ΜΑΘΗΜΑΤΙΚΑ.— **The general solutions of nonlinear differential equations as functions of their arbitrary constants,** *by Demetrios G. Magiros**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰω. Ξανθάκη.

INTRODUCTION

One of the major difficulties encountered in the general solutions of nonlinear differential equations, in contrast to the solutions of linear ones, is the manner in which the arbitrary constants enter these solutions. In this paper information is given and results are found in connection with this subject.

In nonlinear DE, where the nonlinearity character is an essential factor, we have not, according to the present status, much information concerning the general solutions of these DE, as well as the manner in which the arbitrary constants enter to these solutions.

Given such a DE, the first subject for its solution is to attempt to find a general solution expressed in terms of the classical functions. To do this one must discover transformations, which may reduce the DE to some types that are known to have solutions of the desired kind, but this is, in general, either difficult or impossible.

It is customary to regard a linear DE as solved, if its solution can be reduced to the quadrature of a known function even though the quadrature can not be expressed in terms of classical functions.

In the same sense, one may regard a nonlinear DE as solved, if it can be reduced to the solution of a linear DE, even though the solution is not explicitly reducible to the classical functions.

In many cases of the nonlinear DE the classical functions are inadequate to express the solutions. Numerous nonclassical functions have been defined and partially explored in various ways recently, so there exists today an impressible collection of them from which one attempts to construct the general solutions of nonlinear DE.

The arbitrary constants enter the general solutions of nonlinear DE either «linearly» in a few cases, or «rationally linearly» in some other

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cases. Nevertheless, in the majority of the cases, the arbitrary constants enter the general solutions in a «complicated nonlinear way», which characterizes the nature of the general solutions and their corresponding DE.

The singularities, especially the essential singularities, appearing in the general solutions, make these solutions, as well as the manner in which the arbitrary constants enter them, very complicated.

I. GENERAL SOLUTIONS AS LINEAR FUNCTIONS OF THEIR ARBITRARY CONSTANTS

In a linear DE, the general solution is a linear combination of the fundamental set of solutions, and this general solution contains the arbitrary constants linearly. Such a general solution interprets the «principle of superposition», which holds only in linear DE.

As an example, we take the «Bessel DE»:

$$x^2y'' + xy' + (x^2 - n^2)\,y = 0$$

of which the general solution is of the form:

$$y = c_1 J_n(x) + c_2 Y_n(x)$$

where c_1, c_2 are the two arbitrary constants, and $J_n(x)$, $Y_n(x)$ the «Bessel functions» of the first and second kind, respectively, which are «special functions» particular solutions of the DE.

We remark that the principle of superposition is sometimes difficult to apply, as, e.g. the general solution of the Mathieu equation:

$$y'' + (a + b\cos 2x)\,y = 0$$

is not known containing two arbitrary constants in the above way **[2]**.

In the following we find classes of nonlinear DE of which the general solutions are linear functions of their arbitrary constants.

Some special methods are applied, illustrated by proper examples.

I. 1. **The Factorization Method.**

By applying the factorization method to a nonlinear DE, if this method is applicable, one can have several general solutions of the DE, which contain the arbitrary constants linearly.

A general example is the first order DE: $F(x, y, y') = 0$ in case it is a polynomial for y' of order m. In this case, one can write:

$$F(x, y, y') \equiv y'^{m} + P_1(x, y) \cdot y'^{m-1} + \ldots + P_{m-1}(x, y) \cdot y' + P_m(x, y) = 0. \quad (a.1)$$

This DE can be solved under the restriction $F_{y'}(x, y, y') \neq 0$, and if $y_1', \ldots, y_m'$ are the m simple roots, one can get F in the product form:

$$F \equiv [y_1' - p_1(x, y)] \cdot [y_2' - p_2(x, y)] \ldots [y_m' - p_m(x, y)] = 0, \quad (a.2)$$

which is equivalent to the m DE:

$$y_1' = p_1(x, y), \quad y_2' = p_2(x, y), \quad \ldots, \quad y_m' = p_m(x, y). \quad (a.3)$$

Integrating, the m general solutions of (a. 1) are:

$$\varphi_1(x, y, c_1) = 0, \quad \varphi_2(x, y, c_2) = 0, \quad \ldots, \quad \varphi_m(x, y, c_m) = 0. \quad (a.4)$$

There are forms of the functions p_i in (a. 3) which lead to functions φ_i as linear functions of the arbitrary constants; for instance, in the cases where DE (a. 3) are with separable variables, or they are exact DE, or p_i are homogeneous functions of, say, degree n, etc.

