ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **Nonlinear differential equations with several general solutions,** *by Demetrios G. Magiros* *.

᾿Ανεκοινώθη ὑπὸ τοῦ ᾿Ακαδημαϊκοῦ κ. ᾿Ιω. Ξανθάκη.

I. INTRODUCTION

Nonlinear differential equations constitute today a field of scientific knowledge basic for the investigation of the majority of physical phenomena, technological systems, social problems.

The general solutions of these DE are the most desirable, especially in applications, but only for some nonlinear DE the general solutions are known to exist, and only in a few classes of these DE the general solutions can be determined in a closed form.

While any linear ordinary DE has one general solution, a nonlinear ODE may have either one or several general solutions.

It is necessary here to make clear some concepts related to that of the general solution. We consider a nonlinear ODE (F) valid in a region R of the space of its variables, and a function (Φ) of the variables of (F) containing a number of arbitrary constants independent to each other, which can take the values in a region C of the space of these arbitrary constants. A function coming from (Φ) by a specification of all its arbitrary constants, if it satisfies the DE (F), is called a «particular solution» of (F). The function (Φ), a totality of all the particular solutions of (F) coming from (Φ), is called a «general solution» of (F).

Any function, coming from the general solution (Φ) of (F) by a specification of some only arbitrary constants of (Φ), which, then, contains the unspecified constants of (Φ) as arbitrary parameters, is called a «part of the general solution» of (F).

If there are functions (Φ_i), $i = 1, 2, \dots, n$, which have properties similar to that of (Φ) in connection to the DE (F), and, in addition, particular solutions of one of the (Φ_i) are not identical to particular solutions of any of the others (Φ_i), these functions must be considered as «distinct general solutions» of (F), and, in this case, we say that the DE

* Δ. Γ. ΜΑΓΕΙΡΟΥ, **Μὴ γραμμικαὶ διαφορικαὶ ἐξισώσεις μὲ πολλὰς γενικὰς λύσεις.**

(F) has «several general solutions». By this definition, is not excluded that a «common part» among some of (Φ_i) may exist.

In this paper remarks are given concerning the existence and determination of several general solutions of some nonlinear ODE.

The «restriction of the solutions of the DE», and the «factorization of the DE» can be used as methods for investigation. These methods are explained and illustrated by examples from which we draw certain conclusions.

II. THE METHOD OF RESTRICTING THE SOLUTIONS

By this method, if it is applicable, one can determine several general solutions of a given nonlinear DE in a closed form. We give an example.

Example 1: $$y'''(1+y'^2) - 2y'y''^2 = 0 \tag{1.1}$$

One can have two appropriate restrictions of the unknown solution of (1.1), and to each restriction a general solution of (1.1) corresponds.

(a): The solution y of (1.1) is restricted by $y'' = 0$, which implies $y''' = 0$. In this case, (1.1) is satisfied simultaneously by:

$$y'' = 0, \quad y''' = 0 \tag{1.2}$$

when the function

$$y = \alpha_1 x + \alpha_2 \tag{1.3}$$

which satisfies (1.2) is a general solution of (1.1). The α_1 and α_2 are independent to each other and the two-parameter family (1.3) represents «all straight lines» in the x, y - plane.

(b): The solution y of (1.1) is restricted by $y'' \neq 0$, and, in this case, one can determine a general solution of (1.1) which is different than (1.3). The DE (1.1) in this case can be written in the form: [1]

$$\frac{y'''}{y''} = \frac{3y'y''}{1+y'^2} = \frac{3}{2} \cdot \frac{d(1+y'^2)}{1+y'^2}. \tag{1.4}$$

An integration of this equation gives:

$$\frac{y''}{(1+y'^2)^{3/2}} = c_1.$$

A new integration leads to:

$$y' = \frac{c_1x + c_2}{\sqrt{1 - (c_1x + c_2)^2}} \tag{1.5}$$

and a third integration to:

$$c_1y + c_3 = -\sqrt{1 - (c_1x + c_2)^2} \tag{1.6}$$

which can get the form:

$$x^2 + y^2 + C_1y + C_2y + C_3 = 0. \tag{1.7}$$

The C_1, C_2, C_3, functions of c_1, c_2, c_3, are arbitrary constants independent to each other, and the three-parameter family (1. 7), which represents «all circles» in x, y - plane, is a general solution of (1. 1). There is no specification of the arbitrary constants of (1. 3) and (1. 7) by which a member of (1. 3) becomes identical with a member of (1. 7), then (1. 3) and (1. 7) are two distinct general solutions of (1. 1).

