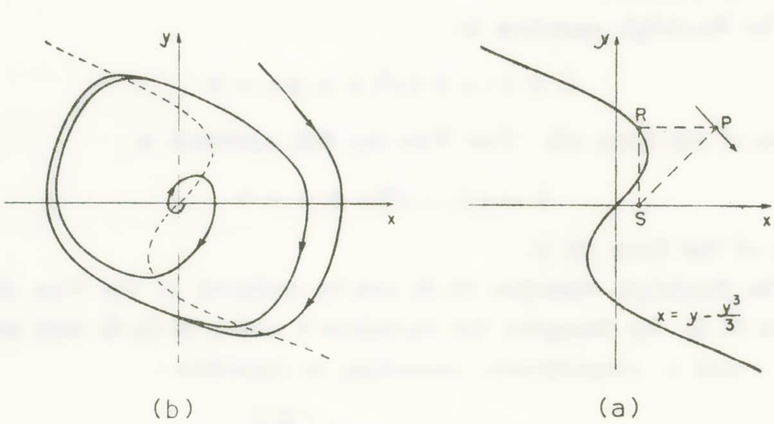


Fig. 3.

Example 2. The Van der Pol equation:

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0 \tag{4.9}$$

is equivalent to the equation:

$$\frac{dy}{dx} = \frac{-x + \varepsilon(1 - x^2)y}{y}. \tag{4.10}$$

For $\varepsilon = 0$, the general solution of (4.10) is the family of concentric circles with center the origin. For $\varepsilon = 1$, application of isocline method shows the limit cycle as in Figure 4, corresponding to self-excited oscillations.

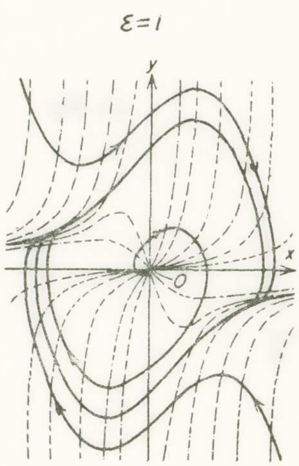


Fig. 4.

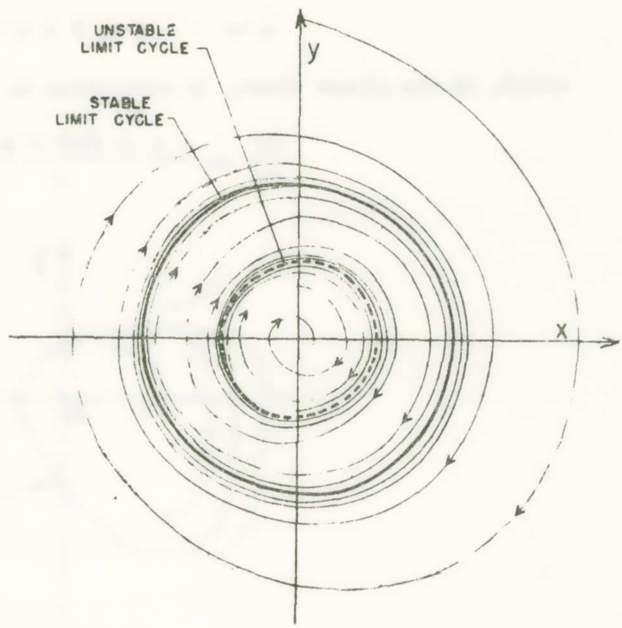


Fig. 5.

Example 3. The equation: [2]

$$\ddot{x} + \varepsilon(1 - \alpha\dot{x}^2 + b\dot{x}^4)\dot{x} + x = 0 \tag{4.11}$$

for $\varepsilon = 1$ is equivalent to:

$$\frac{dy}{dx} = \frac{-x - (1 - \alpha y^2 + by^4)y}{y} \tag{4.12}$$

of which the graphical solution, as shown in Figure 5, has two limit cycles, one unstable and the other stable, corresponding to self-excited oscillations.

Example 4. The equation:

$$\ddot{x} + x = \alpha\dot{x} + bx\dot{x} + c\dot{x}^2 + dx^2 \tag{4.13}$$

can be produced in the theory of a common cathode generator, taking into account the anode reaction, if the value characteristic is represented by a quadratic polynomial. [11]

This equation in case: $\alpha = 0.2$, $b = 1$, $c = -1$, $d = c$ becomes:

$$\ddot{x} = -x + (0.2 + x - \dot{x})\dot{x} \tag{4.14}$$

which, in the phase plane, is equivalent to

$$\frac{dy}{dx} = \frac{-x + (0.2 + x - y)y}{y}. \tag{4.15}$$

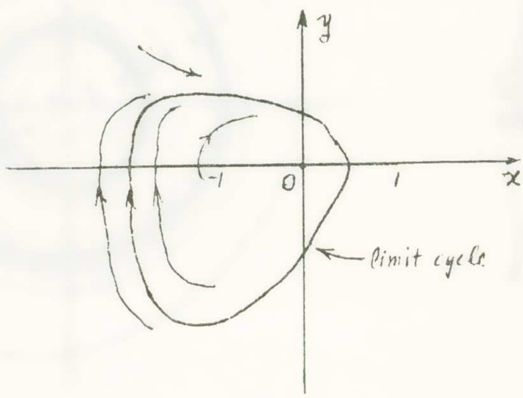


Fig. 6.

Equation (4.15) has one equilibrium point, the origin, which is unstable, and the method of isoclines, applied to (4.15), leads to Figure 6, where we see that a stable limit cycle exists, corresponding to self-excited oscillations. By using a graphical method, we can check that the shape of the limit cycle is distorted with the increase of the parameter α, becoming more and more non-sinusoidal.

