$$\sqrt{3}\cdots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{2}{1},$$

und $1351^2 = 3.780^2 + 1$. Wir haben also, auf Grund der von Theon von Smyrna und Proklos überlieferten Methode, die Beziehungen die Archimedes, ohne Erklärung, als bekannt, angibt.

ΓΕΩΜΕΤΡΙΚΗ ΑΛΓΕΒΡΑ.— Συμβολὴ εἰς τὴν ἔρευναν τῆς γεωμετρικῆς ἀλγέβρας τῶν Πυθαγορείων, *ὑπὸ* ***Εὐαγγ. Σταμάτη**** Ἀνεκοινώθη ὑπὸ τοῦ κ. Μιχαὴλ Στεφανίδου.

I. ΕΙΣΑΓΩΓΗ

1. Εἰς τὸν ἴδιον τὸν Πυθαγόραν ἀποδίδεται κατὰ τὴν παράδοσιν ἡ ἀπόδειξις τοῦ περιφήμου ὁμωνύμου θεωρήματος (Εὐκλείδου I, 47), ὡρισμέναι ἀκέραιαι λύσεις τῆς ἐξισώσεως $z^2 = x^2 + y^2$ καὶ ἡ ἀνακάλυψις τῶν ἀσυμμέτρων [1]. Εἰς τοὺς Πυθαγορείους ἐν γένει ἀποδίδεται μεταξὺ ἄλλων τὸ II Βιβλίον καὶ τὸ πλεῖστον τῶν ἀριθμητικῶν τῶν Στοιχείων τοῦ Εὐκλείδου, καὶ ἡ εὕρεσις τῶν ἀκεραίων λύσεων τῆς ἐξισώσεως $y^2 = 2x^2 \mp 1$, (1), αἵτινες χρησιμεύουσι διὰ τὸν ὑπολογισμὸν τῆς κατὰ προσέγγισιν ἀριθμητικῆς τιμῆς τῆς $\sqrt{2}$. Αἱ ἀκέραιαι λύσεις τῆς ἐξισώσεως (1) ὀνομάζονται, ὡς γνωστόν, πλευρικοὶ καὶ διαμετρικοὶ ἀριθμοί, διότι οἱ μὲν ἐκ τούτων ἀντιστοιχοῦσιν εἰς τὰς πλευράς, οἱ δὲ εἰς τὰς διαγωνίους τετραγώνων [2].

Ὁ νόμος σχηματισμοῦ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν διεσώθη ὑπὸ τοῦ Θέωνος τοῦ Σμυρναίου καὶ ἔχει ὡς κάτωθι:

	Πλευρικοὶ ἀριθμοί	Διαμετρικοὶ ἀριθμοί
1.	1	1
2.	$1 + 1 = 2$	$2 \cdot 2 + 1 = 3$
3.	$2 + 3 = 5$	$2 \cdot 2 + 3 = 7$

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* **EVANGELOS STAMATIS, A contribution to the investigation of the geometrical algebra of the Pythagoreans.**

[1] ΜΙΧΑΗΛ ΣΤΕΦΑΝΙΔΟΥ, Εἰσαγωγὴ εἰς τὴν ἱστορίαν τῶν Φυσικῶν Ἐπιστημῶν, σ. 55-65 καὶ 102-3, Ἀθῆναι 1938.— Πρόκλος εἰς σχόλια Εὐκλείδου I, σ. 65 καὶ 438. Ἔκδ. Friedlein, Teubner. — Ἰαμβλίχου V. P. 246, ἔκδ. L. Deubner, Teubner.

[2] Theonis Smyrnaei, Philosophi Platoniki, ἔκδ. E. Hiller, σ. 43, Teubner. Ε. ΣΤΑΜΑΤΗ, Εὐκλείδου Γεωμετρία - Θεωρία ἀριθμῶν, τόμ. II, σ. 8, (*Ὀργ. Ἐκδ. Σχολικῶν Βιβλίων*, 1953 Ἀθῆναι). – PAUL-HENRI MICHEL, De Pythagore à Euclide. p 438, Paris 1950.— (*Soc. d'éd. Les Belles Lettres*). M. CANTOR Vorlesungen über Geschichte der Mathematik u. Theon von Smyrna.— T. HEATH A history of Greek mathematics I, p. 91, Oxford 1921 at the Clarendon Press. — R. MORIS COHEN and J. E. DRABKIN, A source book in Greek science, p. 43, McGraw-Hill book company inc. New York, Toronto, London, 1984.

4.	$5 + 7 = 12$	$2 \cdot 5 + 7 = 17$
5.	$12 + 17 = 29$	$2 \cdot 12 + 17 = 41$
.	.	.
.	.	.
.	.	.

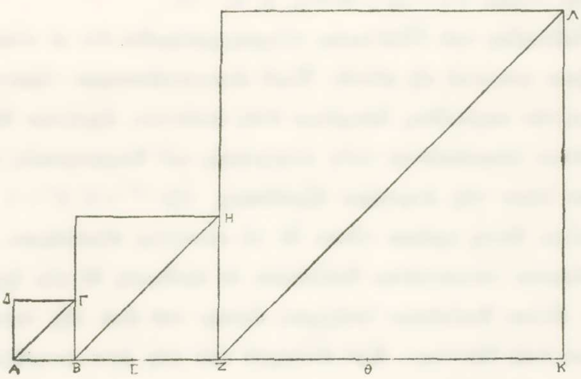
Εἶναι δὲ $\frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \cdots \sqrt{2} \cdots < \frac{17}{12} < \frac{3}{2}$,

καὶ

$$1^2 = 2 \cdot 1^2 - 1$$
$$3^2 = 2 \cdot 2^2 + 1$$
$$7^2 = 2 \cdot 5^2 - 1$$
$$17^2 = 2 \cdot 12^2 + 1$$
$$41^2 = 2 \cdot 29^2 - 1 \quad \text{κλπ.}$$

Ἡ γεωμετρικὴ ἀπόδειξις τοῦ νόμου σχηματισμοῦ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν μνημονεύεται ὑπὸ τοῦ Πρόκλου[1] καὶ ἔχει ὡς ἑξῆς:

Ἔστω τετράγωνον πλευρᾶς $AB = \alpha_1$ καὶ διαγωνίου $A\Gamma = \delta_1$, (σχ. 1), ὅτε εἶναι



Σχ. 1.

$\delta_1^2 = 2\alpha_1^2$. Ἐπὶ τῆς προεκτάσεως τῆς AB λαμβάνομεν τμῆμα $BE = AB = \alpha_1$ καὶ ἐν συνεχείᾳ τμῆμα $EZ = A\Gamma = \delta_1$. Κατὰ τὸν Εὐκλείδην II, 10 θὰ εἶναι $(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2$.

Καὶ ἐπειδὴ $\delta_1^2 = 2\alpha_1^2$, θὰ ἔχωμεν δι' ἀφαιρέσεως τούτου κατὰ μέλη ἐκ τῆς προηγουμένης ἐξισώσεως, $(2\alpha_1 + \delta_1)^2 = 2(\alpha_1 + \delta_1)^2$. Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(\alpha_1 + \delta_1) = BE + EZ$ εἶναι πλευρά, ἡ δὲ $(2\alpha_1 + \delta_1) = AB + BE + EZ$ εἶναι διαγώνιος τετραγώνου, ἡ BH. Ἐὰν ἐπὶ τῆς προεκτάσεως τῆς BZ λάβωμεν τμῆμα

[1] Σχόλια εἰς Πολιτείαν Πλάτωνος, τόμ. II σ. 24 κ.ἑ. 393 κ.ἑ. ὑπὸ F. HULTSCH, ἔκδ. Kroll, Teubner.

ΖΘ=ΒΖ καὶ ἐν συνεχείᾳ τμῆμα ΘΚ=ΒΗ καὶ ἐφαρμόσωμεν τὸ εὐκλείδειον θεώρημα II, 10, θὰ λάβωμεν νέον τετράγωνον, τοῦ ὁποίου ἡ μὲν πλευρὰ θὰ εἶναι ἡ ΖΘ+ΘΚ$=3\alpha_1+2\delta_1$, ἡ δὲ διαγώνιος ἡ ΖΛ=ΒΚ=2ΒΖ+ΘΚ$=4\alpha_1+3\delta_1$.

Ὅθεν λαμβάνομεν τὸ ἑξῆς σχῆμα

Πλευρικοὶ ἀριθμοί	Διαμετρικοὶ ἀριθμοί
α_1	δ_1
$\alpha_2 = \alpha_1 + \delta_1$	$\delta_2 = 2\alpha_1 + \delta_1$
$\alpha_3 = \alpha_2 + \delta_2$	$\delta_3 = 2\alpha_2 + \delta_2$
$\alpha_4 = \alpha_3 + \delta_3$	$\delta_4 = 2\alpha_3 + \delta_3$
.	.
.	.
.	.
$\alpha_\nu = \alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$

Ἐὰν θέσωμεν $\alpha_1=1$ καὶ $\delta_1=1$, λαμβάνομεν τοὺς κατὰ τὸν Θέωνα τὸν Σμυρναῖον πλευρικοὺς καὶ διαμετρικοὺς ἀριθμούς, ἤτοι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2=2x^2\mp 1$, ἢ $\delta_\nu^2=2\alpha_\nu^2+(-1)^\nu$, $(\nu=1, 2, 3, \ldots)$.