III. THE METHOD OF FACTORIZING THE DE

A possible application of this method can give several general solutions of a DE. We give examples.

Example 2: $x^3y''y''' + x^2y''^2 - 2xy'y'' + 2yy'' = 0.$ (2.1)

The factorization gives:

$$y''(x^3y''' + x^2y'' - 2xy' + 2y) = 0 \tag{2.2}$$

then:

$$\left.\begin{array}{ll} (a): & y'' = 0 \\ (b): & x^3y''' + x^2y'' - 2xy' + 2y = 0, \quad x \neq 0 \end{array}\right\} \tag{2.3}$$

the general solutions of which are:

$$\left.\begin{array}{ll} (a): & y = a_1x + a_2 \\ (b): & y = c_1x + c_2x^2 + c_3x^{-1} \end{array}\right\} \tag{2.4}$$

and these functions are the general solutions of (2. 1).

The family of straight lines through the origin is a «common part» of the general solutions (2. 4) of the DE (2. 1).

Example 3: $y'^2 + y(x^2y - xy) - x^3y^2 = 0$ (3.1)

By factorizing one has :

$$(y' - xy) \,.\, (y' + x^2y) = 0 \tag{3.2}$$

then :

$$\left.\begin{aligned} &(a): \quad y' = xy \\ &(b): \quad y' = -x^2y \end{aligned}\right\} \tag{3.3}$$

from which :

$$\left.\begin{aligned} &(a): \quad y = c_1 e^{x^2/2} \\ &(b): \quad y = c_2 e^{-x^3/3} \end{aligned}\right\} \tag{3.4}$$

that is the general solutions of the DE (3. 1) :

We remark that from each point of the x, y - plane two, in general, integral curves of (3. 1) pass with different slopes, one belonging to the family (3. 4. a) corresponding to a specific value of c_1, and one belonging to the family (3. 4. b) for a specific value of c_2, and these values of c_1 and c_2 are different.

The equation $(xy = -x^2y)$ gives the singular lines: $x = 0$, $x = -1$, $y = 0$, at the points of which the integral curves have just one slope, the slope $-y$ at $x = -1$, and the zero-slope at the coordinate axes, which are the «singular solutions» of the DE (3. 1).

Example 4:

«The general nonlinear DE of the first order :

$$F(x, y, y') = 0 \tag{4.1}$$

in case F is a polynomial for y' of degree m».
In this case, (4. 1) can have the form :

$$F(x, y, y') \equiv y'^m + P_1 y'^{m-1} + \ldots + P_{m-1} y' + P_m = 0 \tag{4.2}$$

where $P_1, \ldots, P_m$, functions of x and y, are continuously differentiable in the region of validity of (4. 2).

$$F_{y'}(x, y, y') \neq 0.$$

The DE (4. 2) if $F_{y'}(x, y, y') \neq 0$, solved for y', gives m simple roots : $y'_1, \ldots, y'_m$, when one can write :

$$F(x, y, y') \equiv (y'_1 - p_1) \cdot (y'_2 - p_2) \ldots (y'_m - p_m) = c \tag{4.3}$$

where $p_1, \ldots, p_m$ are functions of x and y.

The DE (4. 3) is equivalent to the m DE :

$$y'_1 = p_1(x, y), \quad y'_2 = p_2(x, y), \ldots, \quad y'_m = p_m(x, y) \tag{4.4}$$

from which, by integration, one can get the functions :

$$\varphi_1(x, y, c_1) = 0, \quad \varphi_2(x, y, c_2) = 0, \ldots \quad \varphi_m = (x, y, c_m = 0 \tag{4.5}$$

which are the m general solutions of the DE (4. 2).