V. THE PHENOMENA OF SUBHARMONIC RESPONSE

Subharmonic phenomena occur in some NLS but, in general, not in LS. They appear when the systems are subjected to external periodic forces, say sinusoidal.

If the frequency of the external force is ω, the system, under the influence of such a force, may exhibit periodic motions with frequency $\frac{\omega}{n}$, where $n = 2, 3, \ldots$, and such motions are, by definition, «subharmonic oscillations» or simply «subharmonics» of order $\frac{1}{n}$, $n = 2, 3, \ldots$

The existence of subharmonics can be found theoretically and checked experimentally.

Mechanical, electrical, acoustical, aerodynamical phenomena, and so on, exhibit subharmonic response. We refer some physical examples.

- The «loudspeaker» can be considered as a physical example of subharmonics of order $\frac{1}{2}$. A sinusoidal current in the coil causes the loudspeaker diaphragm to vibrate axially about a central position. These vibrations may, under certain circumstances, be with frequency half of that of the driving current.
- An aerodynamical model of subharmonics could be based on the fact that cartain parts of an airplane can be excited to violent oscillations by an engine running with a frequency much larger than the natural frequency of the oscillating parts.
- An electrical model of subharmonics might be an electrical oscillatory circuit in which the nonlinear oscillation take place because of a saturablecore inductance under the impression of an alternating electromotive force of sinusoidal type.

In connection with subharmonics of a nonlinear forced system, important problems of current interest are: to find conditions for the existence of one or several subharmonics and their appropriate order, to calculate their amplitudes, to discuss their stability and determine their region of stability. These problems lead to restrictions of the coefficients of the system and of its nonlinearities, and to restrictions of the amplitude and frequency of the external forces.

By the following examples we clarify some of the above statements.

Example 1. The Linear Forced System of One Degree of Freedom.

We consider the linear system of one degree of freedom under the influence of an external sinusoidal force in two cases, one with constant coefficients and the other with variable coefficients.

Case A: Linear System With Constant Coefficients. In this case we have :

$$\ddot{x} + 2r\dot{x} + p^2 x = B\cos(\omega t + \varphi) \tag{5}$$

where r, p, B, ω, φ are constants; B, ω and φ are the amplitude, the frequency and the phase of the external force.

We investigate the possibility of the existence of subharmonics of this system.

The general solution of (5) is of the form :

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + B_1 \cos(\omega t + \varphi - \delta) \tag{5.1}$$

where δ is the phase shift, A_1 and A_2 are arbitrary constants, and the amplitude B_1 of the last term is due to the external force.

The eigenvalues λ_1 and λ_2 are given by :

$$\lambda_{1,2} = -r \pm \sqrt{r^2 - p^2} = -r \pm iq, \qquad q = \sqrt{p^2 - r^2}. \tag{5.2}$$

The subharmonics should come from the first two terms of (5.1), which must be periodic, then we exclude $q = 0$, and q = imaginary, and we accept the case of a real q, that is $p^2 > r^2$, then, by calculating B_1, the solution (5.1) will have the form :

$$x = e^{-rt}(c_1 \cos qt + c_2 \sin qt) + \frac{B\cos(\omega t + \varphi - \delta)}{\sqrt{(p^2 - \omega^2)^2 + 4r^2\omega^2}} \tag{5.3}$$

with c_1, c_2 arbitrary constants.

The first part of (5.3) is the «free oscillations», and the second part the «forced oscillations» of the system (5).

The amplitudes of these two oscillations must be bounded, then we exclude for the first term of (5.3) $r < 0$, and for the second term $p^2 = \omega^2$, $r = 0$. Also, we exclude the case $r > 0$, because, in this case, the free oscillations of (5.3) are damped out and only the forced oscillations can be observed. Therefore, it remains the only case, the undamped case, $r = 0$, when (5.3) assumes the form:

$$x = c_1 \cos pt + c_2 \sin pt + \frac{B \cos(\omega t + \varphi - \delta)}{|p^2 - \omega^2|} \tag{5.4}$$

with the restriction $p \neq \omega$. δ in this formula is either zero (if $\omega < p$), or 2π (if $\omega > p$), when the «free oscillations» are either «in phase», or «180^0 out of phase» with the «forced oscillations». We can select ω and p such that $\omega = np$, $n =$ integer, when the free oscillations of (5.4) should be «subharmonics of order $\frac{1}{n}$» of the undamped system (5).

But in the actual cases no system is undamped, and we can tell that we have «unstable subharmonics», which are not acceptable in practice. As a result, the system (5) with constant coefficients has no subharmonic oscillations.

Case B. Linear Systems With Variable Coefficients. In this case, subharmonics may exist even in the presence of viscous damping, but they rest on hypotheses which are not always met exactly by reality.

The systems with coefficients varying periodically in time are of great interest, and the external forces may depend not only on time, but on displacement and velocity as well.

We give two physical examples.

B.1: The Problem of Transverse Vibrations of a Rod Under the Action of a Longitudinal Periodic Force. [1]

This problem, after some hypotheses and transformations, leads to the «Mathieu equation»

$$\ddot{x} + \omega^2 (1 - h \cos \nu t)\, x = 0 \tag{5.5}$$

where

$$\omega^2 = \frac{g\pi^4 EI}{\gamma A l^4}, \quad h = \frac{Pl^2}{\Pi EI} \ll 1, \tag{5.6}$$

l is the length of the rod, A its cross-section, γ its density, EI its rigidity, ω the free frequency of (5. 5), and the external force is $F(t) = P\cos\nu t$. We can calculate a solution of the equation (5. 5) which is a subharmonic of order $\frac{1}{2}$, of the form :

$$x = \alpha \cos\left(\frac{\nu}{2} t + \theta\right) \tag{5.7}$$

by calculating α and θ as appropriate functions of time.