Ἐκ τῆς Πολιτείας τοῦ Πλάτωνος πληροφορούμεθα ὅτι οἱ πλευρικοὶ καὶ διαμετρικοὶ ἀριθμοὶ ἦσαν γνωστοὶ εἰς αὐτόν. Ἐκεῖ ἀναγιγνώσκομεν «ἑκατὸν μὲν ἀριθμῶν ἀπὸ διαμέτρων ρητῶν πεμπάδος, δεομένων ἑνὸς ἑκάστων, ἀρρήτων δὲ δυοῖν» (546 c). Ἐνταῦθα ὁ Πλάτων ὑπαινίσσεται τοὺς πλευρικοὺς καὶ διαμετρικοὺς ἀριθμοὺς καὶ δὴ καὶ μίαν ἀκεραίαν λύσιν τῆς ἀνωτέρω ἐξισώσεως, τὴν $7^2=2\cdot 5^2-1$. Τοῦτο συνάγεται ἐκ τοῦ Πρόκλου, ὅστις γράφει «ὅπου δὲ τὸ σύνεγγυς ἀγαπῶμεν, οἷον εὑρόντες ἐν γεωμετρίᾳ τετράγωνον τετραγώνου διπλάσιον, ἐν ἀριθμοῖς δὲ οὐκ ἔχοντες ἑνὸς δέοντος φαμὲν ἄλλον ἄλλου διπλάσιον ὑπάρχειν, ὥσπερ τοῦ ἀπὸ τῆς πεντάδος ὁ ἀπὸ τῆς ἑπτάδος διπλάσιον ἑνὸς δέοντος». Καὶ ἀλλαχοῦ «οὐ γάρ ἐστι τετράγωνος ἀριθμὸς τετραγώνου διπλάσιος εἰ μὴ λέγει τις τὸν σύνεγγυς. ὁ γὰρ ἀπὸ τοῦ ζ′ τοῦ ἀπὸ τοῦ ε′ διπλάσιός ἐστιν ἑνὸς δέοντος[1]. (Εἶναι δηλ. $7^2=2\cdot 5^2-1$).

2. Ὁ Ἀρχιμήδης εἰς τὴν πραγματείαν αὐτοῦ «Κύκλου Μέτρησις» χρησιμοποιεῖ ἄνευ ἀποδείξεως τὰς σχέσεις

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}, \text{ καὶ } 265^2=3.153^2-2,\ 1351^2=3.780^2+1.$$

Γεωμετρικὴν ἀπόδειξιν τῶν σχέσεων τούτων ὑπεβάλομεν εἰς τὴν Ἀκαδημίαν Ἀθηνῶν[2].

Αὕτη συνδέεται πρὸς τοὺς πλευρικοὺς καὶ διαμετρικοὺς ἀριθμούς. Θεωροῦμεν

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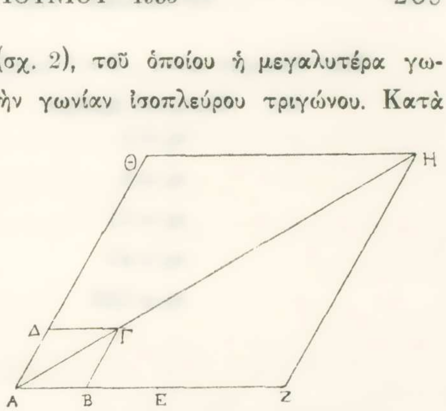
[1] Πρόκλος εἰς Εὐκλείδην I, σ. 61 καὶ 427, ἔκδ. G. FRIEDLEIN, Teubner.

[2] Βλ. Πρακτικά, ἀνωτ., σ. 255 κ. ἑξ.

ἰσοσκελὲς ἀμβλυγώνιον τρίγωνον, τὸ ΑΒΓ (σχ. 2), τοῦ ὁποίου ἡ μεγαλυτέρα γωνία, ἡ ΑΒΓ, νὰ εἶναι ἴση πρὸς τὴν ἐξωτερικὴν γωνίαν ἰσοπλεύρου τριγώνου. Κατὰ τὸν Εὐκλείδην II, 12, ἐὰν καλέσωμεν τὴν πλευρὰν $ΑΒ=\alpha_1$ καὶ τὴν $ΑΓ=\delta_1$, ἥτις βεβαίως εἶναι ἡ μεγαλυτέρα διαγώνιος τοῦ ρόμβου ΑΒΓΔ, θὰ εἶναι $\delta_1^2=3\alpha_1^2$, καὶ συνεπῶς $\delta_1:\alpha_1=\sqrt{3}$. Ἐφαρμόζομεν τώρα ἀκριβῶς τὴν ὑπὸ τοῦ Πρόκλου ὑποδεικνυομένην μέθοδον διὰ τὴν ἀπόδειξιν τῶν ἐκ τετραγώνων σχημάτων προκυπτόντων πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν. Ἐπὶ τῆς προεκτάσεως τῆς ΑΒ λαμβάνομεν τμῆμα $ΒΕ=ΑΒ=\alpha_1$ καὶ ἐν συνεχείᾳ τμῆμα $ΕΖ=ΑΓ=\delta_1$. Κατὰ τὸν Εὐκλείδην II, 10 θὰ ἔχωμεν,



Σχ. 2.

$$(2\alpha_1+\delta_1)^2+\delta_1^2=2\alpha_1^2+2\,(\alpha_1+\delta_1)^2,$$

καὶ ἐκ ταύτης

$$(2\alpha_1+\delta_1)^2=4\alpha_1^2+4\alpha_1\,\delta_1+\delta_1^2\,, \qquad (1).$$

Εἶναι ἄρα καὶ $3(2\alpha_1+\delta_1)^2=12\alpha_1^2+12\alpha_1\delta_1+3\delta_1^2$. Ἀλλὰ $\delta_1^2=3\alpha_1^2$. Ἑπομένως

$$3(2\alpha_1+\delta_1)^2=9\alpha_1^2+12\alpha_1\delta_1+4\delta_1^2=(3\alpha_1+2\delta_1)^2.$$

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1+\delta_1)$ εἶναι πλευρά, ἡ δὲ $(3\alpha_1+2\alpha_1)$ εἶναι ἡ μεγαλυτέρα διαγώνιος ὁμοίου ρόμβου πρὸς τὸν ΑΒΓΔ τοῦ ΑΖΗΘ. Ἐὰν ἐπὶ τῆς προεκτάσεως τῆς ΑΖ λάβωμεν τμῆμα ἴσον πρὸς ΑΖ καὶ ἐν συνεχείᾳ τμῆμα ἴσον πρὸς ΑΗ, τότε ἔχομεν κατὰ τὸν αὐτὸν νόμον τὴν πλευρὰν καὶ τὴν μεγαλυτέραν διαγώνιον νέου ὁμοίου ρόμβου πρὸς τὸν ἀρχικόν, ἤτοι πλευρὰ μὲν εἶναι ἡ $2ΑΖ+ΑΗ$, διαγώνιος δὲ μεγαλυτέρα, ἡ $3ΑΖ+2ΑΗ$, ἢ $7\alpha_1+4\delta_1$ καὶ $12\alpha_1+7\delta_1$ ἀντιστοίχως. Καλοῦντες τὰς τιμὰς τῶν πλευρῶν πλευρικοὺς ἀριθμοὺς καὶ τὰς τιμὰς τῶν μεγαλυτέρων διαγωνίων, τῶν συνεχῶν κατὰ τὸν ἀνωτέρω νόμον κατασκευαζομένων ρόμβων, διαμετρικοὺς ἀριθμούς, θὰ ἔχωμεν

Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
α_1	δ_1
$\alpha_2=2\alpha_1+\delta_1$	$\delta_2=3\alpha_1+2\delta_1$
$\alpha_3=2\alpha_2+\delta_2$	$\delta_3=3\alpha_2+2\delta_2$
$\alpha_4=2\alpha_3+\delta_3$	$\delta_4=3\alpha_3+2\delta_3$
$\alpha_5=2\alpha_4+\delta_4$	$\delta_5=3\alpha_4+2\delta_4$
.	.
.	.
.	.
$\alpha_\nu=2\alpha_{\nu-1}+\delta_{\nu-1}$	$\delta_\nu=3\alpha_{\nu-1}+2\delta_{\nu-1}$

Ἐὰν θέσωμεν $\alpha=1$, $\delta=1$ λαμβάνομεν

(A) Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
$\alpha_1=1$	$\delta_1=1$
$\alpha_2=3$	$\delta_2=5$
$\alpha_3=11$	$\delta_3=19$
$\alpha_4=41$	$\delta_4=71$
$\alpha_5=153$	$\delta_5=265$
.	.
.	.
.	.