We remark that m is the number of the factors of (4. 3), but we restrict ourselves to those factors which lead to real functions of real variables, when the number of the general solutions of (4. 2) is either equal or smaller than m. Simple example for that is the DE :

$$F = y'^4 - 1 = 0 \tag{4.6}$$

for which one can have:

$$F \equiv (y'^2 - 1)(y'^2 + 1) = (y' - 1)(y' + 1)(y'^2 + 1) = 0$$

and since $y'^2 + 1 \neq 0$, only the first two factors count, to which the DE: $y' = 1$, $y' = -1$ correspond, and then the general solutions of (4. 6) are :

$$y = x + c_1, \quad y = -x + c_2 \tag{4.7}$$

Example 5: $$y'^2 y''' + y^2 y''' + y''' = 0 \tag{5.1}$$

The factorization gives :

$$y'''(y'^2 + y^2 + 1) = 0 \tag{5.2}$$

and, since $y'^2 + y^2 + 1 \neq 0$, (5. 2) is equivalent to :

$$y''' = 0 \tag{5.3}$$

of which the general solution is :

$$y = c_1 x^2 + c_2 x + c_3 \tag{5.4}$$

For any set of values of the arbitrary constants for which $c_1 \neq 0$, (5. 4) represents parabolas in the x, y -plane. For $c_1 = 0$, (5. 4) reduces to :

$$y = c_2 x + c_3 \tag{5.5}$$

which represents the straight lines in the x, y -plane.

The functions (5.4) and (5. 5), which contain arbitrary constants and satisfy (5. 1), are not two distinct general solutions of (5. 1), but (5. 5) is «part of the general solution (5. 4)».

Examle 6: «The Emden's Equation:

$$y'' + \frac{2}{x} y' + y^n = 0 ». \tag{6.1}$$

We use this basic in Astrophysics DE in order to give appropriate remarks in connection with the general solutions of a DE and the physical reality from which the DE comes.

Emden (1907) [2] examined the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of Thermodynamics, and found the DE (6.1).

The solutions of (6.1), in a closed form, are:

$$\left.\begin{array}{lll} (a): & n = 0: & y = \alpha + \dfrac{b}{x} - \dfrac{x^2}{6} \\ (b): & n = 1: & y = \alpha \dfrac{\sin x}{x} + b \dfrac{\cos x}{x} \\ (c): & n = 5: & y = \left(\dfrac{3\alpha}{x^2 + 3a^2}\right)^{1/2} \end{array}\right\} \tag{6.2}$$

The first and second of the functions (6.2) are general solutions of the corresponding DE, while the third one, which contains α as an arbitrary constant, is a part of the unknown general solution of (6.1).

The physical situation considered by Emden lead him to the boundary conditions:

$$x = 0, \; y = 1, \; y' = 0 \tag{6.3}$$

to which the following particular solutions correspond:

$$\left.\begin{array}{lll} (a): & n = 0: & y = 1 - \dfrac{x^2}{6} \\ (b): & n = 1: & y = \dfrac{\sin x}{x} \\ (c): & n = 5: & y = \left(1 + \dfrac{x^2}{3}\right)^{-1/2} \end{array}\right\} \tag{6.4}$$

Comparing the theoretical results (6.4) with the physical situation of his problem, Emden found that these results are not acceptable physically, and that, from a physical point of view, n of (6.1) must have values only between 0 and 5.

The general solution of (6.1) must contain two arbitrary constants, but one of these it that called a «constant of homology», which does not help in finding the general solution of (6.1).

For the solution of (6.1) needed, if n is in (0,5), and x takes the values of an adequate interval, the investigation was turned to use a Taylor's expansion about $x = 0$ and, then, an analytic continuation of the series. Tables have been computed and graphical representations.

Astrophysicists use today the tables of the Emden's functions, appropriately modified, in order to estimate the density and the interior temperature of stars and some other things, by taking into account new physical data.

R e m a r k. If one follows the converse prosedure, that is, if a function with arbitrary constants in a variety of forms is given, and, then, one formulates, in the known way, the correspondent DE, one may get more rich results in connection with the subject of the paper.