The calculation shows that oscillations of the form (5. 7) will be automatically excited, if the parameters ω, h of (5. 5) and the frequency ν of the external force are restricted according to :

$$2\omega\left(1 - \frac{h}{4}\right) < \nu < 2\omega\left(1 + \frac{h}{4}\right) \tag{5.8}$$

which represents a zone for the existence of such a subharmonic.

The solution (5. 7) is in its «first approximation».

The solution of (5. 5) in its «second approximation» is :

$$x = \alpha \cos\left(\frac{\nu}{2} t + \theta\right) - \frac{\alpha h \omega}{8\left(\omega + \frac{\nu}{2}\right)} \cos\left(\frac{3}{2}\nu t + \theta\right) \tag{5.9}$$

under the restriction :

$$2\omega\left(1 - \frac{h}{4} - \frac{h^2}{64}\right) < \nu < 2\omega\left(1 + \frac{h}{4} + \frac{h^2}{64}\right) \tag{5.10}$$

We remark that ω and h are taken as parameters of (5. 5), and the subharmonic of (5. 5) may be called «parametric subharmonic».

B. 2. A Conical Loudspeaker Diaphragm. [7a]

The mechanism, shown in Figure 7, of which a simplified version may be a device for a conical loudspeaker diaphragm, can execute a parametric subharmonic oscillation of order $\frac{1}{2}$ of the driving force.

The mass m slides over a frictionless horizontal plane. The links and the spring are massless. The pin-joints at O, A, B are frictionless.

OA and AB are long enough for motion parallel to BD to be negligible in comparison with that along the axis of the spring. The driving force $F_0 = lf_0 \cos 2vt$ is applied to the cross-head B. It may be resolved into

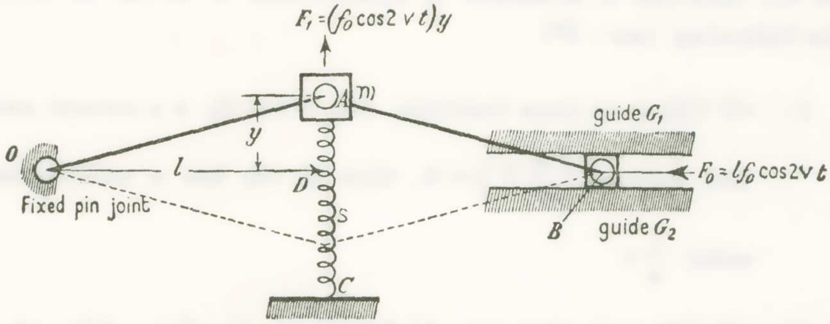


Fig. 7.

two components, one along AB, the other along DA. The latter is nearly $(f_0 \cos 2vt)y$ and it causes m to slide along the line CDA. There are three forces associated with m: one the inertia force $m\ddot{y}$, one the constraint sy due to the spring, and the driving force $(f_0 \cos 2nt)y$. The equation of the motion of m is:

$$m\ddot{y} + sy = (f_0 \cos 2v)\, y \tag{5.11}$$

which, if $\omega^2 = \frac{s}{m}$, $h = \frac{f_0}{s}$, becomes

$$\ddot{y} + \omega(1 - h \cos 2vt)\, y = 0 \tag{5.12}$$

a «Mathieu equation». As we can check, this equation enhibits a parametric subharmonic oscillation of order $\frac{1}{2}$. Figure 7 can be considered as a schematic plan of mechanism illustrating Mathieu equations.

We now refer to some examples of subharmonics of NLS.

E x a m p l e 2. Existence of Subharmonics of NL Nondissipative Systems.

The motion of a material point along a line is under the influence of a nonlinear restoring force, g(x), and an external time-dependent force, f(t), in the absence of resistance, when the equation of its motion is:

$$\ddot{x} + g(x) = f(t) \tag{5.13}$$

f(t) is periodic, $f(t) = f(t+T)$, and $g(x)$ satisfies a Lipschitz condition. From the theorems of existence of subharmonic of (5.13) we refer only to the following two: [9]

I: «If f(t) is an even function, $f(t) = f(-t)$, n a natural number, and $\dot{x}(0) = \dot{x}\left(\frac{n}{2}T\right) = 0$, then (5.13) has a subharmonic of order $\frac{1}{n}$».

II: «If f(t) and g(x) are odd functions, $f(-t) = -f(t)$, $g(-x) = -g(x)$, and $x(0) = x\left(\frac{n}{2}T\right) = 0$, then (5.13) has a subharmonic of order $\frac{1}{n}$».

Example 3. The NLDE:

$$\ddot{y} + \omega_0^2 y - 2\varepsilon(1 - by^2)\dot{y} = -\frac{4\varepsilon\omega}{\sqrt{b}}\cos 3\omega t \tag{5.14}$$

with NL damping has a subharmonic of order $\frac{1}{3}$.

Indeed, by using the trigonometric identity:

$$4\cos^3\omega t = 3\cos\omega t + \cos 3\omega t$$

we can check that the function

$$y = \frac{2}{\sqrt{b}}\sin\omega t$$

satisfies (5.14).