Οἱ ἀριθμοὶ οὗτοι παρέχουσι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2=3x^2-2$, ἤτοι εἶναι:

$$1^2=3\cdot1^2-2$$
$$5^2=3\cdot3^2-2$$
$$19^2=3\cdot11^2-2, \text{ κλπ.}$$

Ἐὰν θέσωμεν $\alpha_1=1$, $\delta_1=2$ λαμβάνομεν:

(B) Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
$\alpha_1 = 1$	$\delta_1 = 2$
$\alpha_2 = 4$	$\delta_2 = 7$
$\alpha_3 = 15$	$\delta_3 = 26$
$\alpha_4 = 56$	$\delta_4 = 97$
$\alpha_5 = 209$	$\delta_5 = 362$
$\alpha_6 = 780$	$\delta_6 = 1351$
.	.
.	.
.	.

Οἱ ἀριθμοὶ οὗτοι παρέχουσι τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως $y^2=3x^2+1$, ἤτοι εἶναι

$$2^2 = 3\cdot1^2 + 1$$
$$7^2 = 3\cdot4^2 + 1$$
$$26^2 = 3\cdot15^2 + 1, \text{ κλπ.}$$

Οἱ λόγοι $\frac{\delta_\nu}{\alpha_\nu}$ τῶν (A) καὶ (B) ἀποτελοῦσι δύο ἀκολουθίας ἐκ τῶν ὁποίων ἡ ἡ μὲν τῶν (A) εἶναι αὔξουσα, ἡ δὲ τῶν (B) φθίνουσα· Τὸ κοινὸν φράγμα τούτων εἶναι ἡ $\sqrt{3}$, ἤτοι εἶναι

$$1)\ \frac{1}{1} < \frac{5}{3} < \frac{19}{11} < \frac{41}{71} < \frac{265}{153} < \cdots\sqrt{3}\cdots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{2}{1},$$

καὶ 2) $265^2=3\cdot153^2-2$, $1351^2=3\cdot780^2+1$, ὡς χρησιμοποιεῖ ταῦτα ἄνευ ἀποδείξεως, ὡς γνωστά, ὁ Ἀρχιμήδης·

II.

Ἐκ τῶν ἀνωτέρω ἐκτεθέντων συνάγομεν τὸ συμπέρασμα ὅτι οἱ Πυθαγόρειοι ἐγνώριζον καὶ τὰς ἀκεραίας λύσεις τῆς ἐξισώσεως

$$\delta_\nu^2 = \lambda\alpha_\nu^2 + (\lambda - 4)^\nu (-1)^\nu, \qquad (\nu = 1, 2, 3 \ldots \text{ καὶ } \lambda \geqq 5,$$

ἀκέραιος μὴ τετράγωνος), καὶ ὅτι ἡ $\sqrt{\lambda}$ εἶναι τὸ κοινὸν φράγμα δύο ἀκολουθιῶν, μιᾶς αὐξούσης καὶ μιᾶς φθινούσης, διότι ἡ ἀπόδειξις τούτων εἶναι ἀκριβῶς ἡ αὐτὴ πρὸς τὰς ἀνωτέρω ἐκτεθείσας.

Παρέχομεν τὴν ἀπόδειξιν διὰ τὰς ἐξισώσεις

$$\delta_\nu^2 = 5\alpha_\nu^2 + (5-4)^\nu (-1)^\nu$$
$$\delta_\nu^2 = 6\alpha_\nu^2 + (6-4)^\nu (-1)^\nu$$
$$\delta_\nu^2 = 7\alpha_\nu^2 + (7-4)^\nu (-1)^\nu$$
$$\delta_\nu^2 = 8\alpha_\nu^2 + (8-4)^\nu (-1)^\nu$$
$$\vdots$$
$$\delta_\nu^2 = 17\alpha_\nu^2 + (17-4)^\nu (-1)^\nu,$$

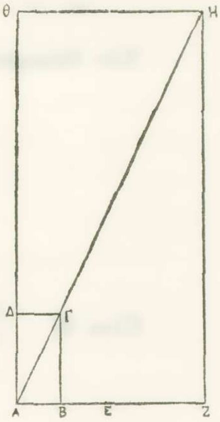
καὶ τὴν $\sqrt{5}, \sqrt{6}, \sqrt{7}, \cdots \sqrt{17}$. Εἶναι δὲ γνωστὸν ἐκ τοῦ Θεαιτήτου τοῦ Πλάτωνος ὅτι ὁ Θεόδωρος[1] (ὁ Κυρηναῖος, ὅστις θεωρεῖται Πυθαγόρειος) ἀπέδειξε τὸ ἀσύμμετρον τῆς $\sqrt{3}, \sqrt{5} \cdots \sqrt{17}$, (Θεαίτητος 147 D – 148 B).

II. 1. $\delta^2_\nu = 5\alpha^2_\nu + (5-4)^\nu (-1)^\nu$, καὶ $\sqrt{5}$ ·

Θεωροῦμεν ὀρθογώνιον παραλληλόγραμμον, τὸ ΑΒΓΔ (σχ. 3), ἔνθα ἔστω $AB = \alpha_1$, $B\Gamma = 2\alpha_1$ καὶ ἡ διαγώνιος $A\Gamma = \delta_1^2$. Εἶναι ἄρα $\delta_1^2 = 5\alpha_1^2$, καὶ $\frac{\delta_1}{\alpha_1} = \sqrt{5}$.

Ἐπὶ τῆς προεκτάσεως τῆς ΑΒ λαμβάνομεν τμῆμα $BE = AB = \alpha_1$ καὶ ἐν συνεχείᾳ τμῆμα $EZ = A\Gamma = \delta_1$. Κατὰ τὸν Εὐκλείδην II, 10 θὰ ἔχωμεν.

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2, \text{ καὶ ἐκ ταύτης}$$



Σχ. 3.

[1] 1. PAULY - WISSOWA, Realenzyklopädie unter Theodoros. Dort Literaturangabe: M. CANTOR, E. FRANK, F. HULTSCH, G. JUNGE, H. VOGT, H. G. ZEUTHEN, EVA SACHS, T. BONNESEN, H. HASSE - H. SCHOLZ, T. HEATH. — 2. Und W. L. WAN DER WAERDEN, Die Arithmetik der Pythagoreer II. Die Theorie des Irrationalen, *Mathem. Annalen*, **120**, 5./6. Heft, 1949, Springer Verlag, Berlin, Göttingen, Heidelberg. — 3. K. REIDEMEISTER, Die Arithmetik der Griechen, 1940, Leipzig. — 4. J. E. HOFMANN, Geschichte der Mathematik I, S. 26 - 27, Berlin, 1953. (*Sammlung Göschen*, 226. Walter de Gruyter und C°,) 5. ROBERT S. BRUMBAUGH, Plato's Mathematical Imagination, p. 146, *Indiana University Press*, Bloomington, 1954.

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$$

Εἶναι ἄρα καὶ $5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 20\alpha_1\delta_1 + 5\delta_1^2$.

Ἀλλὰ $\delta_1^2 = 5\alpha_1^2$. Ἑπομένως

$$5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 5\alpha_1^2 + 20\alpha_1\delta_1 + 4\delta_1^2 = (5\alpha_1 + 2\delta_1)^2.$$

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1) = \mathrm{AZ}$ εἶναι πλευρά, ἡ δὲ $(5\alpha_1 + 2\delta_1) = \mathrm{AH}$ εἶναι διαγώνιος ὁμοίου πρὸς τὸ ἀρχικὸν ὀρθογωνίου παραλληλογράμμου τοῦ ΑΖΗΘ. Κατὰ τὸν προφανῆ νόμον τῆς κατασκευῆς ἐν συνεχείᾳ ὁμοίων ὀρθογωνίων παραλληλογράμμων θὰ ἔχωμεν:

Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 5\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 5\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 5\alpha_3 + 2\delta_3$
.	.
.	.
$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_5 = 5\alpha_{\nu-1} + 2\delta_{\nu-1}$

Ἐὰν θέσωμεν $\alpha_1 = 1,\ \delta_1 = 2$ λαμβάνομεν

$\alpha_2 = 4,\ \delta_2 = 9$

$\alpha_3 = 17,\ \delta_3 = 38$

$\alpha_4 = 72,\ \delta_4 = 161$

. . .

Εἶναι δὲ $\frac{2}{1} < \frac{38}{17} < \cdots \sqrt{5} \cdots < \frac{161}{72} < \frac{9}{4}$, καὶ

$$2^2 = 5 \cdot 1^2 - 1$$
$$9^2 = 5 \cdot 4^2 + 1$$
$$38^2 = 5 \cdot 17^2 - 1$$
$$\cdot \quad \cdot$$
$$\delta_\nu^2 = 5 \cdot \alpha_\nu^2 + (5-4)^\nu (-1)^\nu.$$

ΙΙ. 2. $\delta_\nu^2 = 10\alpha_\nu^2 + (10-4)^\nu (-1)^\nu$, καὶ $\sqrt{10}$ ·

Εἰς τὸ προηγούμενον σχῆμα 3 λαμβάνομεν $\mathrm{AB} = \alpha_1$, $\mathrm{B\Gamma} = 3\alpha_1$, ὁπότε $\delta_1^2 = 10\alpha_1^2$, καὶ $\frac{\delta_1}{\alpha_1} = \sqrt{10}$.