Π Ε Ρ Ι Λ Η Ψ Ι Σ

Αἱ μὴ γραμμικαὶ διοφορικαὶ ἐξισώσεις ἀποτελοῦν σήμερον τὸ πλέον βασικὸν πεδίον ἐπιστημονικῆς γνώσεως διὰ τὴν ἔρευναν φυσικῶν φαινομένων, τεχνολογικῶν συστημάτων, κοινωνικῶν προβλημάτων.

Αἱ γενικαὶ λύσεις τῶν ἐξισώσεων αὐτῶν εἶναι αἱ περισσότερον ἐπιθυμηταὶ λύσεις των, ἀλλὰ μόνον εἰς μερικὰς κλάσεις τῶν ἐξισώσεων αὐτῶν αἱ γενικαὶ λύσεις εἶναι δυνατὸν νὰ προσδιορισθοῦν. Ἐνῷ εἰς τὰς γραμμικὰς διαφορικὰς ἐξισώσεις ὑπάρχει μόνον μία γενικὴ λύσις, εἰς τὰς μὴ γραμμικὰς δύναται κανεὶς νὰ εὕρη μίαν ἢ καὶ περισσοτέρας γενικὰς λύσεις.

Ἡ παροῦσα ἐργασία σχετίζεται μὲ τὴν ὕπαρξιν καὶ τὸν προσδιορισμὸν πολλῶν γενικῶν λύσεων εἰς συνήθεις μὴ γραμμικὰς διαφορικὰς ἐξισώσεις. Ὁ «κατάλληλος περιορισμὸς τῶν λύσεων τῆς ἐξισώσεως», ὅπως καὶ ἡ «ἀναγωγὴ τῆς ἐξισώσεως εἰς γινόμενον παραγόντων» χρησιμοποιοῦνται εἰς τὴν ἐργασίαν αὐτὴν ὡς μέθοδοι ἐρεύνης, ποὺ ἐφαρμόζονται εἰς εἰδικὰ παραδείγματα.

Αἱ γενικαὶ λύσεις, παρὰ τὴν χρησιμότητά των, δὲν δύνανται ἐνίοτε νὰ ἐκφράσουν τὴν φυσικὴν πραγματικότητα, καὶ ὁ προσδιορισμός των, ἀπὸ ἀπόψεως ἐφαρμογῶν, εἶναι ἐνίοτε ἄσκοπος. Τοῦτο ὑποδεικνύεται ἀπὸ τὸ τελευταῖον παράδειγμα.

REFERENCES

1. E. Goursat, «Differential Equations», Ginn and Company, Boston (U. S. A.), 1917, p. 5.
2. H. Davis, «Introduction to Nonlinear Differential and Integral Equations», Dover Publications Inc. N. Y. (1962) pp. 371 - 377.

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Ὁ Ἀκαδημαϊκὸς κ. **Ἰω. Ξανθάκης,** παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν, εἶπε τὰ ἑξῆς :

Ἔχω τὴν τιμὴν νὰ παρουσιάσω τὴν ἐργασίαν τοῦ κ. Δημητρίου Μαγείρου ὑπὸ τὸν τίτλον: «Μὴ γραμμικαὶ διαφορικαὶ Ἐξισώσεις μὲ πολλὰς γενικὰς λύσεις». Αἱ μὴ γραμμικαὶ διαφορικαὶ ἐξισώσεις διαδραματίζουν σπουδαῖον ρόλον εἰς τὴν ἔρευναν τόσον φυσικῶν φαινομένων ὅσον καὶ τεχνολογικῶν συστημάτων καὶ κοινωνικῶν προβλημάτων. Ὑπάρχει ὅμως μία σημαντικὴ διαφορὰ μεταξὺ τῶν γραμμικῶν καὶ μὴ γραμμικῶν διαφορικῶν ἐξισώσεων. Πράγματι ἐνῷ διὰ τὰς γραμμικὰς διαφορικὰς ἐξισώσεις ὑπάρχει πάντοτε μόνον μία γενικὴ λύσις, εἰς τὰς μὴ γραμμικὰς εἶναι δυνατὸν νὰ ὑφίστανται περισσότεραι τῆς μιᾶς γενικαὶ λύσεις. Ἐπιπροσθέτως αἱ γενικαὶ λύσεις τῶν μὴ γραμμικῶν διαφορικῶν ἐξισώσεων δὲν εἶναι δυνατὸν πάντοτε νὰ προσδιορισθοῦν, μόνον δηλαδὴ εἰς ὡρισμένας κλάσεις τῶν ἐξισώσεων τούτων δύναταί τις νὰ ἀνεύρῃ μίαν ἢ περισσοτέρας γενικὰς λύσεις.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν ὁ κ. Δημ. Μάγειρος μελετᾷ τὴν ὕπαρξιν καὶ τὸν προσδιορισμὸν πολλῶν γενικῶν λύσεων συνήθων μορφῶν μὴ γραμμικῶν διαφορικῶν ἐξισώσεων.