Example 1: $$y'^2 + y'(x^2y - xy) - x^3y^2 = 0. \quad (1.1)$$

The factorization gives: $(y' - xy) \cdot (y' + x^2y) = 0$, when the two distinct general solutions of (1. 1), containing the arbitrary constants linearly, are:

$$y = c_1 e^{x^2/2}, \quad y = c_2 e^{-x^3/3}. \quad (1.2)$$

Example 2: $$x^3y''y''' + x^2y''^2 - 2xy'y'' + 2yy'' = 0. \quad (2.1)$$

The factorization gives: $y''(x^3y''' + x^2y'' - 2xy' + 2y) = 0$, when the two general solutions of (2. 1) are:

$$\text{(a)} \quad y = a_1 x + a_2, \qquad \text{(b)} \quad y = c_1 x + c_2 x^2 + c_3 x^{-1}. \quad (2.2)$$

These two general solutions contain the family of straight lines through the origin ($a_2 = c_2 = c_3 = 0$, $a_1 = c_1$) as a «common part».

Example 3: $$y'^2 y''' + y^2 y''' + y''' = 0, \quad (3.1)$$

y''' is a common factor of the terms, then $y'''(y'^2 + y^2 + 1) = 0$, and since $y'^2 + y^2 + 1 \neq 0$, the DE (3.1) is equivalent to $y''' = 0$, of which the general solution is:

$$y = c_1 x^2 + c_2 x + c_3, \tag{3.2}$$

which consists of two different families of curves, the parabolas ($c_1 \neq 0$) and the straight lines ($c_1 = 0$).

Example 4: $$y'^3 - \frac{y^2 - x^2}{2xy} y'^2 - y' + \frac{y^2 - x^2}{2xy} = 0. \tag{4.1}$$

Factorizing, one can get:

$$(y'^2 - 1) \cdot \left(y' - \frac{y^2 - x^2}{2xy} \right) = 0 \tag{4.2}$$

which is equivalent to:

$$\text{(a)} \quad y' = \pm 1, \qquad \text{(b)} \quad y' = \frac{y^2 - x^2}{2xy} \tag{4.3}$$

The first of (4.3) gives two general solutions of (4.1):

$$\text{(a)} \quad y = x + c_1, \qquad \text{(b)} \quad y = -x + c_2. \tag{4.4}$$

The second DE of (4.3) has its right hand member as a homogeneous function of second degree, and can be written as:

$$y' = \frac{(y/x)^2 - 1}{2(y/x)}. \tag{4.5}$$

By using the transformation $y = xu$, the DE (4.5) can be solved and its general solution can be found to be:

$$x^2 + y^2 - cx = 0. \tag{4.6}$$

The three functions (4.4, a, b) and (4.6) are the three general solutions of the DE (4.1), containing the arbitrary constants linearly.

I. 2. **The Method of Restricting Quantities of the DE.**

By restricting quantities of the DE one may have several general solutions of the DE containing the arbitrary constants linearly. We give two examples.

Example 5: $$y'(1 + y'^2) - 2y'y'' = 0. \tag{5.1}$$

We restrict the unknown function y of (5. 1) as follows:

(a): If y is such that $y''=0$, which implies $y'''=0$, then the general solution of (5. 1) is:

$$y = a_1 x + a_2 . \tag{5.2}$$

(b): If y is such that $y''\neq 0$, then one can integrate (5. 1) exactly, and its new general solution is:

$$x^2 + y^2 + c_1 x + c_2 y + c_3 = 0. \tag{5.3}$$

Both functions (5 2) and (5. 3) are distinct general solutions of (5. 1) and contain the arbitrary constants linearly.

Example 6:

$$y'' + \frac{2}{x} y' + y^n = 0. \tag{6.1}$$

This is the famous Emden equation, coming from his investigation on basic problems of astrophysics. The solutions of this DE in a closed form are [**1**]:

$$\left.\begin{aligned} &(a)\quad n = 0: \quad y = a_1 + \frac{a_2}{x} - \frac{x^2}{6} \\ &(b)\quad n = 1: \quad y = c_1 \frac{\sin x}{x} + c_2 \frac{\cos x}{x} \\ &(c)\quad n = 5: \quad y = \left(\frac{3a}{x^2 + 3a^2}\right)^{1/2}. \end{aligned}\right\} \tag{6.2}$$

a, a_1, a_2, c_1, c_2 are arbitrary constants. The first two functions are the only known general solutions of the corresponding linear DE of (6. 1) containing the arbitrary constants linearly. The third function of (6. 2), containing one arbitrary constant nonlinearly, is a «part» of the unknown general solution of (6. 1) in case $n = 5$.