Ἐφαρμόζοντες τὴν προηγουμένην κατασκευὴν (ΙΙ. 1) λαμβάνομεν

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$$

Εἶναι ἄρα καὶ $10(2\alpha_1 + \delta_1)^2 = 40\alpha_1^2 + 40\alpha_1\delta_1 + 10\delta_1^2$

Ἀλλὰ $\delta_1^2 = 10\alpha_1^2$. Ἑπομένως

$$10(2\alpha_1 + \delta_1)^2 = 40\alpha_1^2 + 60\alpha_1^2 + 40\alpha_1\delta_1 + 4\delta_1^2 = (10\alpha_1 + 2\delta_1)^2.$$

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(10\alpha_1 + 2\delta_1)$ διαγώνιος ὁμοίου ὀρθογωνίου παραλληλογράμμου πρὸς τὸ ἀρχικόν. Ὁ νόμος τῆς κατασκευῆς τῶν ὁμοίων ἐν συνεχείᾳ παραλληλογράμμων εἶναι προφανής. Ὅθεν θὰ εἶναι

Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 10\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 10\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 10\alpha_3 + 2\delta_3$
. . .	. . .
$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_\nu = 10\alpha_{\nu-1} + 2\delta_{\nu-1}$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$ λαμβάνομεν

$$\alpha_2 = 4 \qquad \delta_2 = 14$$
$$\alpha_3 = 22 \qquad \delta_3 = 68$$
$$\alpha_4 = 112 \qquad \delta_4 = 356$$

.

Εἶναι δὲ $\frac{2}{1} < \frac{68}{22} < \cdots \sqrt{10} \cdots < \frac{356}{112} < \frac{14}{4}$

καὶ

$$2^2 = 10 \cdot 1^2 - 6$$
$$14^2 = 10 \cdot 4^2 + 6^2$$
$$68^2 = 10 \cdot 22^2 - 6^3$$
$$356^2 = 10 \cdot 112^2 + 6^4$$

$$\delta_\nu^2 = 10 \cdot \alpha_\nu^2 + (10 - 4)^\nu (-1)^\nu$$

II. 3 $\qquad \delta_\nu^2 = 17\alpha^2 + (17\text{-}4)^\nu (-1)^\nu$, καὶ $\sqrt{17}$.

Εἰς τὸ αὐτὸ σχῆμα 3 λαμβάνομεν $AB = \alpha_1$, $B\Gamma = 4\alpha_1$, $A\Gamma = \delta_1$, ὁπότε εἶναι $\delta_1^2 = 17\alpha_1^2$ καὶ $\frac{\delta_1}{\alpha_1} = \sqrt{17}$. Ἐφαρμόζομεν πάλιν τὴν κατασκευὴν (II. 1) ὁπότε ἔχομεν $(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$.

Εἶναι ἄρα καὶ $17(2\alpha_1+\delta_1)^2=68\alpha_1^2+68\alpha_1\delta_1+17\delta_1^2$. Ἀλλὰ $\delta_1^2=17\alpha_1^2$. Ἑπομένως $17(2\alpha_1+\delta_1)^2=68\alpha_1^2+221\alpha_1^2+68\alpha_1\delta_1+4\delta_1^2=(17\alpha_1+2\delta_1)^2$.

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1+\delta_1)$ εἶναι πλευρά, ἡ δὲ $(17\alpha_1+2\delta_1)$ διαγώνιος ὁμοίου ὀρθογωνίου παραλληλογράμμου πρὸς τὸ ἀρχικόν. Ὁ νόμος σχηματισμοῦ τῶν ὁμοίων ὀρθογ. παραλληλογράμμων εἶναι προφανής. Ὅθεν θὰ ἔχωμεν

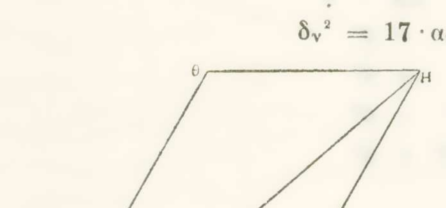
Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 17\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 17\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 17\alpha_3 + 2\delta_3$
.	.
.	.
.	.
$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_\nu = 17\alpha_{\nu-1} + 2\delta_{\nu-1}$.

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$ λαμβάνομεν

$$\alpha_2 = 4 \qquad \delta_2 = 21$$
$$\alpha_3 = 29 \qquad \delta_3 = 110$$
$$\alpha_4 = 168 \qquad \delta_4 = 713$$

Εἶναι δὲ $\frac{2}{1} < \frac{110}{29} < \cdots\sqrt{17}\cdots < \frac{713}{168} < \frac{21}{4}$, καὶ

$$2^2 = 17\cdot 1^2 - 13$$
$$21^2 = 17\cdot 4^2 + 13^2$$
$$110^2 = 17\cdot 29^2 - 13^3$$
$$713^2 = 17\cdot 168^2 + 13^4$$
$$\vdots$$
$$\delta_\nu^2 = 17\cdot\alpha_\nu^2 + (17-4)^\nu\,(-1)^\nu$$



Σχ. 4.

ΙΙ. 4.

Θεωροῦμεν τὸ ρομβοειδὲς παραλληλόγραμμον ΑΒΓΔ (σχ. 4) ἔνθα γωνία $ΑΒΓ=120^\circ$, $ΑΒ=\alpha_1$, $ΒΓ=2\alpha_1$ καὶ ἡ μεγαλυτέρα διαγώνιος $ΑΓ=\delta_1$. Κατὰ τὸν Εὐκλείδην ΙΙ, 12 θὰ εἶναι $\delta_1^2=7\alpha_1^2$ ὁπότε $\frac{\delta_1}{\alpha_1}=\sqrt{7}$. Πάλιν ἐφαρμόζομεν τὴν αὐτὴν κατασκευὴν (ΙΙ. 1), ὁπότε λαμβάνοντες $ΒΕ=\alpha_1$, $ΕΖ=\delta_1$, θὰ ἔχωμεν κατὰ τὸν Εὐκλείδην ΙΙ, 10.

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2, \quad \text{ἐξ ἧς}$$
$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2.$$

Εἶναι ἄρα καὶ $7(2\alpha + \delta_1)^2 = 28\alpha_1^2 + 28\alpha_1\delta_1 + 7\delta_1^2$.

Ἀλλὰ $\delta_1^2 = 7\alpha_1^2$. Ἑπομένως

$$7(2\alpha_1 + \delta_1)^2 = 28\alpha_1^2 + 21\alpha_1^2 + 28\alpha_1\delta_1 + 4\delta_1^2 = (7\alpha_1 + 2\delta_1)^2.$$

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(7\alpha_1 + 2\delta_1)$ διαγώνιος μεγαλυτέρα ὁμοίου πρὸς τὸ ἀρχικὸν ρομβοειδοῦς παραλληλογράμμου. Κατὰ τὸν προφανῆ νόμον κατασκευῆς ὁμοίων ρομβοειδῶν παραλληλογράμμων θὰ ἔχωμεν:

Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 7\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 7\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 7\alpha_3 + 2\delta_3$
.	.
$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_\nu = 7\alpha_{\nu-1} + 2\delta_{\nu-1}$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$, λαμβάνομεν

$\alpha_2 = 4$ $\quad\delta_2 = 11$

$\alpha_3 = 19$ $\quad\delta_3 = 50$

$\alpha_4 = 88$ $\quad\delta_4 = 233$

⋮

Εἶναι δὲ $\frac{2}{1} < \frac{50}{19} < \cdots \sqrt{7} \cdots < \frac{233}{88} < \frac{11}{4}$,

καὶ
$$2^2 = 7 \cdot 1^2 - 3$$
$$11^2 = 7 \cdot 4^2 + 3^2$$
$$50^2 = 7 \cdot 19^2 - 3^3$$
$$233^2 = 7 \cdot 88^2 + 3^4$$

$$\delta_\nu^2 = 7 \cdot \alpha_\nu^2 + (7-4)^\nu (-1)^\nu.$$

II. 5. Εἰς τὸ προηγούμενον σχῆμα 4 λαμβάνομεν $AB = \alpha_1$, $B\Gamma = 3\alpha_1$, $A\Gamma = \delta_1$. Κατὰ τὸν Εὐκλείδην II, 12 εἶναι $\delta_1^2 = 13\alpha_1^2$, καὶ $\frac{\delta_1}{\alpha_1} = \sqrt{13}$. Πάλιν ἐφαρμόζομεν τὴν αὐτὴν κατασκευὴν ὡς καὶ προηγουμένως, ἤτοι λαμβάνομεν ΒΕ = α_1, ΕΖ = δ_1 ὁπότε κατὰ τὸ II, 10 τοῦ Εὐκλείδου θὰ εἶναι

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2 , \quad \text{ἐξ ἧς}$$

$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2 .$$

Εἶναι ἄρα καὶ $13(2\alpha_1 + \delta_1)^2 = 52\alpha_1^2 + 52\alpha_1\delta_1 + 13\delta_1^2$.