Example 4. A generalization of the equation (5.14) is:

$$\ddot{y} + \omega_0^2 y + \varepsilon f(y)\dot{y} = A\cos(n\omega t + \varphi) \tag{5.15}$$

with $\varepsilon \ll 1$. By applying the «Poincaré method» and restricting A, f(y) and the integer n appropriately, we can find stable subharmonics of this equation. [3]

Example 5. We consider the NLS [6a, b]

$$\ddot{Q} + \bar{k}\dot{Q} + \bar{c}_1 Q + \bar{c}_2 Q^2 + \bar{c}_3 Q^3 = A \sin 2t \tag{5.16}$$

coming from electrical problems. By changing the coefficients according to:

$$\bar{k} = \varepsilon k, \quad 1 - \bar{c}_1 = \varepsilon c_1, \quad \bar{c}_2 = \varepsilon c_2, \quad \bar{c}_3 = \varepsilon c_3 \tag{5.17}$$

the system can be written in the useful form:

$$\ddot{Q} + Q = \varepsilon f(Q, \dot{Q}) + A \sin 2t \tag{5.18}$$

where:

$$f(Q, \dot{Q}) = -k\dot{Q} + c_1 Q - c_2 Q^2 - c_3 Q^3 \tag{5.19}$$

Case A: For $\varepsilon = 0$, the solution of (5.18) is:

$$Q = \bar{x}_1 \sin t + \bar{x}_2 \cos t - \frac{A}{3} \sin 2t \tag{5.20}$$

where $\bar{x}_1$ and $\bar{x}_2$ are arbitrary constants. The first two terms of (5.20) give the «subharmonic component» of order $\frac{1}{2}$ of the solution (5.20) with amplitude: $\bar{r} = \sqrt{\bar{x}_1^2 + \bar{x}_2^2}$.

This subharmonic is, as we know, without practical importance.

Case B: For $\varepsilon \neq 0$, we try to establish a solution of (5.18) of the form (5.20), where, instead of the constants $\bar{x}_1$ and $\bar{x}_2$, we calculate appropriate functions $x_1 = x_1(\varepsilon, t)$ and $x_2 = x_2(\varepsilon, t)$, such that their limits, as $\varepsilon \to 0$, are the constants $\bar{x}_1$ and $\bar{x}_2$, respectively.

The calculation of the amplitude $r = \sqrt{x_1^2 + x_2^2}$ of the subharmonic of order $\frac{1}{2}$ in the nonlinear case leads to appropriate restrictions for the existence of a real and stable amplitude r, and for the existence of two subharmonics of order $\frac{1}{2}$ with two amplitudes r_1 and r_2, which can be calculated.

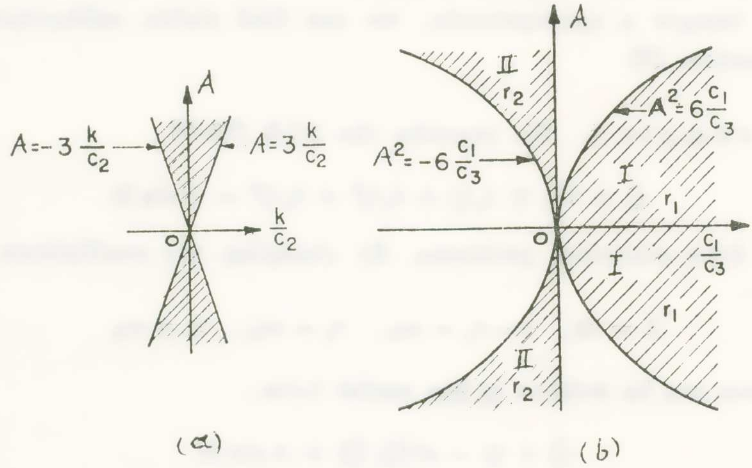


Fig. 8.

In Figure 8(a), the shaded regions in the $\left(\frac{k}{c_2},\ A\right)$ -plane corresponds to a real amplitude r of subharmonics.

In Figure 8(b), the shaded regions I and II in the $\left(\frac{c_1}{c_3},\ A\right)$ -plane correspond to real amplitudes r_1 and r_2, respectively, of the subharmonics.

Example 6. The forced system which is linearly damping and has a nonlinear restoring force expressed by : [5]

$$\ddot{y} + y = \varepsilon\{-k\dot{y} - f(y) + A\sin(\omega t + \varphi) \qquad (5.21)$$

where $\varepsilon \ll 1$, $f(y)/y \geqslant 0$, and ε, A, k positive, can be studied by applying the «Cartwright-Littlewood method», when important results in connection with its subharmonics can be found. Some of these results are:

(a) : Subharmonics of a given order exist in certain frequency bands.

(b) : If, at a given frequency, several states of subharmonics are possible, the amplitude is largest and the frequency band smallest for the lower order of subharmonics.

(c) : For a given form of the non-linearity f(y), the highest order obtainable is a function of the ratio A / k, which also determines the width of the frequency bands.

We remark that the subharmonic oscillations of a NLS belong to a general class of periodic motions characterized by the property that the ratio of the frequencies ω_0 (free frequency) and ω (frequency of the external force) is a rational number, that is : $\frac{\omega_0}{\omega} = \frac{n_0}{n}$, where n_0 and n are mutually prime integers.

We may have the following cases :

(a) If $n_0 = n = 1$, then: $\omega_0 = \omega$: harmonic oscillation of the system

(b) If $n = 1$, then: $\omega_0 = n_0 \omega$: harmonic oscillation of order n_0

(c) If $n_0 = 1$, then: $\omega_0 = \frac{\omega}{n}$: subharmonic oscillation of order $\frac{1}{n}$

(d) If $1 < n_0 < n$, then: $\omega_0 = n_0\left(\frac{\omega}{n}\right)$: n_0 multiple of subharmonic of order $\frac{1}{n}$.