Πρὸς τοῦτο χρησιμοποιεῖ ὡς μεθόδους ἐρεύνης ἀφ' ἑνὸς μὲν τὸν κατάλληλον περιορισμὸν τῆς λύσεως τῆς ἐξισώσεως, ἀφ' ἑτέρου δὲ τὴν ἀναγωγὴν τῆς ἐξισώσεως εἰς γινόμενον παραγόντων. Πρέπει νὰ σημειώσῃ τις ὅτι αἱ γενικαὶ λύσεις τῶν μὴ γραμμικῶν διαφορικῶν ἐξισώσεων δὲν δύνανται ἐνίοτε νὰ ἐκφράσουν τὴν φυσικὴν πραγματικότητα καὶ ὁ προσδιορισμός των, ἀπὸ ἀπόψεως ἐφαρμογῶν, εἶναι ἐνίοτε ἄσκοπος. Τὸ τελευταῖον τοῦτο ἀποδεικνύεται ἀπὸ τὴν διαφορικὴν ἐξίσωσιν τοῦ Emden. Πράγματι ὁ Emden μελετώντας κατὰ τὰς ἀρχὰς τοῦ παρόντος αἰῶνος τὴν συμπεριφορὰν μιᾶς σφαιρικῆς ἀεριώδους μάζης ὑπὸ τὴν ἐπίδρασιν τῶν ἀμοιβαίων ἕλξεων τῶν μορίων της καὶ τὸν κλασσικὸν νόμον τῆς θερμοδυναμικῆς κατέληξε εἰς τὴν μὴ γραμμικὴν διαφορικὴν ἐξίσωσιν :

$$y'' + \frac{2}{x} y' + y^n = 0$$

Ἡ ἐξίσωσις αὕτη διὰ τὰς τιμὰς τοῦ ἐκθέτου $n = 0, 1 \ldots\ldots$ καὶ 5, δέχεται τρεῖς διαφόρους λύσεις ἐκ τῶν ὁποίων αἱ δύο πρῶται διὰ $n = 0$ καὶ $n = 1$ εἶναι γενικαὶ λύσεις, ἐνῷ ἡ τρίτη $n = 5$ εἶναι ἓν μέρος μιᾶς ἀγνώστου γενικῆς λύσεως.

Συγκρίνοντας τὰ θεωρητικὰ ἐξαγόμενα μὲ τὰ φυσικὰ δεδομένα τοῦ προβλήματος ὁ Emden εὗρε ὅτι τὰ θεωρητικὰ ταῦτα ἐξαγόμενα δὲν ἀντιπροσωπεύουν τὴν φυσικὴν πραγματικότητα.

Ἀπὸ φυσικῆς ἀπόψεως, ὁ ἐκθέτης n πρέπει νὰ λαμβάνῃ τιμὰς μεταξὺ 0 καὶ 5.

Οἱ ἀστροφυσικοὶ τῆς σήμερον κάμνουν εὐρεῖαν χρῆσιν τῶν συναρτήσεων Emden καταλλήλως τροποποιημένων ὑπὸ μορφὴν ἀριθμητικῶν πινάκων διὰ τὴν ἐκτίμησιν τῆς πυκνότητος καὶ τῆς θερμοκρασίας εἰς τὸ ἐσωτερικὸν τῶν ἀστέρων.