Ἀλλὰ $\delta_1^2 = 13\alpha_1^2$ · ἑπομένως

$$13(2\alpha_1 + \delta_1)^2 = 52\alpha_1^2 + 117\alpha_1^2 + 52\alpha_1\delta^2 + 4\delta_1 = (13\alpha_1 + 2\delta_1)^2 .$$

Ἡ σχέσις ὅμως αὕτη σημαίνει ὅτι ἡ μὲν $(2\alpha_1 + \delta_1)$ εἶναι πλευρά, ἡ δὲ $(13\alpha_1 + 2\delta_1)$ διαγώνιος μεγαλυτέρα, ὁμοίου ρομβοειδοῦς παραλληλογράμμου πρὸς τὸ ἀρχικόν. Κατὰ τὸν προφανῆ νόμον κατασκευῆς τῶν ὁμοίων ρομβοειδῶν παραλληλογράμμων θὰ ἔχωμεν

Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = 13\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = 13\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = 13\alpha_3 + 2\delta_3$
.	.
.	.
$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_\nu = 13\alpha_{\nu-1} + 2\delta_{\nu-1}$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$ λαμβάνομεν

$\alpha_2 = 4$	$\delta_2 = 17$
$\alpha_3 = 25$	$\delta_3 = 86$
$\alpha_4 = 136$	$\delta_4 = 497$
.	.
.	.

Εἶναι δε $\frac{2}{1} < \frac{86}{25} < \cdots \sqrt{13} < \cdots \frac{497}{136} < \frac{17}{4}$, καὶ

$$2^2 = 13 \cdot 1^2 - 9$$

$$17^2 = 13 \cdot 4^2 + 9^2$$

$$86^2 = 13 \cdot 25^2 - 9^3$$

$$497^2 = 13 \cdot 136^2 + 9^4$$

$$\delta_\nu^2 = 13\alpha_\nu^2 + (13 - 4)^\nu (-1)^\nu .$$

Ἐκ τῶν ἀνωτέρω ἐκτεθεισῶν κατασκευῶν καὶ ἀποδείξεων καθίσταται αὐτονόητος ὁ σχηματισμὸς ἀντιστοίχων πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν δι' ἀλγεβρικοῦ καθαρῶς ὑπολογισμοῦ καὶ οὐχὶ γεωμετρικοῦ, διὰ τὴν $\sqrt{6}$, $\sqrt{8}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{14}$, $\sqrt{15}$, $\cdots$ καὶ τὰς συναφεῖς ἐξισώσεις.

Οὕτω θὰ εἶναι

II. 6. Διὰ $\delta_\nu^2 = 6\alpha_\nu^2 + (6-4)^\nu(-1)^\nu$ καὶ $\sqrt{6}$.

$$
\begin{array}{llll}
\alpha_1 & & \delta_1 & \\
\alpha_2 & = 2\alpha_1 + \delta_1 & \delta_2 & = 6\alpha_1 + 2\delta_1 \\
\alpha_3 & = 2\alpha_2 + \delta_2 & \delta_3 & = 6\alpha_2 + 2\delta_2 \\
\alpha_4 & = 2\alpha_3 + \delta_3 & \delta_4 & = 6\alpha_3 + 2\delta_3 \\
\vdots & & \vdots & \\
\alpha_\nu & = 2\alpha_{\nu-1} + \delta_{\nu-1} & \delta_\nu & = 6\alpha_{\nu-1} + 2\delta_{\nu-1}
\end{array}
$$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$ λαμβάνομεν

$$
\begin{array}{ll}
\alpha_2 = 4 & \delta_2 = 10 \\
\alpha_3 = 18 & \delta_3 = 44 \\
\alpha_4 = 80 & \delta_4 = 196
\end{array}
$$

Εἶναι δὲ $\frac{2}{1} < \frac{44}{18} < \cdots \sqrt{6} \cdots < \frac{196}{80} < \frac{10}{4}$, καὶ

$$
\begin{aligned}
2^2 &= 6\cdot 1^2 - 2 \\
10^2 &= 6\cdot 4^2 + 2^2 \\
44^2 &= 6\cdot 18^2 - 2^3 \\
196^2 &= 6\cdot 80^2 + 2^4 \\
&\vdots \\
\delta_\nu^2 &= 6\cdot\alpha_\nu^2 + (6-4)^\nu(-1)^\nu .
\end{aligned}
$$

II. 7. Διὰ $\delta_\nu^2 = 8\alpha_\nu^2 + (8-4)^\nu(-1)^\nu$ καὶ $\sqrt{8}$.

$$
\begin{array}{llll}
\alpha_1 & & \delta_1 & \\
\alpha_2 & = 2\alpha_1 + \delta_1 & \delta_2 & = 8\alpha_1 + 2\delta_1 \\
\alpha_3 & = 2\alpha_2 + \delta_2 & \delta_3 & = 8\alpha_2 + 2\delta_2 \\
\alpha_4 & = 2\alpha_3 + \delta_3 & \delta_4 & = 8\alpha_3 + 2\delta_3 \\
\vdots & & \vdots & \\
\alpha_\nu & = 2\alpha_{\nu-1} + \delta_{\nu-1} & \delta_\nu & = 8\alpha_{\nu-1} + 2\delta_{\nu-1}
\end{array}
$$

Ἐὰν θέσωμεν $\alpha_1 = 1$ $\delta_1 = 2$ λαμβάνομεν

$$
\begin{array}{ll}
\alpha_2 = 4 & \delta_2 = 12
\end{array}
$$

$$\alpha_3 = 20 \qquad \delta_3 = 56$$
$$\alpha_4 = 96 \qquad \delta_4 = 272$$
$$\vdots \qquad\qquad \vdots$$

Εἶναι δὲ $\frac{2}{1} < \frac{56}{20} < \cdots \sqrt{8} \cdots < \frac{272}{96} < \frac{12}{4}$, καὶ

$$2^2 = 8 \cdot 1^2 - 4$$
$$12^2 = 8 \cdot 4^2 + 4^2$$
$$59^2 = 8 \cdot 20^2 - 4^3$$
$$272^2 = 8 \cdot 96^2 + 4^4$$
$$\vdots$$
$$\delta_\nu^2 = 8 \cdot \alpha_\nu^2 + (8-4)^\nu (-1)^\nu .$$

II. 8. Διὰ $\delta_\nu^2 = 11\alpha_\nu^2 + (11-4)^\nu (-1)^\nu$, καὶ $\sqrt{11}$ ·

$$\alpha_1 \qquad\qquad \delta_1$$
$$\alpha_2 = 2\alpha_1 + \delta_1 \qquad \delta_2 = 11\alpha_1 + 2\delta_1$$
$$\alpha_3 = 2\alpha_2 + \delta_2 \qquad \delta_3 = 11\alpha_2 + 2\delta_2$$
$$\alpha_4 = 2\alpha_3 + \delta_3 \qquad \delta_4 = 11\alpha_3 + 2\delta_3$$
$$\vdots \qquad\qquad \vdots$$
$$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1} \qquad \delta_\nu = 11\alpha_{\nu-1} + 2\delta_{\nu-1}$$

Ἐὰν θέσωμεν $\alpha_1 = 1$ καὶ $\delta_1 = 2$ λαμβάνομεν

$$\alpha_2 = 4 \qquad \delta_2 = 15$$
$$\alpha_3 = 23 \qquad \delta_3 = 74$$
$$\alpha_4 = 120 \qquad \delta_4 = 401 ,$$

Εἶναι δὲ $\frac{2}{1} < \frac{74}{23} < \cdots \sqrt{11} \cdots < \frac{401}{120} < \frac{15}{4}$, καὶ

$$2^2 = 11 \cdot 1^2 - 7$$
$$15^2 = 11 \cdot 4^2 + 7^2$$
$$74^2 = 11 \cdot 23^2 - 7^3$$
$$401^2 = 11 \cdot 120^2 + 7^4$$
$$\vdots$$
$$\delta_\nu^2 = 11\alpha_\nu^2 + (11-4)^\nu (-1)^\nu .$$