VI. AMPLITUDE AND FREQUENCY OF PERIODIC SOLUTIONS OF FREE LINEAR AND NON-LINEAR SYSTEMS

The periodic solutions of free (unforced) LS have amplitude independent of the frequency, and the frequency is the same for all trajectories. On the contrary, in the periodic solutions of free NLS the amplitude depends on the frequency, and the frequency changes from trajectory to trajectory. We see examples.

Example 1. The free LS :

$$\ddot{x} + 2r\dot{x} + p^2 x = 0 \tag{6}$$

has as general oscillatory solution the function :

$$x = e^{-rt}(c_1 \cos \beta t + c_2 \sin \beta t), \quad \beta = \sqrt{p^2 - r^2} \tag{6.1}$$

where c_1 and c_2 arbitrary constants.

(a): In the damped case, $r \neq 0$, the factor e^{-rt} of (6.1), which characterizes the amplitude of the oscillatory motion (6.1), either tends to infinity (for $r < 0$), or tends to zero (for $r > 0$) as $r \rightarrow \infty$, and the motion is not periodic.

(b): In the undamped case, $r = 0$, (6.1) becomes:

$$x = c_1 \cos pt + c_2 \sin pt \tag{6.2}$$

which is a family of periodic motions with the same frequency p, but amplitude $c = \sqrt{c_1^2 + c_2^2}$ calculated from the initial conditions. These amplitudes are independent of the frequency.

Example 2. The free NLS : [10 a]

$$\ddot{x} + p^2 x + b x^3 = 0 \tag{6.3}$$

can be solved exactly, and the period of the closed trajectories can be calculated. (6.3) is equivalent to:

$$\dot{x} = y, \qquad \dot{y} = -(p^2 x + b x^3)$$

then to:

$$\frac{dy}{dx} = -\frac{p^2 x + b x^3}{y} \tag{6.4}$$

of which the solution is the family of the closed trajectories:

$$y^2 + p^2 x^2 + \frac{b}{2} x^4 = c \tag{6.5}$$

c is the integration constant.

For the period T of the trajectories (6.5), we insert $y = \dot{x}$ and we take into account the symmetry of these trajectories, when:

$$T = 4 \int_0^{\alpha} \frac{dx}{\sqrt{c - \left(p^2 x^2 + \frac{b}{2} x^4\right)}} . \tag{6.6}$$

Transforming according to:

$$x = A \sin \theta \tag{6.7}$$

we can find:

$$T = 4\sqrt{2} \int_0^{\pi/2} \frac{dx}{\sqrt{2p^2 + bA^2 + bA^2 \sin^2 \theta}} . \tag{6.8}$$

As a result, the period T, or the frequency (T^{-1}), corresponding to the periodic motion on a trajectory is dependent on the amplitude of this motion, and the period, or the frequency, changes from trajectory to trajectory. In the special case $b = 0$ the system (6.3) becomes linear, and its period, which comes from (6.8) for $b = 0$, is a constant for all trajectories.

VII. THE RESONANCE PHENOMENA

The resonance phenomena occur in forced LS and NLS, in case the free frequency of the system is equal or very close to the frequency of the external force.

The damping in a LS and the nonlinearities in a NLS play a very important role in the resonance phenomena.

By the following examples is shown the influence of the damping and of the nonlinearities to the resonance phenomena.

Example 1. The general oscillatory solution of a damped forced LS:

$$\ddot{x} + 2r\dot{x} + p^2 x = \alpha \cos \omega t \tag{7}$$

is the function:

$$x = e^{-rt}(c_1 \cos qt + c_2 \sin qt) + A \cos(\omega t + \varphi) \tag{7.1}$$

where:

$$q = \sqrt{p^2 - r^2}, \qquad A = \frac{\alpha}{\sqrt{(p^2 - \omega^2)^2 + 4r^2\omega^2}} \tag{7.2}$$

c_1 and c_2 in (7.1) are arbitrary constants.

The system (7) is in «resonance» with the external force, if $\omega = p$, and «near resonance» is $\omega - p \ll 1$.

We distinguish two cases:

(a): If (7) is «undamped», $r = 0$, then the solution (7.1) becomes:

$$x = c_1 \cos pt + c_2 \sin pt + \frac{\alpha}{|p^2 - \omega^2|} \cos(\omega t + \varphi). \tag{7.3}$$

If, in (7.3), ω is equal or close to p, the amplitude of the term of forced oscillations of (7.3) becomes infinite or very large, when the undamped system (7) is in «resonance» or «near resonance» with the external force.

(b): If (7) is positively damped, $r > 0$, the free oscillations of (7.1) are oscillatory motions but not periodic, since for $t \to \infty$, are damped out, when in the case of «resonance», $p = \omega$, the solution (7.1) becomes:

$$x = \frac{\alpha}{2\omega r} \cos \omega t. \tag{7.4}$$

This is a periodic motion with finite amplitude, which becomes very large if the damping coefficient r becomes very small.

By the above example is shown that the damping in LS can prevent resonance, and that a weak damping force can be capable of sustaining oscillations of large amplitude.

Example 2. Consider the LS and NLS:

$$(a): \ \ddot{x} + x = 0, \qquad (b): \ \ddot{x} + x + \frac{1}{6} x^3 = 0 \tag{7.5}$$

of which the general solutions are:

$$(a): \ x^2 + y^2 = c^2, \qquad (b): \ x^2 + y^2 + \frac{1}{12} x^4 = c^2 \tag{7.6}$$

All solutions of the LS are periodic with the same period.

But the period of the solutions of the NLS changes from trajectory to trajectory, and the period varies with amplitude. A periodic disturbance in the NLS will become out of phase with the free motion, and the forcing function should be an obstacle to increasing amplitude. The period varies with amplitude and non periodic solutions are possible.