On the occasion of this DE, we remark that any attempt to find the general solutions of (6. 1) for other values of n will be governed rather by a theoretical curiosity than by its usefulness, since, even if we know it, it has, according to Emden, no physical meaning in the Emden problems of astrophysics. Emden and his followers in astrophysics found a solution of (6. 1) in the form of a Taylor series, which interprets the reality very adequately.

I. 3. A class of nonlinear DE with general solutions containing the arbitrary constants linearly.

We can find a class of nonlinear DE of which the general solution contains the arbitrary constants linearly.

Starting from the simple primitive:

$$y^2 = 4x\,; \quad y \neq 0, \quad x > 0$$

by differentiation one gets the DE: $yy' - 2 = 0$ of which the general solution is $y^2 = 4x + c_1$. Another differentiation gives: $yy'' + y'^2 = 0$ with general solution: $y^2 = 4x + c_1 x + c_2$. Continuing the differentiation up, say, to the order n, a nonlinear DE of order n results of which the general solution is:

$$y^2 = 4x + c_1 x^{n-1} + \ldots + c_{n-1} x + c_n\,,$$

where the arbitrary constants $c_1, \ldots, c_n$ enter linearly.

Generalizing the above, one can see that a nonlinear DE of order **n**:

$$F(y, y', \ldots, y^{(n)}) = \Phi(x), \tag{7.1}$$

where:

$$F = \frac{d^n}{dx^n} f(y), \qquad \Phi(x) = \frac{d^n}{dx^n} \varphi(x) \tag{7.2}$$

has as general solution the function:

$$f(y) = \varphi(x) + c_1 x^{n-1} + \ldots + c_{n-1} x + c_n\,. \tag{7.3}$$

We remark that the «principle of superposition» may be applicable to some nonlinear DE of which the general solutions are linear functions of the arbitrary constants, and this is an important problem.

II. GENERAL SOLUTIONS AS RATIONALLY LINEAR FUNCTIONS OF THE ARBITRARY CONSTANTS

We distinguish a class of nonlinear DE of which the general solutions are rationally linear functions of the arbitrary constants, that is the general solutions are ratios of functions, where the arbitrary constants enter linearly in the numerators and denominators.

This class of nonlinear DE is the class of «Riccati equations of any order».

II. 1. Let us start from the simple case:

$$y = \frac{a_1 + cb_1}{a_2 + cb_2} \tag{8.1}$$

where c is the arbitrary constant, and a_1, a_2, b_1, b_2 functions of x.

Eliminating c between (8.1) and its derivative, the resulting DE has (8.1) as a general solution. The differentiation gives:

$$y' = \frac{a_1' + cb_1' - y(a_2' + cb_2')}{a_2 + cb_2}. \tag{8.2}$$

From (8.1) and (8.2) one can get, respectively:

$$c = -\frac{a_1 - ya_2}{b_1 - yb_2}, \qquad c = -\frac{a_1' - ya_2' - y'a_2}{b_1 - yb_2' - y'b_2} \tag{8.2}$$

and equating these values of c, one has:

$$(b_1 a_2 - a_1 b_2) y' + (a_1' b_2 - a_1 b_2' + b_1 a_2' - b_1' a_2) y + \\ + (a_2 b_2' - a_2' b_2) y^2 = a_1' b_1 - a_1 b_1',$$

which is of the form:

$$y' = A_0(x) + A_1(x) y + A_2(x) y^2, \tag{8.3}$$

where:

$$\left.\begin{array}{ll} A_0 = (a_1' b_1 - a_1 b_1') / D, & A_1 = (a_1 b_2' - a_1' b_2 + b_1' a_2 - b_1 a_2') / D \\ A_2 = (a_2' b_2 - a_2 b_2') / D, & D = b_1 a_2 - a_1 b_2 \neq 0. \end{array}\right\} \tag{8.4}$$

The DE (8.3) is the «Riccati DE of order first», and the primitive (8.1) is its general solution, which contains the arbitrary constant c rationally linearly. We remark that the transformation:

$$y = \frac{u'}{A_3 u}, \tag{8.5}$$

applied to (8.3), leads to a linear DE of order two in u, when the DE (8.3) can be regarded as a solved DE.