II. 9. Διὰ $\delta_\nu^2 = 12\alpha_\nu^2 + (12-4)^\nu(-1)^\nu$, καὶ $\sqrt{12}$.

$$
\begin{array}{llll}
\alpha_1 & & \delta_1 & \\
\alpha_2 = 2\alpha_1 + \delta_1 & & \delta_2 = 12\alpha_1 + 2\delta_1 & \\
\alpha_3 = 2\alpha_2 + \delta_2 & & \delta_3 = 12\alpha_2 + 2\delta_2 & \\
\alpha_4 = 2\alpha_3 + \delta_3 & & \delta_4 = 12\alpha_3 + 2\delta_3 & \\
\cdot & & \cdot & \\
\cdot & & \cdot & \\
\cdot & & \cdot & \\
\cdot & & \cdot & \\
\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1} & & \delta_\nu = 12\alpha_{\nu-1} + 2\delta_{\nu-1}\,. &
\end{array}
$$

Ἐὰν θέσωμεν $\alpha_1 = 1$ $\delta_1 = 2$ λαμβάνομεν

$$
\begin{array}{ll}
\alpha_2 = 4 & \delta_2 = 16 \\
\alpha_3 = 24 & \delta_3 = 80 \\
\alpha_4 = 128 & \delta_4 = 448\,.
\end{array}
$$

Εἶναι δὲ $\frac{2}{1} < \frac{80}{24} < \cdots \sqrt{12} \cdots < \frac{448}{128} < \frac{16}{4}$, καὶ

$$
\begin{array}{l}
2^2 = 12 \cdot 1^2 - 8 \\
16^2 = 12 \cdot 4^2 + 8^2 \\
80^2 = 12 \cdot 24^2 - 8^3 \\
448^2 = 12 \cdot 128^2 + 8^4 \\
\cdot \\
\cdot \\
\cdot \\
\delta_\nu^2 = 12 \cdot \alpha_\nu^2 + (12-4)^\nu(-1)^\nu\,.
\end{array}
$$

II. 10. Διὰ $\delta_\nu^2 = 14\alpha_\nu^2 + (14-4)^\nu(-1)^\nu$, καὶ $\sqrt{14}$.

$$
\begin{array}{llll}
\alpha_1 & & \delta_1 & \\
\alpha_2 = 2\alpha_1 + \delta_1 & & \delta_2 = 14\alpha_1 + 2\delta_1 & \\
\alpha_3 = 2\alpha_2 + \delta_2 & & \delta_3 = 14\alpha_2 + 2\delta_2 & \\
\alpha_4 = 2\alpha_3 + \delta_3 & & \delta_4 = 14\alpha_3 + 2\delta_3 & \\
\cdot & & \cdot & \\
\cdot & & \cdot & \\
\cdot & & \cdot & \\
\cdot & & \cdot & \\
\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1} & & \delta_\nu = 14\alpha_{\nu-1} + 2\delta_{\nu-1} &
\end{array}
$$

Ἐὰν θέσωμεν $\alpha_1 = 1$ $\delta_1 = 2$ λαμβάνομεν

$$
\begin{array}{ll}
\alpha_2 = 4 & \delta_2 = 18 \\
\alpha_3 = 26 & \delta_3 = 92 \\
\alpha_4 = 144 & \delta_4 = 548
\end{array}
$$

Εἶναι δὲ $\frac{2}{1} < \frac{92}{26} < \cdots \sqrt{14} \cdots < \frac{548}{144} < \frac{18}{4}$, καὶ

$$2^2 = 14 \cdot 1^2 - 10$$
$$18^2 = 14 \cdot 4^2 + 10^2$$
$$92^2 = 14 \cdot 26^2 - 10^3$$
$$548^2 = 14 \cdot 144^2 + 10^4$$
$$\vdots$$
$$\delta_\nu^2 = 14 \cdot \alpha_\nu^2 + (14-4)^\nu (-1)^\nu .$$

II. 11. Διὰ $\delta_\nu^2 = 15\alpha_\nu^2 + (15-4)(-1)^\nu$, καὶ $\sqrt{15}$.

$$\alpha_1 \qquad\qquad \delta_1$$
$$\alpha_2 = 2\alpha_1 + \delta_1 \qquad \delta_2 = 15\alpha_1 + 2\delta_1$$
$$\alpha_3 = 2\alpha_2 + \delta_2 \qquad \delta_3 = 15\alpha_2 + 2\delta_2$$
$$\alpha_4 = 2\alpha_3 + \delta_3 \qquad \delta_4 = 15\alpha_3 + 2\delta_3$$
$$\vdots \qquad\qquad \vdots$$
$$\alpha^\nu = 2\alpha_{\nu-1} + \delta_{\nu-1} \qquad \delta_\nu = 15\alpha_{\nu-1} + 2\delta_{\nu-1}$$

Ἐὰν θέσωμεν $\alpha_1 = 1$, $\delta_1 = 2$ λαμβάνομεν

$$\alpha_2 = 4 \qquad \delta_2 = 19$$
$$\alpha_3 = 27 \qquad \delta_3 = 98$$
$$\alpha_4 = 152 \qquad \delta_4 = 601$$
$$\vdots$$

Εἶναι δὲ $\frac{2}{1} < \frac{98}{27} < \cdots \sqrt{15} \cdots < \frac{601}{152} < \frac{19}{4}$, καὶ

$$2^2 = 15 \cdot 1^2 - 11$$
$$19^2 = 15 \cdot 4^2 + 11^2$$
$$98^2 = 15 \cdot 27^2 - 11^3$$
$$601^2 = 15 \cdot 152^2 + 11^4$$
$$\vdots$$
$$\delta_\nu^2 = 15 \cdot \alpha_\nu^2 + (15-4)^\nu (-1)^\nu .$$

III. 1. Ἐκ τῶν προηγουμένων παρατηροῦμεν ὅτι ἡ κατὰ τὸ εὐκλείδειον θεώρημα II, 10 γεωμετρικὴ κατασκευή, τὴν ὁποίαν μνημονεύει ὁ Πρόκλος, ἄγει εἰς τὴν ταυτότητα (1), $(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2$, τὴν ἀποδεικνυομένην κατὰ τὸ εὐκλείδειον θεώρημα II, 4. Ἐὰν $\delta_1^2 = \lambda\alpha_1^2$, ἔνθα

$\lambda \geqq 5$, ἀκέραιος μὴ τετράγωνος, θὰ εἶναι καὶ $(\lambda-4)\delta_1^2 = (\lambda-4)\lambda\alpha_1^2$, ὁπότε ἐκ τῆς (1) ἔχομεν

$$\lambda(2\alpha_1+\delta_1)^2 = 4\lambda\alpha_1^2 + 4\lambda\alpha_1\delta_1 + (\lambda-4)\lambda\alpha_1^2 + 4\delta_1^2 = (\lambda\alpha_1+2\delta_1)^2 .$$

Ἑπομένως θὰ εἶναι

	Πλευρικοὶ ἀριθμοί.	Διαμετρικοὶ ἀριθμοί.
1)	α_1	δ_1
	$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = \lambda\alpha_1 + 2\delta_1$
	$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = \lambda\alpha_2 + 2\delta_2$
	$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = \lambda\alpha_3 + 2\delta_3$
	. . . .	. . . .
	$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_\nu = \lambda\alpha_{\nu-1} + 2\delta_{\nu-1}$,

2) $$\frac{\delta_1}{\alpha_1} < \frac{\delta_3}{\alpha_3} < \frac{\delta_5}{\alpha_5} < \cdots \sqrt{\lambda} \cdots < \frac{\delta_6}{\alpha_6} < \frac{\delta_4}{\alpha_4} < \frac{\delta_2}{\alpha_2}, \quad (\alpha_1=1, \delta_1=2).$$

3) $$\delta_\nu^2 = \lambda\alpha_\nu^2 + (\lambda-4)^\nu(-1)^\nu$$

III. 2.

Εἶναι δυνατὸν ἡ $\sqrt{2}$ νὰ ὑπολογισθῇ καὶ ἐκ τῶν πλευρικῶν καὶ διαμετρικῶν ἀριθμῶν τῆς μορφῆς $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, $\delta_\nu = \lambda\alpha_{\nu-1} + 2\delta_{\nu-1}$, ὅταν $\lambda = 2$. Τὴν μέθοδον ταύτην καλοῦμεν γενικὴν πρὸς διάκρισιν ἀπὸ τῆς μεθόδου τῆς διασωθείσης ὑπὸ τοῦ Θέωνος τοῦ Σμυρναίου καὶ τοῦ Πρόκλου, τὴν ὁποίαν καλοῦμεν εἰδικήν.

Πρὸς σύγκρισιν παραθέτομεν τὰ ἐξαγόμενα καὶ τῶν δύο μεθόδων.

Α΄. Μέθοδος εἰδική, $\alpha_\nu = \alpha_{\nu-1} + \delta_{\nu-1}$, $\delta_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$.

$\alpha_1 = 1$	$\delta_1 = 1$
$\alpha_2 = 2$	$\delta_2 = 3$
$\alpha_3 = 5$	$\delta_3 = 7$
$\alpha_4 = 12$	$\delta_4 = 17$,
. . . .	