As a result, the nonlinearity can prevent resonance, even in the absence of damping.

Due to the nonlinearity, the frequency will be changed, then resonance will be stopped.

The nonlinear terms exert, in general, a stabilizing influence until the motion has passed.

«Resonance phenomena» are in many cases unavoidable. They are dangerous, but sometimes controllable, and, although uncomfortable, they are not in all cases undesirable.

We refer to some physical examples related to resonance.

- If an elastic machine part vibrates in resonance with a sinusoidal force, it may become the source of vibrations with large amplitudes, which, in turn, may produce excessive stresses and lead to possible failure. It is, then, vital to design machine parts, or other engineering structures, in such a way as to avoid resonance with periodic forces.

• Tacoma Narrows bridge offers an example of a big failure in engineering history, due to resonance. This suspension bridge, just after its opening, started to exhibit a marked flexibility and a series of torsicnal oscillations, the amplitude of which steadily increased until the convolutions tore several suspenders loose, and the span of the bridge broke up (November 7, 1940) four months after its building. The wind created aerodynamical forces, which, at the time, were insufficiently understood.

• When a group of soldiers marches in step over a suspension bridge, the feet of the group exert a periodic force on the road bed. If the period of marching is equal to the natural period of the bridge resonance occurs and the sustained bridge oscillations may become dangerous.

• Resonance is sometimes not undesirable. One can in fact utilize it to produce large vibrations by means of small forces. E.g., the vibrations of a string can be sustained by means of an electro-magnet which is activated from a weak alternating current.

VIII. JUMP, OR HYSTERESIS, PHENOMENA

Jump discontinuities are phenomena in damped forced NLS and not of LS. They are found and explained mathematically and checked experimentally, especially in electrical and mechanical systems.

There are frequency regions where the amplitude of the oscillations jumps discontinuously and, in these regions, the oscillations have a kind of instability.

Let us take, as an example, the system : **[10b]**

$$\ddot{x} + c\dot{x} + p^2 x + b x^3 = F\cos(\omega t + \varphi) \tag{8}$$

The investigation of a periodic solution of this equation of the form : $x = A\cos\omega t$ loads to the formula :

$$\left\{(p^2 - \omega^2) A + \frac{3}{4} b A^3\right\}^2 + c A^2 \omega^2 = F^2 \tag{8. 1}$$

where we consider A versus ω, p and c constants, and F as a parameter.

In the undamped case, $c = 0$, Figure 9 shows the amplitude curves which are curves without closed branches.

In the damped case, $c > 0$, Figure 10 shows these curves having a single branch for each value of F.

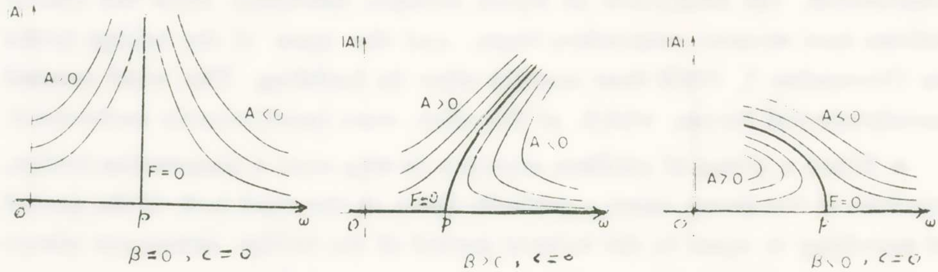


Fig. 9.

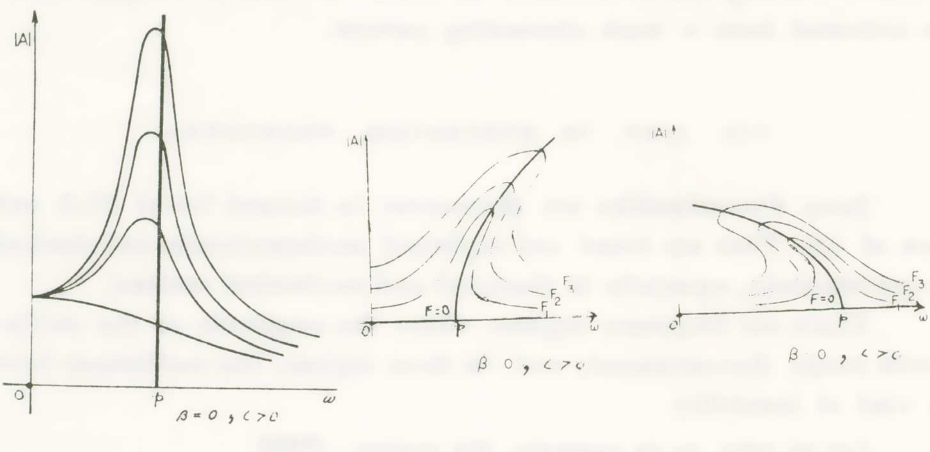
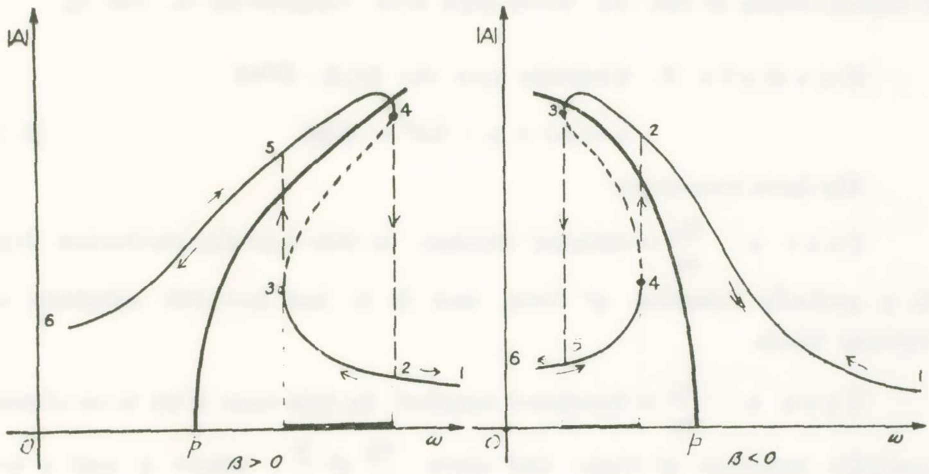


Fig. 10.