As a simple example of the above is the DE: $y' = -2xy + xy^2$, which is of the form (8. 3), and which has as a general solution the function: $y = 2/(1 + ce^{x^2})$ of the form (8. 1).

II. 2. A natural generalization of (8. 1) is the function:

$$y = \frac{c_1 v_1 + \ldots + c_n v_n}{c_1 w_1 + \ldots + c_n w_2} \tag{8. 6}$$

where c_i are the arbitrary constants, and v_i and w_i arbitrary functions of x. This generalization was introduced by E. Vessiot (1895) and G. Wallenberg (1899) [**1**]. The elimination of c_i gives a nonlinear DE of order n, called a «Riccati DE of order n», of which (8. 6) is its general solution.

By a proper transformation, the solution (8. 6) of the Riccati DE of order n can be expressed in terms of the solutions of a linear DE of order $(n+1)$, which corresponds to this Riccati DE.

The function (8. 6) in case all of w_i are zero, except one of them, say $w_n \neq 0$, becomes:

$$y = \bar{c}_1 \frac{v_1}{w_n} + \ldots + \bar{c}_{n-1} \frac{v_{n-1}}{w_n} + \frac{v_n}{w_n}, \tag{8. 7}$$

which contains the arbitrary constants $\bar{c}_i = (c_i / c_n)$, $i = 1, \ldots, (n-1)$ linearly, and it is the general solution of a linear DE of order $(n-1)$.

The polynomial DE:

$$y' = A_0(x) + A_1(x) y + \ldots + A_n(x) y^n, \tag{8. 8}$$

which is a natural generalization of the Riccati DE (8. 3), in case all of A's are constants, can be integrated exactly and its general solution contains the arbitrary constant linearly.

III. THE GENERAL CASE OF THE GENERAL SOLUTIONS

In the general case of nonlinear DE the arbitrary constants enter into their general solutions in a «complicated nonlinear way», which characterizes the nature of the general solutions and their corresponding DE.

The singularities appearing in the general solutions make these solutions, as well as the manner in which the arbitrary constants enter them, very complicated.

An investigation on this line of problems is of theoretical and practical interest.

The writer has arrived at some results on these problems, but he considers these results as not yet sufficiently decisive to be communicated.

ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν παροῦσαν ἐργασίαν ἐρευνᾶται ὁ τρόπος μὲ τὸν ὁποῖον αἱ αὐθαίρετοι σταθεραὶ εἰσέρχονται εἰς τὰς γενικὰς λύσεις τῶν μὴ γραμμικῶν διαφορικῶν ἐξισώσεων.

Αἱ μὴ γραμμικαὶ ΔΕ δύνανται νὰ ὑπαχθοῦν εἰς τρεῖς κατηγορίας.

Ἡ πρώτη ἀποτελεῖται ἀπὸ τὰς κλάσεις τῶν ΔΕ τῶν ὁποίων αἱ γενικαὶ λύσεις περιέχουν τὰς αὐθαιρέτους σταθερὰς «γραμμικῶς».

Ἡ δευτέρα περιέχει τὰς ΔΕ εἰς τῶν ὁποίων τὰς γενικὰς λύσεις αἱ αὐθαίρετοι σταθεραὶ ὑπεισέρχονται «ρητῶς - γραμμικῶς».

Ἡ τρίτη περιέχει ὅλας τὰς ἄλλας ΔΕ εἰς τῶν ὁποίων τὰς γενικὰς λύσεις αἱ αὐθαίρετοι σταθεραὶ ὑπεισέρχονται κατὰ «πολύπλοκον μὴ γραμμικὸν τρόπον».

Ἡ διερεύνησις τῶν συνθηκῶν, ὑπὸ τὰς ὁποίας ἡ «ἀρχὴ τῆς ἐπιπροσθέσεως» (principle of superposition) τῶν λύσεων τῶν μὴ γραμμικῶν ΔΕ δύναται νὰ ἐφαρμοσθῇ, ἕνα σημαντικὸν πρόβλημα, ὑπάγεται εἰς τὴν πρώτην κατηγορίαν.

Ἡ δευτέρα κατηγορία τῶν ΔΕ ἀναφέρεται εἰς τὰς γενικὰς λύσεις τῶν ἐξισώσεων τοῦ Riccati οἱασδήποτε τάξεως.

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