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_3}{\alpha_3} < \frac{\delta_5}{\alpha_5} < \cdots \sqrt{2} \cdots < \frac{\delta_6}{\alpha_6} < \frac{\delta_4}{\alpha_4} < \frac{\delta_2}{\alpha_2}$$

$1^2 = 2\cdot1^2 - 1$, $3^2 = 2\cdot2^2 + 1$, $7^2 = 2\cdot5^2 - 1$, $\cdots\cdot$ $\delta_\nu^2 = 2\alpha_\nu^2 + (-1)^\nu$.

[Διὰ $\alpha_1 = 1$, $\delta_1 = 2$.

$\alpha_1 = 1$	$\delta_1 = 2$
$\alpha_2 = 3$	$\delta_2 = 4$
$\alpha_3 = 7$	$\delta_3 = 10$

$\alpha_4 = 17 \qquad \delta_4 = 24$

$\vdots$

$$\frac{\delta_1}{\alpha_1} > \frac{\delta_3}{\alpha_3} > \frac{\delta_5}{\alpha_5} > \cdots \sqrt{2} \cdots > \frac{\delta_6}{\alpha_6} > \frac{\delta_4}{\alpha_4} > \frac{\delta_2}{\alpha_2}$$

$$2^2 = 2 \cdot 1^2 + 2$$
$$4^2 = 2 \cdot 3^2 - 2$$
$$10^2 = 2 \cdot 7^2 + 2$$
$$24^2 = 2 \cdot 17^2 - 2$$
$$\vdots$$
$$\delta_\nu^2 = 2\alpha_\nu^2 + (2-4)(-1)^\nu].$$

Β′. Μέθοδος γενική, $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, $\delta_\nu = 2\alpha_{\nu-1} + 2\delta_{\nu-1}$.

Διὰ $\alpha_1 = 1$, $\delta_1 = 1$.

$\alpha_1 = 1 \qquad \delta_1 = 1$

$\alpha_2 = 3 \qquad \delta_2 = 4$

$\alpha_3 = 10 \qquad \delta_3 = 14$

$\alpha_4 = 34 \qquad \delta_4 = 48$

$\vdots$

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_2}{\alpha_2} < \frac{\delta_3}{\alpha_3} < \cdots \sqrt{2}$$

$$1^2 = 2 \cdot 1^2 - 1$$
$$4^2 = 2 \cdot 3^2 - 2$$
$$14^2 = 2 \cdot 10^2 - 2^2$$
$$48^2 = 2 \cdot 34^2 - 2^3$$
$$\vdots$$
$$\delta_\nu^2 = 2 \cdot \alpha_\nu^2 + (2-4)^{\nu-1} . (-1)^\nu .$$

Διὰ $\alpha_1 = 1$, $\delta_1 = 2$,

$\alpha_1 = 1 \qquad \delta_1 = 2$

$\alpha_2 = 4 \qquad \delta_2 = 6$

$\alpha_3 = 14 \qquad \delta_3 = 20$

$\alpha_4 = 48 \qquad \delta_4 = 68$,

$\vdots$

$$\sqrt{2} \cdots < \frac{\delta_3}{\alpha_3} < \frac{\delta_2}{\alpha_2} < \frac{\delta_1}{\alpha_1} .$$

$$2^2 = 2\cdot 1^2 + 2$$
$$6^2 = 2\cdot 4^2 + 2^2$$
$$20^2 = 2\cdot 14^2 + 2^3$$
$$68^2 = 2\cdot 48^2 + 2^4$$
$$\vdots$$
$$\delta_\nu^2 = 2\cdot\alpha_\nu^2 + (2-4)^\nu.(-1)^\nu .$$

Κατ' ἀντιστοιχίαν πρὸς τὴν ὑπὸ τοῦ Ἀρχιμήδους παρεχομένην τιμὴν τῆς $\sqrt{3}$ θὰ εἴχομεν, διὰ τὴν $\sqrt{2}$,

$$\frac{1}{1} < \frac{4}{3} < \frac{14}{10} < \frac{48}{34} < \frac{164}{116} < \cdots \sqrt{2} \cdots < \frac{792}{560} < \frac{232}{164} < \frac{68}{48} < \frac{20}{14} < \frac{6}{4} < \frac{2}{1} .$$

Εἶναι φανερόν, ὅτι εἶναι προτιμοτέρα ἡ ὑπὸ τοῦ Θέωνος τοῦ Σμυρναίου καὶ τοῦ Πρόκλου διασωθεῖσα μέθοδος διὰ τὸν ὑπολογισμὸν τῆς $\sqrt{2}$.

SUMMARY

I 1. The law of formation of the side- and diameter-(diagonal-) numbers is explained by Theon of Smyrna. According to Proclus the related identity is proved by Euclid book II, proposition 10.

Side numbers $\alpha_\nu = \alpha_{\nu-1} + \delta_{\nu-1}$, diagonal numbers $\delta_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, $\delta_\nu^2 = 2\alpha_\nu^2 \mp 1$. For $\alpha_1 = 1$, $\delta_1 = 1$ we have $\frac{1}{1} < \frac{7}{5} < \cdots \sqrt{2} \cdots < \frac{17}{12} < \frac{3}{2}$.

2. Archimedes for the arithmetical approximation to π starts from a greater and a lesser limit to the value of $\sqrt{3}$, which without remark as known,

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}, \quad 265^2 = 3\cdot 153^2 - 2, \quad 1351^2 = 3\cdot 780^2 + 1 .$$

We give the following interpretation on the archimedean formula, with pythagorean method of the side-and diagonal-numbers. In the figure 2 is $AB = \alpha_1$ the side, $A\Gamma = \delta_1$ the greater diagonal of the rhomb $AB\Gamma\Delta$, and the greater angle $AB\Gamma = 120^\circ$. Then $\delta_1^2 = 3\alpha_1^2$, $\delta_1 : \alpha_1 = \sqrt{3}$. According to Euclid II Prop. 10 we have

$$(2\alpha_1 + \delta_1)^2 + \delta_1^2 = 2\alpha_1^2 + 2(\alpha_1 + \delta_1)^2$$
$$(2\alpha_1 + \delta_1)^2 = 4\alpha_1^2 + 4\alpha_1\delta_1 + \delta_1^2, \quad (1)$$

It is also $3(2\alpha_1 + \delta_1)^2 = 12\alpha_1^2 + 12\alpha_1\delta_1 + 3\delta_1^2$, and because $\delta_1^2 = 3\alpha_1^2$

$$3(2\alpha_1 + \delta_1)^2 = 9\alpha_1^2 + 12\alpha_1\delta_1 + 4\delta_1^2, = (3\alpha_1 + 2\delta_1)^2 .$$

The law of formation of the corresponding side-and diagonal-numbers is evidently. Side numbers $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, diagonal numbers $\delta_\nu = 3\alpha_{\nu-1} + 2\delta_{\nu-1}$.

When	α_1	=	1	δ_1	=	1	When	α_1	=	1	δ_1	=	2
	α_2	=	3	δ_2	=	5		α_2	=	4	δ_2	=	7
	α_3	=	11	δ_3	=	19		α_3	=	15	δ_3	=	26
	α_4	=	41	δ_4	=	71		α_4	=	56	δ_4	=	97
	α_5	=	153	δ_5	=	265		α_5	=	209	δ_5	=	362
								α_6	=	780	δ_6	=	1351

and $\delta_\nu^2 = 3\alpha_\nu^2 - 2$ and $\delta_\nu^2 = 3\alpha_\nu^2 + 1$.

$$\frac{1}{1} < \frac{5}{3} < \frac{19}{11} < \frac{71}{41} < \frac{265}{153} < \cdots < \sqrt{3} \cdots < \frac{1351}{780} < \frac{362}{209} < \frac{97}{56} < \frac{26}{15} < \frac{7}{4} < \frac{2}{1}$$

II. In the following we start from the identity (1).

1. In the figure 3 we take $AB = \alpha_1$, $B\Gamma = 2\alpha_1$, $A\Gamma = \delta_1$. Then $\delta_1^2 = 5\alpha_1^2$, $\delta_1 : \alpha_1 = \sqrt{5}$, and $5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 20\alpha_1\delta_1 + 5\delta_1^2$. Because $\delta_1^2 = 5\alpha_1^2$ is $5(2\alpha_1 + \delta_1)^2 = 20\alpha_1^2 + 5\alpha_1^2 + 20\alpha_1\delta_1 + 4\delta_1^2 = (5\alpha_1 + 2\delta_1)^2$.

Side numbers $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, diagonal numbers, $\delta_\nu = 5\alpha_{\nu-1} + 2\delta_{\nu-1}$, $\delta_\nu^2 = 5\alpha_\nu^2 + (5-4)^\nu (-1)^\nu$.

When $\alpha_1 = 1$, $\delta_1 = 2$, $\frac{2}{1} < \frac{38}{14} < \cdots \sqrt{5} \cdots < \frac{161}{72} < \frac{9}{4}$.