Suppose we keep F = constant and vary ω from large values to smaller values, starting, say, at point 1, Figure 11 (a). As ω decreases, A increases slowly until point 3 (tangent point to the curve), when a further decrease of ω causes a jump up to the amplitude from point 3 to point 5 of the curve, and after that the amplitude decreases with ω.

If we start increasing ω from a value corresponding to point 6, the amplitude follows the portion $6 \rightarrow 5 \rightarrow 4$ of the curve, when, if 4 is the tangent point to the curve, the amplitude jumps down to point 2, and after that decreases slowly with increasing ω.



(a) (b)

Fig. 11.

The same phenomenon occurs in Figure 11 (b), but in the reverse direction. The above is the jump, a hysteresis, phenomenon, corresponding to an interval of ω, in which the oscillations are unstable.

The portion $3 \rightarrow 4$ of the curve is a «dead portion».

IX. COMBINATION FREQUENCIES

Helmholtz in his acoustical studies, and Poincaré in his studies of NLDE established that, in addition to certain fundamental frequencies ω_1 and ω_2 in a NLS, there exist solutions of the same DE with the frequencies: $\omega = m\omega_1 + n\omega_2$, where m and n are integers.

These are called «combination frequencies», or «combination tones» of the system, and the phenomena of combination frequencies are phenomena of NLS.

Example 1. The oscillations of the LS:

$$\ddot{x} + c\dot{x} + x = F_1 \cos \omega_1 t + F_2 \cos \omega_2 t \equiv H(t) \tag{9}$$

where $\omega_1 \neq \omega_2 \neq 1$, are superposition of a damped free oscillation and a forced oscillation, which, in turn, is a superposition of the fundamental oscillations due to each of the two separate tones of the excitation H(t) individually. That is essentially the oscillations of the LS (9) are simply a superposition of the two harmonies with frequencies ω_1 and ω_2.

Example 2. Consider now the NLS: [10c]

$$\ddot{x} + c\dot{x} + x - bx^3 = H(t). \tag{9.1}$$

We have two cases:

Case a: $\frac{\omega_1}{\omega_2}$ = rational number. In this case the excitation H(t) is a periodic function of time, and (9.1) has periodic solutions of various kinds.

Case b: $\frac{\omega_1}{\omega_2}$ = irrational number. In this case H(t) is an almost periodic function of time, and since $\frac{\omega_1}{\omega_2} \neq \frac{p}{q}$, where p and q are mutually prime integers, we will have solutions with frequency: $\omega = q\omega_1 - p\omega_2 \neq 0$.

In this case, by using approximate methods for calculation of approximations of the solutions of (9.1), we find that these approximations of the solutions contain terms with denominators powers of $(\pm m\omega_1 \pm n\omega_2)$, where m and n are integers.

But, by virtue of the «Kronecker theorem» it is known that the expressions $(\pm m\omega_1 \pm n\omega_2)$ are arbitrarily close to zero for infinitely many different integers m and n, when the approximations of the solutions of (9.1) are divergent, because of the «difficulty of small divisors», which was first pointed out by Poincaré in discussing perturbations methods with problems in Celestial Mechanics.

The «difficulty of small divisors» can be circumvented in some cases if we use viscous damping in the system.

In case the excitation of the NLS has a single frequency, which is an irrational multiple of the free frequency of the system, we have the same situation as above.

RESULTS AND REMARKS

• The linear and nonlinear systems have striking properties which characterize them. Some properties are properties of LS and some others only of NLS.

• The principle of superposition of the solutions, the global property, the independence of amplitude and frequency of an oscillation, characterize LS.

The existence of limit cycles, the change of frequency from trajectory to trajectory, the jump discontinuities of the amplitude of oscillations, the development of stable self-excited oscillations, the phenomenon of combination tones, characterize NLS.

The resonance phenomena are related to LS as well as to NLS. The damping in LS can prevent resonance, and the nonlinearities in NLS can stop resonance, and by a proper selection of nonlinearities the system can become stabilized.

• The reduction of a NLS to a LS implies that some properties of NLS will be lost, and, therefore, if a problem in connection with the NLS is related to a property which will be lost by linearization, then the linearization as a method to solve the problem is not applicable.

ΠΕΡΙΛΗΨΙΣ

Τὰ γραμμικὰ (ΓΣ) καὶ μὴ - γραμμικὰ (ΜΓΣ) συστήματα ἔχουν ἰδιότητας αἱ ὁποῖαι ἄλλαι μὲν χαρακτηρίζουν τὰ ΓΣ ἄλλαι τὰ ΜΓΣ.

• Εἰς τὰ ΓΣ ἡ γενικὴ λύσις δύναται νὰ δοθῇ ὡς γραμμικὸς συνδυασμὸς μερικῶν ἁπλῶν λύσεων, καὶ αἱ σταθεραὶ ὁλοκληρώσεως εἰσέρχονται εἰς τὰς γενικὰς λύσεις γραμμικῶς. Εἰς τὰ ΜΓΣ ἢ δὲν ὑπάρχουν γενικαὶ λύσεις, ἤ, ἐὰν ὑπάρχουν, ἔχουν πολύπλοκον μορφήν, ὅπου αἱ σταθεραὶ εἰσέρχονται ἐν γένει μὴ - γραμμικῶς.

• Εἰς τὰ ΓΣ αἱ λύσεις ἔχουν ὁλικὸν χαρακτῆρα, ἐνῷ εἰς τὰ ΜΓΣ ἔχουν τοπικὸν χαρακτῆρα.

• Εἰς τὸ ΓΣ τὸ πλάτος περιοδικῶν κινήσεων εἶναι ἀνεξάρτητον τῆς συχνότητος, καὶ εἰς ἓν συνεχὲς πεδίον περιοδικῶν κινήσεων αἱ τροχιαὶ ἀντιστοιχοῦν εἰς τὴν αὐτὴν συχνότητα. Τοὐναντίον, εἰς τὰ ΜΓΣ τὸ πλάτος ἐξαρτᾶται ἀπὸ τὴν συχνότητα, καὶ ἡ συχνότης ἀλλάζει ἀπὸ τροχιᾶς εἰς τροχιάν.

● Ἡ ὕπαρξις ἀνωμαλιῶν εἰς τὰ πλάτη περιοδικῶν κινήσεων συναρτήσει τῆς συχνότητος, εἶναι φαινόμενον τῶν ΜΓΣ, καὶ τοιοῦτον φαινόμενον δὲν ὑπάρχει εἰς τὰ ΓΣ.

● Ἡ ὕπαρξις μεμονωμένων περιοδικῶν κινήσεων (limit cycles), ὅπως καὶ «εὐσταθῶν ταλαντώσεων αὐτοαναπτυσσομένων» χαρακτηρίζει τὰ ΜΓΣ, καὶ τοιαῦτα φαινόμενα δὲν ὑφίστανται εἰς τὰ ΓΣ.

● Τὸ φαινόμενον τοῦ «συνδυασμοῦ συχνοτήτων» (combination frequencies) εἶναι φαινόμενον τῶν ΜΓΣ καὶ ὄχι τῶν ΓΣ.

● Αἱ δυνάμεις «damping» εἰς τὰ ΓΣ δύνανται νὰ ἐμποδίσουν φαινόμενα «resonance», ἐνῷ εἰς τὰ ΜΓΣ αἱ μὴ-γραμμικότητες δύνανται νὰ χρησιμοποιηθοῦν δι' εὐστάθειαν ἔναντι «resonance».

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★

Ὁ Ἀκαδημαϊκὸς κ. **Ἰωάννης Ξανθάκης,** παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν, εἶπε τὰ ἑξῆς :

Εἰς τὴν παροῦσαν ἀνακοίνωσιν ὁ κ. Μάγειρος μελετᾷ τὰς χαρακτηριστικὰς ἰδιότητας τῶν γραμμικῶν καὶ μὴ γραμμικῶν συστημάτων. Ἡ διερεύνησις αὕτη τὸν ὡδήγησεν εἰς τὰ ἑξῆς γενικὰ συμπεράσματα :

1. Εἰς τὰ γραμμικὰ συστήματα ἡ γενικὴ λύσις δύναται νὰ εἶναι ἕνας γραμμικὸς συνδυασμὸς μερικῶν ἁπλῶν λύσεων, ὅπου αἱ σταθεραὶ ὁλοκληρώσεως εἰσέρχονται εἰς τὰς γενικὰς λύσεις γραμμικῶς. Ἀντιθέτως εἰς τὰ μὴ γραμμικὰ συστήματα ἢ δὲν ὑπάρχουν γενικαὶ λύσεις ἢ ἐὰν ὑπάρχουν ἔχουν πολύπλοκον μορφήν, αἱ δὲ σταθεραὶ εἰσέρχονται μὴ γραμμικῶς.

2. Εἰς τὰ γραμμικὰ συστήματα τὸ πλάτος τῶν περιοδικῶν κινήσεων εἶναι ἀνεξάρτητον τῆς συχνότητος, αἱ τροχιαὶ δέ, εἰς ἓν συνεχὲς πεδίον περιοδικῶν κινήσεων, ἀντιστοιχοῦν εἰς τὴν αὐτὴν συχνότητα. Ἀντιθέτως εἰς τὰ μὴ γραμμικὰ συστήματα τὸ πλάτος ἐξαρτᾶται ἀπὸ τὴν συχνότητα, ἡ ὁποία ἀλλάζει ἀπὸ τροχιᾶς εἰς τροχιάν.

3. Ἡ ὕπαρξις ἀνωμαλιῶν εἰς τὰ πλάτη περιοδικῶν κινήσεων συναρτήσει τῆς συχνότητος παρατηρεῖται μόνον εἰς τὰ μὴ γραμμικὰ συστήματα, οὐδόλως δὲ εἰς τὰ γραμμικὰ τοιαῦτα.

4. Ἡ ὕπαρξις μεμονωμένων περιοδικῶν κινήσεων καθὼς καὶ εὐσταθῶν ταλαντώσεων αὐτοαναπτυσσομένων εἶναι χαρακτηριστικὸν τῶν μὴ γραμμικῶν συστημάτων, δὲν παρατηρεῖται δὲ τὸ φαινόμενον τοῦτο εἰς τὰ γραμμικὰ συστήματα.

Τέλος τὸ φαινόμενον τοῦ «Συνδυασμοῦ συχνοτήτων» παρατηρεῖται μόνον εἰς τὰ μὴ γραμμικὰ συστήματα.