2. In the same figure 3 we take $AB = \alpha_1$, $B\Gamma = 3\alpha_1$, $A\Gamma = \delta_1$. Then $\delta_1^2 = 10\alpha_1^2$, $\delta_1 : \alpha_1 = \sqrt{10}$, and $10(2\alpha_1 + \delta_1)^2 = 40\alpha_1^2 + 40\alpha_1\delta_1 + 10\delta_1^2$. Because $\delta_1^2 = 10\alpha_1^2$ is $10(2\alpha_1 + \delta_1)^2 = 100\alpha_1^2 + 40\alpha_1\delta_1 + 4\delta_1^2 = (10\alpha_1 + 2\delta_1)^2$.

Side numbers $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, diagonal numbers $\delta_\nu = 10\alpha_{\nu-1} + 2\delta_{\nu-1}$, $\delta_\nu^2 = 10\alpha_\nu^2 + (10-4)^\nu (-1)^\nu$. When $\alpha_1 = 1$, $\delta_1 = 2$,

$$\frac{2}{1} < \frac{68}{22} < \cdots \sqrt{10} \cdots < \frac{356}{112} < \frac{14}{4}.$$

3. In the same figure 3 we take $AB = \alpha_1$, $B\Gamma = 4\alpha_1$, $A\Gamma = \delta_1$ Then $\delta_1^2 = 17\alpha_1^2$, $\delta_1 : \alpha_1 = \sqrt{17}$, and $17(2\alpha_1 + \delta_1)^2 = 68\alpha_1^2 + 68\alpha_1\delta_1 + 17\delta_1^2$. Because $\delta_1^2 = 17\delta_1^2 = 289\alpha_1^2 + 68\alpha_1\delta_1 + 4\delta_1^2 = (17\alpha_1 + 2\delta_1)^2$.

Side numbers $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, D. N. $\delta_\nu = 17\alpha_{\nu-1} + 2\delta_{\nu-1}$,

$$\delta_\nu^2 = 17\alpha_\nu^2 + (17-4)^\nu (-1)^\nu.$$

When $\alpha_1 = 1$ $\delta_1 = 2$, $\frac{2}{1} < \frac{110}{29} < \cdots \sqrt{17} \cdots < \frac{713}{168} < \frac{21}{14}$.

4. In the figure 4 we take $AB = \alpha_1$, $B\Gamma = 2\alpha_1$, $A\Gamma = \delta_1$. The angle $AB\Gamma = 120^\circ$. Then $\delta_1^2 = 7\alpha_1^2$, $\delta_1 : \alpha_1 = \sqrt{7}$, and $7(2\alpha_1 + \delta_1)^2 = 28\alpha_1^2 + 28\alpha_1\delta_1 + 7\delta_1^2$. Because $\delta_1^2 = 7\alpha_1^2$, $7(2\alpha_1 + \delta_1)^2 = 49\alpha_1^2 + 28\alpha_1\delta_1 + 4\delta_1^2 = (7\alpha_1 + 2\delta_1)^2$.

S.N., $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, D.N., $\delta_\nu = 7\alpha_{\nu-1} + 2\delta_{\nu-1}$, $\delta_\nu^2 = 7\alpha_\nu^2 + (7-4)^\nu (-1)^\nu$.

When $\alpha_1 = 1$, $\delta_1 = 2$, $\frac{2}{1} < \frac{50}{19} < \cdots \sqrt{7} \cdots < \frac{233}{88} < \frac{11}{4}$.

5. Jn the same figure 4 we take $AB = \alpha_1$, $B\Gamma = 3\alpha_1$, $A\Gamma = \delta_1$. Then $\delta_1^2 = 13\alpha_1^2$, $\delta_1 : \alpha_1 = \sqrt{13}$, $13(2\alpha_1 + \delta_1)^2 = 52\alpha_1^2 + 52\alpha_1\delta_1 + 13\delta_1^2$.

Because $\delta_1^2 = 13\alpha_1^2$ is $13(2\alpha_1+\delta_1)^2 = 169\alpha_1^2 + 52\alpha_1\delta_1 + 4\delta_1^2 = (13\alpha_1 + 2\delta_1)^2$.
S.N., $\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$, D. N., $\delta_\nu = 13\alpha_{\nu-1} + 2\delta_{\nu-1}$, $\delta_\nu^2 = 13\alpha_1^2 + (13-4)^\nu(-1)^\nu$.

$$\frac{2}{1} < \frac{86}{25} < \cdots \sqrt{13} \cdots < \frac{497}{136} < \frac{17}{4}$$

In the same way we take the side–und the diagonal–numbers for the $\sqrt{6}$, $\sqrt{8}$, $\sqrt{11}$, $\sqrt{14}$, $\sqrt{15}$. (We mention Theaetetus of Plato 147 D).

III. If $\lambda \geqq 5$, integer no square number and $\delta_1^2 = \lambda\alpha_1^2$, then $(\lambda-4)\delta_1^2 = (\lambda-4)\lambda\alpha_1^2$, and according to the identity (1)

$$\lambda(2\alpha_1+\delta_1)^2 = 4\lambda\alpha_1^2 + 4\lambda\,\alpha_1\delta_1 + \lambda\delta_1^2,$$
$$= 4\lambda\alpha_1^2 + (\lambda-4)\lambda\alpha_1^2 + 4\lambda\alpha_1\delta_1 + 4\delta_1^2 = (\lambda\alpha_1 + 2\delta_1)^2.$$

Side numbers	Diagonal numbers
α_1	δ_1
$\alpha_2 = 2\alpha_1 + \delta_1$	$\delta_2 = \lambda\alpha_1 + 2\delta_1$
$\alpha_3 = 2\alpha_2 + \delta_2$	$\delta_3 = \lambda\alpha_2 + 2\delta_2$
$\alpha_4 = 2\alpha_3 + \delta_3$	$\delta_4 = \lambda\alpha_3 + 2\delta_3$
.	.
.	.
.	.
$\alpha_\nu = 2\alpha_{\nu-1} + \delta_{\nu-1}$	$\delta_\nu = \lambda\alpha_{\nu-1} + 2\delta_{\nu-1}$

$$\frac{\delta_1}{\alpha_1} < \frac{\delta_3}{\alpha_3} < \frac{\delta_5}{\alpha_5} < \cdots \sqrt{\lambda} \cdots < \frac{\delta_6}{\alpha_6} < \frac{\delta_4}{\alpha_4} < \frac{\delta_2}{\alpha_2}, \quad \text{and}$$

$\delta_\nu^2 = \lambda\alpha_\nu^2 + (\lambda-4)^\nu(-1)^\nu$. We take here always $\alpha_1 = 1$, $\delta_1 = 2$. $\lambda = 2$ and $\lambda = 3$ are special cases.

ΒΙΒΛΙΟΓΡΑΦΙΑ

PAULY-WISSOWA: *Real Encyclopädie der classischen Altertumswissenschaft*: unter Arithmetica, Theodoros.

J. E. HOFMANN: Geschichte der Mathematik I, Walter de Gruyter und Co, B. 226, Berlin, 1953.

ΜΙΧΑΗΛ ΣΤΕΦΑΝΙΔΗΣ: Εἰσαγωγὴ εἰς τὴν ἱστορίαν τῶν φυσικῶν Ἐπιστημῶν. Ἀθῆναι, 1938.

M. R. COHEN-J. E. DRABKIN: A source book in Greek science. Mc Graw-Hill book company, inc. New York, Toronto, London, 1948.

A. REHM-K. VOGEL: Einleitung in die Altertumswissenschaft II, 5; Exakte Naturwissenschaften, Leipzig-Berlin, 1933.

ROBERT S. BRUMBAUGH: Plato's mathematical imagination. Indiana, University Press, Bloomington, 1954.

PAUL-HENRI MICHEL: De Pythagore à Euclide. *Soc. d' édit. Les Belles Lettres*, Paris, 1950.

B. L. van der WAERDEN: Die Arithmetik der Pythagoreer II, Die Theorie des Irrationalen. *Mathem. Annalen* B. **120**, Heft 5/6, 1949, Springer Verlag, Berlin, Göttingen, Heidelberg.

K. Reidemeister: Die Arithmetik der Griechen. Leipzig - Berlin, 1940.
O. Neugebauer: The exact sciences in antiquity. Kopenhagen, 1951.
M. Cantor: Vorlesungen über Geschichte der Mathematik I, 1907, Teubner.
T. Heath: A history of Greek mathematics, I, II. Oxford at the Clarendon Press 1921.
Κ. Δ. Γεωργουλης: Ἡ ἑλληνικὴ ἐπιστήμη, Ἐγκ. Λεξικὸν «Ἥλιος», Τόμος: Ἑλλὰς σ.;
W. A. Heidel: The heroic age of science. Baltimore 1933.
F. Enriques - G. Santillana: Storia del pensiero scientifico. Bologna 1932.
G. Loria: Le scienze esatte nell' Antica Grecia. Milano, 1895, 1914.
A. Rey: La science grecque. Paris, 1933.
P. Tannery: Mémoires scientifiques I, II, III. Toulouse, Paris.
H. Hasse - H. Scholz: Die Grundlagenkrisis der griechischen Mathematik, Berlin-Charlottenburg 1928.